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APPROXIMATING POSITIVE SOLUTIONS OF QUADRATIC FUNCTIONAL INTEGRAL EQUATIONS

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Abstract. In this paper we prove the existence as well as approximations of the positive solutions for a nonlinear quadratic functional integral equation. An algorithm for the solutions is developed and it is shown that the sequence of successive approximations converges monotonically to the positive solution of related quadratic functional integral equation under some suitable mixed hybrid conditions. We rely our results on Dhage iteration method embodied in a recent hybrid fixed point theorem of Dhage (2014) in partially ordered normed linear spaces. An example is also provided to illustrate the abstract theory developed in the paper.

Keywords: quadratic functional integral equation; approximate positive solution; fixed point theorem.

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1. Introduction

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The quadratic integral equations have been a topic of interest since long time because of their occurrence in the problems of some natural and physical processes of the universe. See Argyros [1], Deimling [3], Chandrasekher [2] and the references therein. The study gained momentum after the formulation of fixed point principles in Banach algebras due to Dhage [4, 5, 6, 7]. The existence results for such equations are generally proved under the mixed Lipschitz and compactness type conditions together with a certain growth condition on the nonlinearities of the quadratic integral equations. See Dhage [5, 6, 7] and the references therein. The Lipschitz and compactness hypotheses are considered to be very strong conditions in the theory of nonlinear differential and integral equations which do not yield any algorithm to determine the numerical solutions. Therefore, it is of interest to relax or weaken these conditions in the existence and approximation theory of quadratic integral equations. This is the main motivation of the present paper. In this paper we prove the existence as well as approximations of the positive solutions of a certain quadratic integral equation via an algorithm based on successive approximations under partially Lipschitz and compactness conditions.

Given a closed and bounded interval $J = [0;T]$; of the real line $\mathbb{R}$, $T > 0$, we consider the quadratic functional integral equation (in short QFIE)

$$x(t) = [f(t,x(t))]\left(q(t) + \int_0^t g(s,x(s))ds\right) \tag{1}$$

where $f,g : J\times\mathbb{R} \to \mathbb{R}$ and $q : J \to \mathbb{R}$ are continuous functions.

By a solution of the QFIE (1) we mean a function $x \in C(J;\mathbb{R})$ that satisfies the equation (1) on J, where $C(J;\mathbb{R})$ is the space of continuous real-valued functions defined on J.

The QFIE (1) is well-known in the literature and studied earlier in the work of Dhage [4]. If $g(t;x) = 0$ for all $t \in J$ and $x \in \mathbb{R}$ the QFIE (1) reduces to nonlinear functional equation.

$$x(t) = f(t,x(t)), t \to J \tag{2}$$

and if $f(t;x) = 1$ for all $t \in J$ and $x \in \mathbb{R}$, it is reduced to nonlinear usual Volterra integral equation.

$$x(t) = q(t) + \int_0^t g(s, x(s))ds, t \in J \tag{3}$$

Therefore, the QFIE (1) is general and the results of this paper include the existence and approximations results for above nonlinear functional and Volterra integral equations as special cases.

The paper is organized as follows. In the following section we give the preliminaries and auxiliary results needed in the subsequent part of the paper. The main result is included in Section 3. In Section 4 some concluding remarks are presented

2. Preliminaries

Unless otherwise mentioned, throughout this paper that follows, let X denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm $\|.\|$.It is known that X is regular if $x_n \in \mathbb{N}$ is a nondecreasing (resp. nonincreasing) sequence in X such that $x_n \to x^*$,as $n \to \infty$ then $x_n \preceq x^*$ resp. $x_n \succeq x^*$ for all $n \in \mathbb{N}$. Clearly, the partially ordered Banach space $C(J;\mathbb{R})$ is regular and the conditions guaranteeing the regularity of any partially ordered normed linear space X may be found in Heikkilla and Lakshmikantham [11] and the references therein.

We need the following notion and results.

