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SOME FIXED POINT THEOREM IN GENERALIZED METRIC SPACES

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Abstract. The purpose of this paper is to obtain a new version of fixed point theorem in standard metric spaces using the notion of generalized metric space introduced by Jleli and Samet [M. Jleli and B. Samet, A *generalized metric space* and related fixed point theorems, *Fixed Point Theory and Applications*, (2015), 2012:61]. Moreover we have provided some examples which validates the introduced concept.

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1. Introduction

Let X be a Banach space and K a closed subset of X. In many applications the domain of a considered function is K, but the codomain is not entirely included in K. So it is of interest to amplify a class of such mappings which have a fixed point. Rhoades [2] introduced a class of

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non-self mappings T of K into X which satisfy the following contractive definition

$$d(Tx,Ty) \leq h \, \max\{\frac{d(x,y)}{2}, d(x,Tx), d(y,Ty), \frac{[d(x,Ty)+d(y,Tx)]}{q}\},$$

where h and q are reals satisfying $0 < h < 1, q \geq 1+2h$. Rhoades proved the fixed point result using this contraction. As pointed out by Rhoades [2, p. 459], the method of proof used in his Theorem 1 does not extend to more general contractive definitions in metric space. The results of Rhoades [2] type contraction has not only get considerable attention from mathematicians because of the development of the fixed point theory in standard metric spaces but also have been extended and improved in various ways and by now there exists a considerable literature in this direction for non-self mappings. To mention a few we cite [2, 3, 4].

In 2015, Jleli and Samet [1] introduced the new concept of generalized metric spaces and also shown that this new concept of generalized metric spaces recover various topological spaces including standard metric space, b-metric spaces, dislocate metric space, and modular spaces.

Due to the applications point of view, in this paper we establish fixed point theorems for generalized metric spaces using the Rhoades [2] type contraction. The presented theorem is an generalization of the results of Rhoades [2] by imposing the condition of self mapping.

Definition 0.1. Let X be a nonempty set and $\mathscr{D} : X \times X \to [0,+\infty]$ be a given mapping. For every $x \in X$, let us define the set

$$C(\mathscr{D},X,x) = \{\{x_n\} \subset X : \lim_{n\to\infty} \mathscr{D}(x_n,x) = 0\}$$

The formal definition of the generalized metric space introduced by Jleli and Samet [1] is as follows :

Definition 0.2. [1] We say $\mathscr{D}$ is generalized metric on X if it satisfying the following conditions:

($\mathscr{D}_1$) for every $(x,y) \in X \times X$, we have

$$\mathscr{D}(x,y) = 0 \;\Rightarrow\; x = y;$$

($\mathscr{D}_2$) for every $(x,y) \in X \times X$, we have

$$\mathscr{D}(x,y) = \mathscr{D}(y,x);$$

($\mathcal{D}_3$) there exist $\mathcal{C} > 0$ such that

$$if\ (x,y) \in X \times X, \{x_n\} \in C(\mathcal{D},X,x),\ then\ \mathcal{D}(x,y) \leq \mathcal{C} \lim_{n\to\infty} \sup \mathcal{D}(x_n,y)$$

Then the pair $(X,\mathcal{D})$ is a generalized metric spaces.

Note that a (usual) metric space is evidently a generalized metric space. However, the converse does not hold. The following example shows that a generalized metric space on X need not be a metric on X

Example 0.1. Let $X = [0,\infty)$ and $\mathcal{D}(x,y) = |x-y|^2$ for all $x,y \in X$. Clearly $\mathcal{D}$ is a generalized metric space with $\mathcal{C} = 2$. However $\mathcal{D}$ is not a metric space.

Remark 0.1. Clearly, if the set $C(\mathcal{D},X,x)$ is empty for every $x \in X$, then $(X,\mathcal{D})$ is a generalized metric spaces if and only if ($\mathcal{D}_1$) and ($\mathcal{D}_2$) are satisfied.

Definition 0.3. Let $(X,\mathcal{D})$ be a generalized metric spaces and let $\{x_n\}$ be a sequence in X. We say

- $\{x_n\}$ is $\mathcal{D}$-converges to x if $\{x_n\} \in C(\mathcal{D},X,x)$.
- $\{x_n\}$ is $\mathcal{D}$-Cauchy sequence if $\lim_{m,n\to\infty} \mathcal{D}(x_n,x_{m+n}) = 0$.

