Available online at http://scik.org
Adv. Fixed Point Theory, 6 (2016), No. 3, 287-294
ISSN: 1927-6303

ON ω-LIMIT SETS OF NON-AUTONOMOUS DISCRETE DYNAMICAL SYSTEM

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Abstract. In this paper we study ω-limit sets of non-autonomous discrete dynamical systems. Some basic concepts are introduced for non-autonomous discrete systems, including ω-limit set, Lyapunov stable set, asymptotically stable set. We give some sufficient conditions for non-autonomous discrete dynamical systems to have asymptotically stable sets.

Keywords: Non-autonomous discrete dynamical systems; ω-limit set; Lyapunov stable set; Asymptotically stable set.

2010 AMS Subject Classification: 54H20, 37B55.

1. Introduction

Throughout this paper, $\mathbb{N}$ denotes natural number set and let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let X be a topological space, $f_n : X \to X$ for each $n \in \mathbb{N}$ be a continuous map and $f_{1,\infty}$ denotes the sequence $(f_1, f_2, \cdots, f_n, \cdots)$. The pair $(X, f_{1,\infty})$ is referred to as a non-autonomous discrete dynamical system [7]. If X is compact, then $(X, f_{1,\infty})$ is called a compact non-autonomous system. Define

$$f_1^n := f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1 \text{ for all } n \in \mathbb{N},$$

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Received March 6, 2016

and $f_1^0 := id_X$, the identity on X. In particular, when $f_{1,\infty}$ is a constant sequence $(f,\cdots,f,\cdots)$, the pair $(X,f_{1,\infty})$ is just classical discrete dynamical system (autonomous discrete dynamical system) (X,f). The orbit initiated from $x \in X$ under $f_{1,\infty}$ is defined by the set

$$\gamma(x,f_{1,\infty}) = \{x, f_1(x), f_1^2(x), \cdots, f_1^n(x), \cdots\}.$$

Its long-term behaviors are determined by its limit sets.

In past ten years, a large number of papers have been devoted to dynamical properties for non-autonomous discrete systems. Kolyada and Snoha [7] gave definition of topological entropy in non-autonomous discrete systems, Kolyada, Snoha and Trofimchuk [8] discussed minimality of non-autonomous dynamical systems, Kempf [6] and Canovas [2] studied ω-limit sets in non-autonomous discrete systems respectively. Krabs [9] discussed stability in non-autonomous discrete systems, Huang, Wen and Zeng ([4, 5]) studied topological pressure and pre-image entropy of non-autonomous discrete systems, Shi and Chen [13] and Oprocha and Wilczynski [12] discussed chaos in non-autonomous discrete systems respectively.

The concept of asymptotically stable set for classical discrete dynamical system (autonomous discrete dynamical system) was introduced by Block and Coppel [1]. Mimna and Steele [10] discussed ω-limit sets and asymptotically stable sets for semi-homeomorphisms, Oprocha [11] studied asymptotically stable sets in continuous dynamical systems. In this paper we give the notions of ω-limit set and asymptotically stable set for a non-autonomous discrete system. Our purpose is to study the properties of asymptotically stable sets for non-autonomous discrete dynamical systems. In particularly, we give necessary and sufficient conditions for non-autonomous discrete systems to have asymptotically stable sets.

2. Preliminaries

Definition 2.1. *Let $(X,f_{1,\infty})$ be a non-autonomous discrete system. For every $x \in X$ and $m \in \mathbb{Z}_+$, the set $\gamma_m(x,f_{1,\infty}) = \{f_1^n(x) : n \geq m\}$ is called positive orbit through x starting at time m. If $m = 0$, we will omit time index.*

Definition 2.2. *Let $(X, f_{1,\infty})$ be a non-autonomous discrete system and let $x \in X$. Define $\omega(x, f_{1,\infty})$ as the set of limit points of the orbit $\gamma(x, f_{1,\infty})$, i.e., $\omega(x, f_{1,\infty}) = \bigcap_{m\in\mathbb{Z}_+} \overline{\gamma_m(x, f_{1,\infty})}$, where $\overline{\gamma_m(x, f_{1,\infty})}$ denotes the closure of $\gamma_m(x, f_{1,\infty})$.*