Definition 1.1. A mapping $\mathscr{A} : X \to X$ is called *isotone* or *monotone nondecreasing* if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathscr{A}x \preceq \mathscr{A}y$ for all $x, y \in X$.

Definition 1.2. An operator $\mathscr{A}$ on a normed linear space X into itself is called *compact* if $\mathscr{A}(X)$ is a relatively compact subset of X. $\mathscr{A}$ is called *totally bounded* if for any bounded subset S of X, $\mathscr{A}(S)$ is a relatively compact subset of X. If $\mathscr{A}$ is continuous and totally bounded, then it is called *completely continuous* on X.

Definition 1.3. [Dhage 4] A mapping $\mathscr{A} : X \to X$ is called *partially continuous* at a point $a \in X$ if for $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|\mathscr{A}x - \mathscr{A}a\| < \varepsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathscr{A}$ called partially continuous on X if it is partially continuous at every point of it.

It is clear that if $\mathscr{A}$ is partially continuous on X, then it is continuous on every chain C contained in X.

Definition 1.4. [Dhage 4] An operator $\mathscr{A}$ on a partially normed linear space X into itself is called *partially bounded* if $A(C)$ is bounded for every chain C in X. $\mathscr{A}$ is called *uniformly partially bounded* if all chains $\mathscr{A}(C)$ in X are bounded by a unique constant. $\mathscr{A}$ is called *partially compact* if $\mathscr{A}(C)$ is a relatively compact subset of X for all totally ordered sets or chains C in X. $\mathscr{A}$ is called *partially totally bounded* if for any totally ordered and bounded subset C of X, $\mathscr{T}(C)$ is a relatively compact subset of X. If $\mathscr{A}$ is partially continuous and partially totally bounded, then it is called *partially completely continuous* on X.

Definition 1.5.[Dhage 4] The order relation $\preceq$ and the metric d on a non-empty set X are said to be *compatible* if $\{x_n\}$ is a monotone, that is, monotone nondecreasing or monotone nondecreasing sequence in X and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the whole sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(X, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible.

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\preceq$ and the norm defined by the absolute value function $|.|$ has this property. Similarly, the finite dimensional Euclidean space $\mathbb{R}_n$ with usual componentwise order relation and the standard norm possesses the compatibility property

Definition 1.6. Let $(X, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $\mathscr{A} : X \to X$ is called partially nonlinear $\mathscr{D}$-Lipschitz if there exists a D-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|\mathscr{A}x - \mathscr{A}y\| \leq \psi(\|x - y\|) \tag{4}$$

for all comparable elements $x, y \in X$, where $\psi(0) = 0$. If $\psi(r) = kr$, $k > 0$, then $\mathscr{A}$ is called a partially Lipschitz with a Lipschitz constant k. If $k < 1$, $\mathscr{A}$ is called a partially contraction with contraction constant k. Finally, $\mathscr{A}$ is called nonlinear $\mathscr{D}$-contraction if it is a nonlinear $\mathscr{D}$-Lipschitz with $\psi(r) < r$ for $r > 0$. Let $(X, \preceq, \|.\|)$ be a partially ordered normed linear algebra. Denote

$X^+ = \{x \in X \mid x \succeq \theta$ whereθ is zero element of X $\}$

and

$$K = \{X^+ \subset X \mid uv \in X^+ \quad for \quad all \quad u,v \in X^+\} \tag{5}$$

The elements of the set K are called the positive vectors in X.

Lemma (Dhage [7]) If $u_1, u_2, v_1, v_2 \in K$ are such that $u_1 \preceq v_1$ and $u_2 \preceq v_2$, then $u_1u_2 \preceq v_1v_2$

Definition 1.7 An operator $T : X \to X$ is said to be positive if the range R(T) of T is such that $R(T) \subseteq K$.

The Dhage iteration method is embodied in the following hybrid fixed point theorem proved in Dhage [9] which is a useful tool in what follows. A few other hybrid fixed point theorems involving the Dhage iteration method may be found in Dhage [10].