Definition 0.4. Let $(X,\mathcal{D})$ be a generalized metric spaces and let $f : X \to X$ be a mapping. We can say f is $\mathcal{R}$-type contraction if

$$\mathcal{D}(fx,fy) \leq h \max\left\{ \tfrac{1}{2}\mathcal{D}(x,y), \mathcal{D}(x,fx), \mathcal{D}(y,fy), \tfrac{\mathcal{D}(x,fy)+\mathcal{D}(y,fx)}{q} \right\} \tag{1}$$

for all $x \neq y$, where h and q are reals such that $h \in (0,1)$ and $q \geq 1+2h$.

Proposition 0.1. *Let f be a $\mathcal{R}$-type contraction for some $h \in (0,1)$ and $q \geq 1+2h$. Then any fixed point $z \in X$ of f satisfies*

$$\mathcal{D}(z,z) < \infty \Rightarrow \mathcal{D}(z,z) = 0.$$

Proof. Let $z \in X$ be a fixed point of f such that $\mathscr{D}(z,z) < \infty$. Since f is a $\mathscr{R}$-type contraction, we have

$$\begin{aligned}\mathscr{D}(z,z) &= \mathscr{D}(f(z), f(z))\\ &\leq h\max\{\frac{1}{2}\mathscr{D}(z,z), \mathscr{D}(z,f(z)), \mathscr{D}(f(z),z), \frac{\mathscr{D}(z,f(z))+\mathscr{D}(z,f(z))}{q}\}\\ &\leq h\,\mathscr{D}(z,z).\end{aligned}$$

Since $h \in (0,1)$ and $\mathscr{D}(z,z) < \infty$, this implies that $\mathscr{D}(z,z) = 0$. ■

2. Main Theorems

Theorem 0.1. *Let $(X,\mathscr{D})$ be a complete generalized metric spaces and let $f : X \to X$ satisfies the contraction (1). Let us assume that there exist a point $x_0 \in X$ such that $\delta(\mathscr{D}, f, x_0) < \infty$. Then the sequence $\{f^n(x_0)\}$ converges to $z \in X$, if $\mathscr{D}(f(z),x_0) < \infty$ and $\mathscr{D}(z, f(z)) < \infty$. Then z is fixed point of f.*

Proof. Let $n \in \mathbb{N}$. Since f is $\mathscr{R}$-type contraction, so one have

$$\begin{aligned}\mathscr{D}(f^{n+i}(x_0), f^{n+j}(x_0)) \leq h\max\{&\frac{1}{2}\mathscr{D}(f^{n-1+i}(x_0), f^{n-1+j}(x_0)), \mathscr{D}(f^{n-1+i}(x_0), f^{n+i}(x_0)),\\ &\mathscr{D}(f^{n-1+j}(x_0), f^{n+j}(x_0)), \frac{\mathscr{D}(f^{n-1+i}(x_0), f^{n+j}(x_0)) + \mathscr{D}(f^{n-1+j}(x_0), f^{n+i}(x_0))}{q}\}, \ \forall\, i,j \in \mathbb{N}\end{aligned}$$

This implies that

$$\begin{aligned}\delta(\mathscr{D}, f, f^n(x_0)) &\leq h\delta(\mathscr{D}, f, f^{n-1}(x_0))\\ &\leq h^2\delta(\mathscr{D}, f, f^{n-2}(x_0))\\ &\dots\\ &\dots\\ &\dots\\ &\leq h^n\delta(\mathscr{D}, f, x_0), \ \forall\, i,j \in \mathbb{N}. \qquad (2)\end{aligned}$$

On making use of the above inequality, we get

$$\mathscr{D}(f^n(x_0), f^{n+m}(x_0)) \le \delta(\mathscr{D}, f, f^n(x_0))$$

Using the Inequality (1), we get

$$\mathscr{D}(f^n(x_0), f^{n+m}(x_0)) \le h^n \delta(\mathscr{D}, f, x_0)$$

Also, $\delta(\mathscr{D}, f, x_0) < \infty$ and $h \in (0,1)$, thus we get

$$\lim_{m,n\to\infty} \mathscr{D}(f^n(x_0), f^{n+m}(x_0)) = 0$$

It follows that $\{f^n(x_0)\}$ is a $\mathscr{D}$-Cauchy sequence.

Since $(X, \mathscr{D})$ is complete, so there exist $z \in X$ such that $\{f^n(x_0)\}$ is $\mathscr{D}$-convergent.

