Available online at http://scik.org
Adv. Fixed Point Theory, 6 (2016), No. 4, 341-351
ISSN: 1927-6303

COUPLED FIXED POINT THEOREMS IN COMPLEX VALUED G_b-METRIC SPACES

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Abstract. In this paper, we introduce the notion of coupled fixed point for a mapping in complex valued Gb metric space and prove some coupled fixed point theorems in this space and provide an example in support of our main theorem.

Keywords: complex valued G_b metric space; coupled fixed point theorem; contractive type mapping.

2010 AMS Subject Classification: 47H10, 54H25.

1. Introduction

The concept of a metric space was introduced by Fréchet [16]. The first important result on fixed-point for contractive-type mappings was the well-known Banach fixed point theorem, published for the first time in 1922. After that many researchers proved the Banach fixed point theorem in a number of generalized metric spaces. Bakhtin [7] presented b-metric spaces as a generalization of metric spaces. In 2011, Azam et al. [4] introduced the notion of complex

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Received March 2, 2016

valued metric space which is a generalization of the classical metric space. Rao et al. [12] introduced the concept of complex valued b-metric space.

Mustafa and Sims [9] presented the notion of g-metric spaces, many researchers [1, 2, 3, 10, 11] obtained common fixed point results for G-metric spaces. The concept of G_b-metric space was given in [6].

E. Ozgur [15] presented the notion of complex valued G_b-metric space. In 2006, Bhaskar et al. [5] introduced the notion of coupled fixed point and proved some fixed point results in this context. Similarly, we introduced the notion of coupled fixed point for a mapping in complex valued G_b-metric spaces.

2. Preliminaries

In this section will recall some properties of G_b-metric spaces.

Definition 2.1 ([6]). Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a mapping $G : X \times X \times X \to R^+$ satisfies:

(Gb_1) $G(x,y,z) = 0$ if $x = y = z$;

(Gb_2) $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq y$;

(Gb_3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$;

(Gb_4) $G(x,y,z) = G(\rho\{x,y,z\})$, where ρ is a permutation of x,y,z;

(Gb_5) $G(x,y,z) \leq s(G(x,a,a) + G(a,y,z))$ for all $x,y,z,a \in X$ (rectangle inequality).

Then, G is called a generalized b-metric space and (X,G) is called a generalized b-metric or a G_b-metric space.

Note that each G_b-metric space is a G-metric space with $s = 1$.

Proposition 2.2 ([6]). *Let X be a G_b-metric space. Then for each $x,y,z,a \in X$, it follows that:*

(1) *If $G(x,y,z) = 0$ then $x = y = z$;*

(2) $G(x,y,z) \leq s(G(x,x,y) + G(x,x,z))$,

(3) $G(x,y,y) \leq 2sG(y,x,x)$*;*

(4) $G(x,y,z) \leq s(G(x,a,z) + G(a,y,z))$.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$.

Define a partial order on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $\mathrm{Re}(z_1) \leq \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) \leq \mathrm{Im}(z_2)$.

It follows that $z_1 \precsim z_2$ if one of the following condition is satisfied.

(1) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,

(2) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$,

(3) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,

(4) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ If $z_1 \neq z_2$ and one of (i), (ii) and (iii) is satisfied and we will write $z_1 \prec z_2$ iff (iii) is satisfied.

The followed statements hold:

(i) If $a, b \in R$ with $a \leq b$, then $az \prec bz$, for all $z \in \mathbb{C}$.

(ii) If $0 \precsim z_1 \precnsim z_2$, then $|z_1| < |z_2|$.

(iii) If $z_1 \precsim z_2$, $z_2 \prec z_3$, them $z_1 \prec z_3$.