Theorem 2.1. Let $\big(X, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear space such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in X. Let $A, B : X \to X$ be two nondecreasing operators such that

(a) A is partially bounded and partially nonlinear $\mathscr{D}$-Lipschitz with $\mathscr{D}$-function ψ_A,
(b) B is partially continuous and uniformly partially compact, and
(c) M $\psi_{a<r,r>0}$,where $M = \sup\{\|B(C)\|$ C is chain in x $\}$
(d) there exists an element $x_0 \in X$ such that $x_0 \preceq Ax_0Bx_0$ or $x_0 \succeq Ax_0Bx_0$.

Then the operator equation

$$AxBx = x \tag{6}$$

has a solution x^* in X and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = Ax_nBx_n$, $n = 0, 1, \ldots$, converges monotonically to x^*.

Remark . The compatibility of the order relation $\preceq$ and the norm $\|.\|$ in every compact chain of X holds if every partially compact subset of X possesses the compatibility property with respect to $\preceq$ and $\|.\|$

3. Main results

The QFIE (1) is considered in the function space $C(J;\mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\preceq$ in $C(J;\mathbb{R})$ by

$$\|x\| = \sup_{t\in J} |x(t)| \tag{7}$$

and

$$x \leq y \Leftrightarrow x(t) \leq y(t) \tag{8}$$

for all $t \in J$ respectively. Clearly, $C(J;\mathbb{R})$ is a Banach algebra with respect to above supremum norm and is also partially ordered w.r.t. the above partially order relation $\leq$. It is known that the partially ordered Banach algebra $C(J;\mathbb{R})$ has some nice properties w.r.t. the above order relation in it. The following lemma follows by an application of Arzella-Ascolli theorem.

Lemma 3.1. Let $C((J,\mathbb{R}),\leq,\|.\|)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (7) and (8) respectively. Then $\|\cdot\|$ and $\leq$ are compatible in every partially compact subset of $C(I;\mathbb{R})$.

Proof. Let S be a partially compact subset of $C(J,\mathbb{R})$ and let $\{x_n\}_{n\in\mathbb{N}}$ be a monotone nondecreasing sequence of points in S. Then we have

$$x_1(t) \leq x_2(t) \leq x_3(t) \cdots$$

for each $t \in \mathbb{R}_+$ Suppose that a subsequence $\{x_{nk}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges to a point $x$in S. Then the subsequence $\{x_{nk}\}_{n\in\mathbb{N}}$ of the monotone real sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent. By monotone characterization, the whole sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges to a point $x(t)$ in $\mathbb{R}$ for each $t \in \mathbb{R}_+$. This shows that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges point-wise in S. To show the convergence is uniform, it is enough to show that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is equicontinuous. Since S is partially compact, every chain or totally ordered set and consequently $\{x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence by Arzella-Ascoli theorem. Hence $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges uniformly to x. As a result $\leq$ and $\|.\|$ are compatible in S. This completes the proof. We need the following definition in what follows.

Definition 3.2. A function $u \in C(J,\mathbb{R})$ is said to be a lower solution of the QFIE (1) if it satisfies

$$u(t) = [f(t,u(t))]\left(q(t) + \int_0^t g(s,u(s))ds\right) \tag{9}$$

for all $t \in J$ Similarly, a function $v \in C(J;R)$ is said to be a lower solution of the QFIE (1) if it satisfies the above inequalities with reverse sign.