Suppose $D(x_0, f(z)) < \infty$, and using the inequality

$$\mathscr{D}(f^n(x_0), f^{n+m}(x_0)) \le h^n \delta(\mathscr{D}, f, x_0) \tag{3}$$

for each $m, n \in \mathbb{N}$. By the property of $\mathscr{D}_3$ there exist $\mathscr{C} > 0$ such that

$$\begin{aligned} \mathscr{D}(z, f^n(x_0)) &\le \mathscr{C} \lim_{m\to\infty} \sup \mathscr{D}(f^n(x_0), f^{n+m}(x_0)) \\ &\le \mathscr{C}\ h^n \delta(\mathscr{D}, f, x_0) \end{aligned} \tag{4}$$

On the other hand, it is observed that

$$\begin{aligned} \mathscr{D}(f(x_0), f(z)) \le h \max\{&\frac{1}{2}\mathscr{D}(x_0, z), \mathscr{D}(x_0, f(x_0)), \\ &\mathscr{D}(z, f(z)), \frac{\mathscr{D}(f(x_0), z) + \mathscr{D}(x_0, f(z))}{q}\} \end{aligned}$$

Using (2) and (3) we get

$$\begin{aligned} \mathscr{D}(f(x_0), f(z)) \le \max\{&\frac{h}{2}\mathscr{C}\ \delta(\mathscr{D}, f, x_0), h\delta(\mathscr{D}, f, x_0), \\ &h\mathscr{D}(z, f(z)), \frac{h}{q}\ (\mathscr{D}(x_0, f(z)) + \mathscr{D}(z, f(x_0)))\} \end{aligned}$$

Using the above inequality, we get

$$\begin{aligned} \mathscr{D}(f^2(x_0), f(z)) \le \max\{&(\frac{h}{2})^2\ \mathscr{C}\ \delta(\mathscr{D}, f, x_0), h^2\delta(\mathscr{D}, f, x_0), \\ &h\ \mathscr{D}(z, f(z)), (\frac{h}{q})^2(\mathscr{D}(x_0, f(z)) + \mathscr{D}(z, f(x_0)))\} \end{aligned}$$

Similarly

$$\mathscr{D}(f^n(x_0), f(z)) \leq \max\{(\frac{h}{2})^n \mathscr{C}\ \delta(\mathscr{D}, f, x_0), h^n\delta(\mathscr{D}, f, x_0),$$
$$h\ \mathscr{D}(z, f(z)), (\frac{h}{q})^n(\mathscr{D}(x_0, f(z)) + \mathscr{D}(z, f(x_0)))\}\ \forall\, n \geq 1.$$

Therefore,

$$\limsup_{n\to\infty} \mathscr{D}(f^n(x_0), f(z)) \leq h\mathscr{D}(z, f(z)) \tag{5}$$

Note that $\mathscr{D}(x_0, f(z)) < \infty$ and $\delta(\mathscr{D}, f, x_0) < \infty$, this gives

$$\mathscr{D}(f(z), z) \leq \limsup_{n\to\infty} \mathscr{D}(f^n(x_0), f(z))$$

Now with the help of (5) the expression becomes,

$$\begin{aligned}\mathscr{D}(f(z), z) &\leq \limsup_{n\to\infty} \mathscr{D}(f^n(x_0), f(z))\\ &\leq h\mathscr{D}(z, f(z))\end{aligned}$$

$$\Rightarrow \mathscr{D}(f(z), z) \leq h\mathscr{D}(z, f(z))$$

$$\text{Consequently, } \mathscr{D}(f(z), z) = 0.$$

Also $\mathscr{D}(f(z), z) < \infty$ and $h \in (0,1)$.

This implies that z is a fixed point of f.

Therefore, $\mathscr{D}(z, z) = 0$ (due to Proposition 0.1). ■

Theorem 0.2. *If we additionally assume that $\mathscr{D}(z, z') < \infty$ and $\mathscr{D}(z', z') < \infty$ for a fixed point z' of X in Theorem 0.1. Then z is the unique fixed point of f.*

Proof. Using the proof of Theorem 0.1, we found that z is a fixed point of f. Now it remains to show that z is unique. To do this, first suppose that z' is another fixed point of X such that $\mathscr{D}(z, z') < \infty$ and $\mathscr{D}(z', z') < \infty$. Again by the proposition $\mathscr{D}(z', z') < \infty$ and z' is fixed point of f. This implies $\mathscr{D}(z', z') = 0$. Now our aim is to show that $\mathscr{D}(z, z') = 0$.
$\mathscr{D}(z, z') = \mathscr{D}(f(z), f(z'))$.