Definition 2.3 ([15]). *Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a mapping $G : X \times X \times X \to \mathbb{C}$ satisfies:*

(CG$_b$1) $G(x,y,z) = 0$ *if* $x = y = z$*;*

(CG$_b$2) $0 \prec G(x,x,y)$ *for all* $x, y \in X$ *with* $x \neq y$*;*

(CG$_b$3) $G(x,x,y) \precsim G(x,y,z)$ *for all* $x,y,z \in X$ *with* $y \neq z$*;*

(CG$_b$4) $G(x,y,z) = G(\rho(x,y,z))$*, where* ρ *is a permutation of* x,y,z*;*

(CG$_b$5) $G(x,y,z) \precsim s(G(x,a,a) + G(a,y,z))$ *for all* $x,y,z,a \in X$ *(rectangle inequality).*

Then, G is called a complex valued G_b-metric and (X,G) is called a complex valued G_b-metric space.

Proposition 2.4 ([15]). *Let (X,G) be a complex valued G_b-metric space. Then for any $x,y,z \in X$,*

- $G(x,y,z) \precsim s(G(x,x,y) + G(x,x,z))$,
- $G(x,y,y) \precsim 2sG(y,x,y)$.

Definition 2.5 ([15]). *Let (X,G) be a complex valued G_b-metric space, let $\{x_n\}$ be a sequence in X.*

(i) $\{x_n\}$ *is complex valued* G_b*-convergent to* x *if for every* $a \in \mathbb{C}$ *with* $0 \prec a$*, there exists* $N \in \mathbb{N}$ *such that* $G(x, x_n, x_m) \prec a$ *for all* $n, m \geq N$*.*

(ii) *A sequence* $\{x_n\}$ *is called complex valued* G_b*-Cauchy if for every* $a \in \mathbb{C}$ *with* $0 \prec a$*, there exists* $N \in \mathbb{N}$ *such that* $G(x_n, x_m, x_\ell) \prec a$ *for all* $n, m, \ell \geq N$*.*

(iii) *If every complex valued* G_b*-Cauchy sequence is complex valued* G_b*-convergent in* (X, G)*, then* (X, G) *is said to be complex valued* G_b*-complete.*

3. Main Results

Theorem 3.1. *Let* (X, G) *be a complete complex valued* G_b*-metric space with coefficient* $s > 1$ *and* $F : X \times X \to X$ *be a mapping satisfying:*

$$G(F(x,y), F(u,v), F(z,w)) \precsim \lambda G(x,u,z) + \mu G(y,v,w) \tag{3.1}$$

for all $x, y, u, v, z, w \in X$*, where* λ *and* μ *are non-negative constants with* $s\lambda + \mu < 1$*. Then* F *has a unique coupled fixed point.*

Proof. Choose $x_0, y_0 \in X$ and set

$$x_1 = F(x_0, y_0), \quad y_1 = F(y_0, x_0)$$
$$\vdots$$
$$x_{n+1} = F(x_n, y_n), \quad y_{n+1} = F(y_n, x_n)$$

From (3.1), we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(F(x_{n-1}, y_{n-1}), F(x_n, y_n), F(x_n, y_n)) \\ &\precsim \lambda G(x_{n-1}, x_n, x_n) + \mu G(y_{n-1}, y_n, y_n) \end{aligned}$$

and similarly

$$\begin{aligned} G(y_n, y_{n+1}, y_{n+1}) &= G(F(y_{n-1}, x_{n-1}), F(y_n, x_n), F(y_n, x_n)) \\ &\precsim \lambda G(y_{n-1}, y_n, y_n) + \mu G(x_{n-1}, x_n, x_n) \end{aligned}$$

Therefore, by letting $G_n = G(x_n,x_{n+1},x_{n+1}) + G(y_n,y_{n+1},y_{n+1})$, we have

$$\begin{aligned} G_n &= G(x_n,x_{n+1},x_{n+1}) + G(y_n,y_{n+1},y_{n+1}) \\ &\precsim \lambda G(x_{n-1},x_n,x_n) + \mu G(y_{n-1},y_n,y_n) + \lambda G(y_{n-1},y_n,y_n) + \mu G(x_{n-1},x_n,x_n) \\ &= (\lambda+\mu)[G(x_{n-1},x_n,x_n) + G(y_{n-1},y_n,y_n)] \\ &= (\lambda+\mu)G_{n-1}\,. \end{aligned}$$

That is $G_n \precsim PG_{n-1}$, where $P = \lambda+\mu < 1$.