Definition 2.3. [8] *Let $(X, f_{1,\infty})$ be a non-autonomous discrete system. Set $A \subseteq X$ is said to be invariant if $f_1^n(A) \subseteq A$ for every $n \in \mathbb{N}$.*

For an autonomous system (X, f), by Block and Coppel [1], if X is a compact space, then $\omega(x, f)$ is invariant for every $x \in X$. However, for a non-autonomous system $(X, f_{1,\infty})$, we have $\omega(x, f_{1,\infty})$ can not be invariant for some $x \in X$. We give the following example which is from [6] to show $\omega(x, f_{1,\infty})$ is not invariant.

Example 2.1. *Let $X = [0,1]$, $f_n : [0,1] \to [0,1]$ be a sequence of continuous maps and*

$$f_n(x) = \begin{cases} 1 - \frac{1}{n+1}, & \text{for } x \in X \text{ and } n \text{ even}, \\ \frac{1}{n+1}, & \text{for } x \in X \text{ and } n \text{ odd}, \end{cases}$$

for every $n \in \mathbb{N}$. Then $\omega(0, f_{1,\infty})$ is not invariant.

From the definition of $f_n(x)$, we have

$$f_1^n(0) = \begin{cases} \frac{1}{n+1}, & \text{for } n \text{ odd}, \\ \frac{n}{n+1}, & \text{for } n \text{ even}. \end{cases}$$

Hence, $\omega(0, f_{1,\infty}) = \{0, 1\}$. Since $f_1^1(\omega(0, f_{1,\infty})) = \{0, \frac{1}{2}\}$, $\omega(0, f_{1,\infty})$ is not invariant.

Definition 2.4. [13] *Let $(X, f_{1,\infty})$ be a non-autonomous discrete system. $f_{1,\infty}$ is said to be k-periodic discrete system if there exists $k \in \mathbb{N}$ such that $f_{n+k}(x) = f_n(x)$ for every $x \in X$ and $n \in \mathbb{N}$.*

Let $(X, f_{1,\infty})$ be a k-periodic discrete system for a $k \in \mathbb{N}$. Define $g =: f_k \circ f_{k-1} \circ \cdots \circ f_1$, we say that (X, g) is induced an autonomous discrete system by k-periodic discrete system $(X, f_{1,\infty})$.

Definition 2.5. [3] *Let X be a topological space and $\{Y_i\}_{i\in I}$ be a family of subsets of X. The family $\{Y_i\}_{i\in I}$ has the finite intersection property if, for every finite subset J of I, the intersection $\bigcap_{j\in J} Y_j$ is a nonempty set.*

Theorem 2.1. [3] *Let X be a metric space and let K be a compact set in X and C be a closed set in X with $K \cap C = \emptyset$. Then there exist two open sets U and V in X, with $K \subseteq U, C \subseteq V$ and $U \cap V = \emptyset$.*

3. Asymptotically stable sets of non-autonomous discrete dynamical systems

In this section, we assume that X is a compact metric space, and we will discuss asymptotically stable sets of non-autonomous discrete system $(X, f_{1,\infty})$.

Definition 3.1. *Let X be a compact metric space and $(X, f_{1,\infty})$ be a non-autonomous discrete system. A is a nonempty closed set in X.*

- **(1):** *A is said to be Lyapunov stable if for each open set U containing A there exists an open set V containing A such that $\gamma(x, f_{1,\infty}) \subseteq U$ for every $x \in V$.*
- **(2):** *A is said to be asymptotically stable if A is Lyapunov stable and there exists an open set U_0 containing A such that $\omega(x, f_{1,\infty}) \subseteq A$ for every $x \in U_0$.*