From the contraction condition (1), we have

$$\mathscr{D}(z,z') = \mathscr{D}(f(z),f(z')) \leq h\max\{\frac{1}{2}\mathscr{D}(z,z'),\mathscr{D}(z,f(z)),\mathscr{D}(f(z'),z'),$$
$$\frac{\mathscr{D}(z',f(z))+\mathscr{D}(z,f(z'))}{q}\}$$

This implies that

$$\mathscr{D}(z,z') \leq \mathscr{D}(z,z')$$
$$\Rightarrow \mathscr{D}(z',z') = 0.$$
$$z = z'.$$

Hence the proof is seen to be complete. ■

We now give two examples to illustrate the above results.

Example 0.2. Let $X = [0,2]$ and $\mathscr{D}(x,y) = |x-y|$ with $\mathscr{C} = 1$. Define the function $f : X \to X$ such that

$$f(x) = \begin{cases} \frac{x}{8},\ 0 \leq x < 2; \\ \frac{1}{4},\ x = 2. \end{cases}$$

If $0 \leq x,y < 2$, then $\mathscr{D}(fx,fy) = |fx-fy| = \frac{1}{8}|x-y| = \frac{1}{4}(\frac{1}{2}\mathscr{D}(x,y))$. This implies that $\mathscr{D}(fx,fy) < \frac{1}{2}\mathscr{D}(x,y)$. Therefore

$$\mathscr{D}(fx,fy) \leq h\ \max\{\frac{1}{2}\mathscr{D}(x,y),\mathscr{D}(x,f(x)),\mathscr{D}(f(y),y),\frac{\mathscr{D}(y,f(x))+\mathscr{D}(x,f(y))}{q}\}$$

It follows that the contraction Condition (1) is satisfied in this case.

Now if $0 \leq x < 2$ and $y = 2$ then,

$$\begin{aligned} \mathscr{D}(fx,fy) = |fx-fy| &= \frac{1}{8}x - \frac{1}{4} \\ &= \frac{1}{8}|x-2| = \frac{1}{8}|x-y| \\ &= \frac{1}{4}(\frac{1}{2}\mathscr{D}(x,y)) \\ \Rightarrow &\leq h\ \max\{\frac{1}{2}\mathscr{D}(x,y),\mathscr{D}(x,f(x)),\mathscr{D}(f(y),y),\frac{\mathscr{D}(y,f(x))+\mathscr{D}(x,f(y))}{q}\} \end{aligned}$$

Again the contraction Condition (1) is satisfied. It is easy to verify that all the conditions of Theorem 0.1 and Theorem 0.2 are satisfied and '0' is the unique fixed point of f.

and '0' is the unique fixed point of f.

Example 0.3. Let $X=[0,1]$ and $\mathscr{D}(x,y)=|x-y|^2$ for all $x,y\in X$. Clearly $\mathscr{D}$ is generalized metric space with $\mathscr{C}=2$. However it can be easily verified that $\mathscr{D}$ is not a metric space. Let us define a function $f:X\to X$ such that

$$f(x)=\begin{cases}\frac{x}{2},\ 0\leq x<1;\\ \frac{1}{2},\ x=1.\end{cases}$$

If $0\leq x,y<1$, then $\mathscr{D}(fx,fy)=|fx-fy|^2=\frac{1}{4}|x-y|^2=\frac{1}{2}(\frac{1}{2}\mathscr{D}(x,y))$. This implies that

$$\mathscr{D}(fx,fy)\leq h\ \max\{\frac{1}{2}\mathscr{D}(x,y),\mathscr{D}(x,f(x)),\mathscr{D}(f(y),y),\frac{\mathscr{D}(y,f(x))+\mathscr{D}(x,f(y))}{q}\}$$

It follows that the contraction condition is satisfied in this case.

On the other hand, if $0\leq x<1$ and $y=1$, then

$$\begin{aligned}\mathscr{D}(fx,fy)&=|fx-fy|^2=|\frac{1}{2}x-\frac{1}{2}|^2\\&=\frac{1}{4}|x-1|^2=\frac{1}{4}|x-y|^2\\&=\frac{1}{2}(\frac{1}{2}\mathscr{D}(x,y))\\ \Rightarrow&\leq h\ \max\{\frac{1}{2}\mathscr{D}(x,y),\mathscr{D}(x,f(x)),\mathscr{D}(f(y),y),\frac{\mathscr{D}(y,f(x))+\mathscr{D}(x,f(y))}{q}\}\end{aligned}$$

Again the contraction condition is satisfied. All the conditions of Theorem 0.1 and Theorem 0.2 are satisfied and '0' is the unique fixed point of f.

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