In general, we have for $n = 0,1,2,\dots$

$$G_n \precsim PG_{n-1} \precsim P^2G_{n-2} \precsim \cdots \precsim P^nG_0.$$

Now, for all $m > n$

$$\begin{aligned} G(x_n,x_m,x_m) &\precsim s[G(x_n,x_{n+1},x_{n+1}) + G(x_{n+1},x_m,x_m)] \\ &\precsim sG(x_n,x_{n+1},x_{n+1}) + s^2[G(x_{n+1},x_{n+2},x_{n+2}) + G(x_{n+2},x_m,x_m)] \\ &\precsim sG(x_n,x_{n+1},x_{n+1}) + s^2G(x_{n+1},x_{n+2},x_{n+2}) + \cdots + s^{m-n}G(x_{m-1},x_m,x_m) \end{aligned}$$

and

$$G(y_n,y_m,y_m) \precsim sG(y_n,y_{n+1},y_{n+1}) + s^2G(y_{n+1},y_{n+2},y_{n+2}) + \cdots + s^{m-n}G(y_{m-1},y_m,y_m)$$

Therefore, have

$$\begin{aligned} G(x_n,x_m,x_m) + G(y_n,y_m,y_m) &\precsim sG_n + s^2G_{n+1} + \cdots + s^{m-n}G_{m-1} \\ &\precsim sP^nG_0 + s^2P^{n+1}G_0 + \cdots + s^{m-n}P^{m-1}G_0 \\ &= sP^n[1 + sP + (sP)^2 + \cdots + (sP)^{m-n-1}]G_0 \\ &\prec \frac{sP^n}{1-sP}G_0\,. \end{aligned}$$

Thus, we obtain

$$|G(x_n,x_m,x_m) + G(y_n,y_m,y_m)| \le \frac{sP^n}{1-sP}|G_0|\,.$$

Since $P < 1$, taking limit as $n \to \infty$, then

$$\frac{sP^n}{1-sP}|G_0| \to 0\,.$$

This means that $|G(x_n,x_m,x_m))| \to 0$ and $(G(y_n,y_m,y_m)) \to 0$.

By Proposition 2.4, we get

$$G(x_n,x_m,x_\ell)+G(y_n,y_m,y_\ell)$$
$$\precsim G(x_n,x_m,x_m)+G(x_\ell,x_m,x_m)+G(y_n,y_m,y_m)+G(y_\ell,y_m,y_m) \text{ for } \ell,m,n\in\mathbb{N}.$$

Thus

$$|G(x_n,x_m,x_\ell)+G(y_n,y_m,y_\ell)|$$
$$\leq |G(x_n,x_m,x_m)+G(y_n,y_m,y_m)|+|(G(x_\ell,x_m,x_m)+G(y_\ell,y_m,y_m)|\,.$$

If we take limit as $n,m,\ell\to\infty$, we obtain

$$|G(x_n,x_m,x_\ell)|\to 0 \quad \text{and} \quad |G(y_n,y_m,y_\ell)|\to 0.$$

which implies that $\{x_n\}$ and $\{y_n\}$ are complex valued G_b-Cauchy sequences in X. By X is complete, there exists $x',y'\in X$ such that

$$\lim_{n\to\infty} x_n = x' \quad \text{and} \quad \lim_{n\to\infty} y_n = y'\,.$$

Let $c\in\mathbb{C}$ with $0\prec c$. For every $m\in\mathbb{N}$, there exits $N\in\mathbb{N}$ such that

$$G(x_n,x_m,x_\ell)\prec c \ \text{ and } \ G(y_n,y_m,y_\ell)\prec c \quad \forall\, n,m,\ell>N\,.$$

Thus, we have

$$G(x_{n+1},F(x',y'),F(x',y')) = G(F(x_n,y_n),F(x',y'),F(x',y'))$$
$$\precsim \lambda G(x_n,x',x')+\mu G(y_n,y',y')$$

This implies that

$$|(G(x_{n+1},F(x',y'),F(x',y'))|\leq \lambda|(G(x_n,x',x')|+\mu|G(y_n,y',y')|$$

Taking limit as $n\to\infty$, we get

$$|G(x',F(x',y'),F(x',y')|\to 0$$

that is $G(x',F(x',y'),F(x',y'))=0$ and hence $F(x',y')=x'$.