Proposition 3.1. *Let $(X, f_{1,\infty})$ be a non-autonomous discrete system, where X is a compact metric space. Let $A \subseteq X$ be a Lyapunov stable set of $(X, f_{1,\infty})$. Then A is invariant, i.e., $f_1^n(A) \subseteq A$ for every $n \in \mathbb{N}$.*

Proof. Suppose that set A is not invariant. Then there exists $n_0 \in \mathbb{N}$ such that $f_1^{n_0}(A) \nsubseteq A$. Furthermore, there exists a point $x_0 \in A$ such that $f_1^{n_0}(x_0) \notin A$. Since X is a compact metric space and A is closed, it follows that X is a Hausdorff space and A is a compact subset in X. By Theorem 2.1, there exist an open neighborhood U_1 of $f_1^{n_0}(x_0)$ and an open neighborhood U_2 of A such that $U_1 \cap U_2 = \emptyset$, which implies $f_1^{n_0}(x_0) \notin U_2$. Hence, for every open set V containing A, we have $x_0 \in A \subseteq V$ and $f_1^{n_0}(x_0) \notin U_2$, which implies $\gamma(x_0, f_{1,\infty}) \nsubseteq U_2$. This is a contradiction.

Theorem 3.1. *Let $(X, f_{1,\infty})$ be a non-autonomous discrete system, where X is a compact metric space, and let subset A in X is an asymptotically stable set in $(X, f_{1,\infty})$. Then there exists an open set U_0 containing A such that for each open set U containing A there exists a finite* *set $P = \{n_1(U), n_2(U), \cdots, n_p(U)\}$, where $n_1(U), n_2(U), \cdots, n_p(U) \in \mathbb{N}$, such that for every $x \in \overline{U_0}$, there exists a positive integer $m \in P$ satisfying $f_1^n(f_1^m(x)) \in U$ for every $n \in \mathbb{N}$.*

Proof. Let A be an asymptotically stable set. Then there exists an open neighborhood W of A such that $\omega(x, f_{1,\infty}) \subseteq A$ for every $x \in W$. Since A and $X \setminus W$ are two closed subsets of X and X is a compact metric space, then A and $X \setminus W$ are two compact subsets. Furthermore, there exists an open set U_0 of X satisfying $A \subseteq U_0 \subseteq \overline{U_0} \subseteq W$.

Notice that $\omega(x, f_{1,\infty}) \subseteq A$ for every point $x \in \overline{U_0}$. Take any open neighborhood U of A, we may also assume that $U \subseteq U_0$. There exists an open neighborhood V of A such that $\gamma(x, f_{1,\infty}) = \{f_1^n(x) : n = 0, 1, \cdots\} \subseteq U$ for every $x \in V$. Moreover, for every $x \in \overline{U_0}$, there exists a $n = n(x) \in \mathbb{N}$ such that $f_1^n(x) \in V$ because $\omega(x, f_{1,\infty}) \subseteq A$. As $f_1^n(x)$ is continuous, there exists an open neighborhood W_x of x such that $f_1^n(W_x) \subseteq V$. Set $\overline{U_0}$ is compact because X is compact, and family $\{W_x\}$ is its open cover. Hence, we may choose finite subcover $\{W_{x_1}, W_{x_2}, \cdots, W_{x_p}\}$ of $\overline{U_0}$. Furthermore, for each W_{x_i}, there exists a $n_i = n(x_i) \in \mathbb{N}$ such that $f_1^{n_i}(W_{x_i}) \subseteq V$ for $i = 1, 2, \cdots, p$. Take $P = \{n(x_1), n(x_2), \cdots, n(x_p)\}$. Hence, for every $x \in \overline{U_0}$, there exists $m \in P$ such that $f_1^m(x) \in V$. Therefore, we have $f_1^n(f_1^m(x)) \in U$ for every $n \in \mathbb{N}$.