Similarly, we have $F(y',x')=y'$.

Hence (x', y') is a coupled fixed point of F.

Now, if (x'', y'') is another coupled fixed point of F, then

$$\begin{aligned} G(x', x'', x'') &= G(F(x', y'), F(x'', y''), F(x'', y'')) \\ &\precsim \lambda G(x', x'', x'') + \mu G(y', y'', y'') \end{aligned}$$

and

$$\begin{aligned} G(y', y'', y'') &= G(F(y', x'), F(y'', x''), F(y'', x'') \\ &\precsim \lambda G((y', y'', y'') + \mu G(x', x'', x'') \end{aligned}$$

Thus we have

$$G(x', x'', x'') + G(y', y'', y'') \precsim (\lambda + \mu)[G(x', x'', x'') + F(y', y'', y'')]$$

which implies that

$$|G(x', x'', x'') + G(y', y'', y'')| \precsim (\lambda + \mu)|G(x', x'', x'') + G(y', y'', y'')|$$

Since $s\lambda + \mu < 1$, we have $|G(x', x'', x'') + G(y', y'', y'')| = 0$.

That is $G(x', x'', x'') + G(y', y'', y'') = 0$.

Thus we have $(x', y') = (x'', y'')$.

Therefore F has a unique coupled fixed point. □

From Theorem 3.1 with $\mu = \lambda$, we have the following corollary:

Corollary 3.2. *Let (X, G) be a complete complex valued G_b-metric space with coefficient $s \geq 1$ and $F : X \times X \to X$ be mapping satisfying:*

$$G(F(x, y), F(u, v), F(z, w)) \precsim \lambda[G(x, u, z) + G(y, v, w)] \tag{3.2}$$

for all $x, y, z, u, v, w \in X$, where λ is a non-negative constant with $\lambda < \frac{1}{2}$ then F has a unique coupled fixed point.

Example 3.3. Let $X = [-1, 1]$ and $G : X \times X \times X \to \mathbb{C}$ be defined as follows:

$$G(x, y, z) = |x - y| + |y - z| + |z - x| \tag{3.3}$$

for all $x,y,z \in X$. (X,G) is complex valued G-metric space. Define

$$G_*(x,y,z) = G(x,y,z)^2.$$

G_* is a complex valued G_b-metric with $s = 2$ (see [6]).

If we define $F : X \times X \to X$ with $F(x,y) = \frac{x+y}{3}i$. Then F satisfied the contractive condition (3.2) for $\frac{1}{9} \leq \lambda < \frac{1}{2}$ that is

$$G(F(x,y),F(u,v),F(z,w) \leq \lambda\,[G(x,u,z)+G(y,v,w)].$$

Here $(0,0)$ is the unique coupled fixed point of F.

Theorem 3.4. *Let (X,G) be a complete complex valued $G_b$metric space. Suppose that the mapping $F : X \times X \to X$ satisfies*

$$G(F(x,y),F(u,v),F(u,v) \precsim \lambda\,[G(x,F(x,y),F(x,y)+G(u,F(u,v),F(u,v)]. \tag{3.4}$$

where $\lambda \in \left[0,\frac{1}{2}\right)$. Then F has a unique coupled fixed point.

Proof. Choose $x_0, y_0 \in X$ and set

$$\begin{aligned} x_1 &= F(x_0,y_0), \quad y_1 = F(y_0,x_0) \\ &\vdots \\ x_{n+1} &= F(x_n,y_n), \quad y_{n+1} = F(y_n,x_n) \end{aligned}$$

From (3.4), we have

$$\begin{aligned} G(x_n,x_{n+1},x_{n+1}) &= G(F(x_{n-1},y_{n-1}),F(x_n,y_n),F(x_n,y_n)) \\ &\precsim \lambda\,[G(x_{n-1},F(x_{n-1},y_{n-1}),F(x_{n-1},y_{n-1})+G(x_n,F(x_n,y_n),F(x_n,y_n))] \\ &\precsim \lambda\,[G(x_{n-1},x_n,x_n)+G(x_n,x_{n+1},x_{n+1})] \end{aligned}$$

which implies

$$G(x_n,x_{n+1},x_{n+1}) \precsim \frac{\lambda}{1-\lambda} G(x_{n-1},x_n,x_n)$$

and similarly

$$G(y_n,y_{n+1},y_{n+1}) = G(F(y_{n-1},x_{n-1}),F(y_n,x_n),F(y_n,x_n)]$$

$$\precsim \lambda[G(y_{n-1},F(y_{n-1},x_{n-1}),F(y_{n-1},x_{n-1}))+G(y_n,F(y_n,x_n),F(y_n,x_n))$$

which implies that

$$G(y_n,y_{n+1},y_{n+1})\precsim \frac{\lambda}{1-\lambda}G(y_{n-1},y_n,y_n)$$

Now, by setting $G_n=G(x_n,x_{n+1},x_{n+1})+G(y_n,y_{n+1},y_{n+1})$, we have

$$\begin{aligned}G_n &= G(x_n,x_{n+1},x_{n+1})+G(y_n,y_{n+1},y_{n+1})\\ &\precsim \frac{\lambda}{1-\lambda}[G(x_{n-1},x_n,x_n)+G(y_{n-1},y_n,y_n)]\end{aligned}$$

that is

$$G_n\precsim PG_{n-1}\quad \text{where } P=\frac{\lambda}{1-\lambda}<1\,.$$

In general, we have for $n=0,1,2,\cdots$

$$G_n\precsim PG_{n-1}\precsim P^2G_{n-2}\precsim\cdots\precsim P^nG_0\,.$$

This implies that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequence in (X,G). and therefore, by completeness of X, there exists $x',y'\in X$ such that $\lim\limits_{n\to\infty}x_n=x'$ and $\lim\limits_{n\to\infty}y_n=y'$.

Let $c\in\mathbb{C}$ with $0\prec c$. For every $m\in\mathbb{N}$, there exists $N\in\mathbb{N}$ such that

$$G(x_n,x_m,x_\ell)\prec c \text{ and } G(y_n,y_m,y_\ell)\prec c\quad \forall\, m,n,\ell>N.$$

Thus, we have

$$\begin{aligned}G(x_{n+1},F(x',y'),F(x',y')) &= G(F(x_n,y_n),F(x',y'),F(x',y')]\\ &\precsim \lambda[G(x_n,F(x_n,y_n),F(x_n,y_n)+G(x',F(x',y'),F(x',y')]\end{aligned}$$

which implies that

$$|G(x_{n+1},F(x',y'),F(x',y')|\le\lambda|G(x_n,x_{n+1},x_{n+1})+G(x',F(x',y'),F(x',y')|\,.$$

Taking limit as $n\to\infty$, we get

$$|G(x',F(x',y'),F(x',y')|\le\lambda|G(x',F(x',y'),F(x',y'))|\,.$$

This implies that $|G(x',F(x',y'),F(x',y')|\to 0$ and therefore $F(x',y')=x'$, similarly $F(y',x')=y'$.

Hence (x',y') is a coupled fixed point of G.

Now if (x'', y'') is another coupled fixed point of F, then

$$G(x', x'', x'') = G(F((x', y'), F(x'', y''), (F(x'', y''))$$
$$\precsim \lambda[G(x', F(x', y'), F(x', y') + G(x'', F(x'', y''), F(x'', y'')]$$

Thus we have $x' = x''$ similarly, we get $y' = y''$.

Therefore F has a unique coupled fixed point.

Conflict of Interests

The authors declare that there is no conflict of interests.

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