Theorem 3.2. *Let $(X, f_{1,\infty})$ be a non-autonomous discrete system, where (X, d) is a compact metric space. Let A be a closed invariant set and U_0 be an open neighborhood of A, if for every open neighborhood U of A, there exists a $N = N(U) \in \mathbb{N}$ such that $f_1^n(\overline{U_0}) \subseteq U$ for all $n \geq N$. Then A is an asymptotically stable set.*

Proof. Firstly, we show that A is Lyapunov stable. Suppose that A is not Lyapunov stable. Then there exists an open neighborhood U of A satisfying for every open set V containing A, there exists a point $x \in V$ such that $\gamma(x, f_{1,\infty}) \nsubseteq U$.

Since U_0 be an open neighborhood of A, we can take points $x_k \in \overline{U_0}$ such that $x_k \to x \in A$ when $k \to \infty$ and integers $n_k \in \mathbb{N}$ such that $f_1^{n_k}(x_k) \notin U$ for all $k \in \mathbb{N}$. As $f_1^n(\overline{U_0}) \subseteq U$ for all $n \geq N$, then $n_k \leq N$ for all k, where $N = N(U)$. Therefore, by drawer principle, there exists some $m < N$ such that $n_k = m$ for infinitely many k. Since A is invariant. Hence $f_1^m(x) \in A$. Furthermore, $f_1^m(x_k) \in U$ for sufficiently large k. This is a contradiction.

Secondly, we prove $\omega(x, f_{1,\infty}) \subseteq A$ for every $x \in \overline{U_0}$. Let $U = B_\varepsilon(A)$ where $B_\varepsilon(A) = \{x \in X : d(x,A) < \varepsilon\}$. Since $U = B_\varepsilon(A)$ is an open neighborhood of A, then there exists a $N = N(U) \in \mathbb{N}$ such that $f_1^n(\overline{U_0}) \subseteq B_\varepsilon(A)$ for every $n \geq N$. Furthermore, we have $f_1^n(x) \in B_\varepsilon(A)$ for every $x \in \overline{U_0}$ and $n \geq N$. Moreover, for every $y \in \omega(x, f_{1,\infty})$, there exists an increasing sequence $\{n_i\}$ such that $y = \lim_{i\to\infty} f_1^{n_i}(x)$. Take

- **(1):** $\varepsilon_1 = 1$. Then there exists a $N_1 = N(B_1(A)) \in \mathbb{N}$ such that $f_1^{n_{i_1}}(x) \in B_1(A)$ when $n_{i_1} \geq N_1$;
- **(2):** $\varepsilon_2 = \frac{1}{2}$. Then there exists a $N_2 = N(B_{\frac{1}{2}}(A)) \in \mathbb{N}$ such that $f_1^{n_{i_2}}(x) \in B_{\frac{1}{2}}(A)$ and $n_{i_2} > n_{i_1}$ when $n_{i_2} \geq N_2$;
- **(3):** $\varepsilon_3 = \frac{1}{3}$. Then there exists a $N_3 = N(B_{\frac{1}{3}}(A)) \in \mathbb{N}$ such that $f_1^{n_{i_3}}(x) \in B_{\frac{1}{3}}(A)$ and $n_{i_3} > n_{i_2}$ when $n_{i_3} \geq N_3$, and so on.

Since $\{n_{i_j} : j = 1, 2, \cdots\}$ is a subsequence of $\{n_i : i = 1, 2, \cdots\}$, it follows that $\lim_{j\to\infty} f_1^{n_{i_j}}(x) = y$. Moreover, $f_1^{n_{i_j}}(x) \in N(B_{\frac{1}{j}}(A))$, i.e., $d(f_1^{n_{i_j}}(x), A) < \frac{1}{j}$. Furthermore,

$$d(y,A) \leq d(y, f_1^{n_{i_j}}(x)) + d(f_1^{n_{i_j}}(x), A).$$

Since A is a closed set of X, thus, when $j \to \infty$, we have $y \in A$. Hence, $\omega(x, f_{1,\infty}) \subseteq A$ for every $x \in \overline{U_0}$.

Theorem 3.3. *Let $(X, f_{1,\infty})$ be a non-autonomous discrete system, where X is a compact metric space . Let A be a closed invariant set and there exists an open set V containing A such that*

- **(1):** $f_1^n(\overline{V}) \subseteq f_1^{n-1}(\overline{V}) \subseteq V$ *for every* $n \in \mathbb{N}$*;*
- **(2):** $\bigcap_{n\in\mathbb{Z}_+} f_1^n(\overline{V}) \subseteq A$.

Then A is an asymptotically stable set.

Proof. Since X is a compact space, thus $\overline{V}$ is a compact subset of X. Moreover, f_1^n is a continuous map for every $n \in \mathbb{N}$. Hence, $f_1^n(\overline{V})$ is a compact subset of X for every $n \in \mathbb{Z}_+$. By condition (1), the compact sets $f_1^n(\overline{V})$ form a decreasing sequence. By Definition 2.5, the family $\{f_1^n(\overline{V})\}_{n\in\mathbb{Z}_+}$ has the finite intersection property. Furthermore, we have $\bigcap_{n\in\mathbb{Z}_+} f_1^n(\overline{V}) \neq \emptyset$. Let $U_0 = V$. Then for every open neighborhood U of A, there exists a positive integer $N = N(U)$

such that $f_1^n(\overline{U_0}) \subseteq U$ for all $n \geq N$. Therefore, by Theorem 3.2, A is an asymptotically stable set of $(X, f_{1,\infty})$.

Theorem 3.4. *Let $(X, f_{1,\infty})$ be a k-periodic discrete system, $g = f_k \circ f_{k-1} \circ \cdots \circ f_1$, (X, g) is its induce autonomous discrete system. If A is an asymptotically stable set of $(X, f_{1,\infty})$, then A is an asymptotically stable set of (X, g).*

Proof. Firstly, we show A is Lyapunov stable in (X, g). Let U be any open set which containing A. Since A is an asymptotically stable set of $(X, f_{1,\infty})$, it follows that there exists an open set V containing A such that $\gamma(x, f_{1,\infty}) \subseteq U$ for every $x \in V$. As $(X, f_{1,\infty})$ be a k-periodic discrete system and $g = f_k \circ f_{k-1} \circ \cdots \circ f_1 = f_1^k$, we have $f_{n+k}(x) = f_n(x)$ for every $x \in X$. Furthermore, $g^m(x) = (f_1^k)^m(x) = f_1^{mk}(x)$. Moreover, for every $x \in V$, $\gamma(x, g) = \{x, g(x), g^2(x), \cdots\} = \{x, f_1^k(x), f_1^{2k}(x), \cdots\}$, thus, $\gamma(x, g) \subseteq \gamma(x, f_{1,\infty})$. Hence, we have $\gamma(x, g) \subseteq U$ for every $x \in V$. This shows A is Lyapunov stable in (X, g).

Secondly, we prove that there exists an open set U_0 containing A such that $\omega(x, g) \subseteq A$ for every $x \in U_0$. Since A is an asymptotically stable set of $(X, f_{1,\infty})$, then there exists an open set U_0 containing A such that $\omega(x, f_{1,\infty}) \subseteq A$ for every $x \in U_0$. Moreover, for every $m \in \mathbb{N}$, we have $\gamma_m(x, g) \subseteq \gamma_m(x, f_{1,\infty})$. Furthermore, we have $\bigcap_{m \in \mathbb{Z}_+} \overline{\gamma_m(x, g)} \subseteq \bigcap_{m \in \mathbb{Z}_+} \overline{\gamma_m(x, f_{1,\infty})}$, which implies $\omega(x, g) \subseteq \omega(x, f_{1,\infty})$. Hence, $\omega(x, g) \subseteq A$ for every $x \in U_0$. This shows A is an asymptotically stable set of (X, g).

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgements. The work was supported by the Natural Science Foundation of Henan Province(162300410075), PR China. and by the National Natural Science Foundation of China (11401363).

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