We consider the following set of assumptions in what follows

- (C_0) q defines a continuous function $q : J \to \mathbb{R}_+$
- (C_1) f defines a function $f : J \times \mathbb{R} \to \mathbb{R}_+$.
- (C_2) There exists a real number $M_f > 0$ such that $f(t;x) \leq M_f$ for all $t \in J$ and $x \in \mathbb{R}$.
- (C_3) There exists a D-function ϕ such that

  $0 \leq f(t;x) - f(t;y) \leq \phi(x-y)$ for all $t \in J$ and $x;y \in \mathbb{R}$, $x \geq y$.
- (C_4) g defines a function $g : J \times \mathbb{R} \to \mathbb{R}_+$
- (C_5) There exists a real number $M_g > 0$ such that $g(t;x) \leq M_g$ for all $t \in J$ and $x \in \mathbb{R}$.
- (C_6) g(t; x) is nondecreasing in x for all $t \in J$.
- (C_7) The QFIE (1) has a lower solution $u \in C(J;\mathbb{R})$

These are the main results of the paper.

Theorem 3.1. Assume that hypotheses(C_1)-(C_7) hold.Furthermore, assume that

$$(\|q\| + M_g T)\phi(r) < r, r > 0 \tag{10}$$

then the QFIE (1) has a positive solution x^* defined on J and the sequence $x_{nn\in N}$ of successive approximations defined by

$$x_{n+1}(t) = [f(t,x_n(t))]\left(q(t) + \int_{t_0}^t g(s,x_n(s))ds\right), t \in J \tag{11}$$

where $x_0 = u$, converges monotonically to x^*.

Proof. Set $X = C(J;R)$. Then, from Lemma 3.1 it follows that every compact chain in X possesses the compatibility property with respect to the norm $\|.\|$ and the order relation $\leq$ in X.

and define two operators A and B on X by

$$Ax(t) = f(t, x(t)), t \in J \tag{12}$$

and

$$Bx(t) = q(t) + \int_{t_0}^{t} g(s, x(s))ds, t \in J \tag{13}$$

From the continuity of the integral and the hypotheses (C_0)-(C_1) and (C_1), it follows that A and B define the maps $A, B : X \to K$. Now by definitions of the operators A and B, the QFIE (1) is equivalent to the operator equation

$$Ax(t)Bx(t) = x(t), t \in J \tag{14}$$

We shall show that the operators A and A satisfy all the conditions of Theorem (2.1). This is achieved in the series of following steps.

Step I: *A and B are nondecreasing on X.*

Let $x, y \in X$ be such that $x \geq y$. Then by hypothesis (C_2), we obtain

$$Ax(t) = f(t, x(t)) \geq f(t, y(t)) = Ay(t)$$

for all $t \in J$. This shows that A is nondecreasing operator on X into X. Similarly using hypothesis (C_7), it is shown that the operator B is also nondecreasing on X into itself. Thus, A and B are nondecreasing positive operators on X into itself

Step II: *A is partially bounded and partially D-Lipschitz on X.*

Let $x \in X$ be arbitrary. Then by (C2), $|Ax(t)| \leq |f(t, x(t))| \leq M_f$ for all $t \in J$. Taking supremum over t, we obtain $\|Ax\| \leq M_f$ and so, A is bounded. This further implies that A is partially bounded on X.

Next, let $x, y \in X$ be such that $x \geq y$. Then

$$\begin{aligned}
|Ax(t) - Ay(t)| &= |f(t, x(t)) - f(t, y(t))| \\
&\leq \phi(|x(t) - y(t)|) \leq \phi\|x - y\|
\end{aligned}$$

for all $t \in J$. Taking supremum over t, we obtain $\|Ax - Ay\| \leq \phi(\|x - y\|)$, for all $x, y \in X$, $x \geq y$. Hence, A is a partial nonlinear D-Lipschitz on X which further implies that A is a partially continuous on X.

Step III: *B is a partially continuous on X.*

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a chain C of X such that $x_n \to x$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} Bx_n(t) &= \lim_{n\to\infty} q(t) + \int_0^t g(s, x_n(s))ds \\
&= q(t) + \int_0^t [\lim_{x\to\infty} g(s, x_n(s))]ds \\
&= q(t) + \int_0^t g(s, x_n(s))ds \\
&= Bx(t).
\end{aligned}$$

for all $t \in J$. This shows that Bx_n converges monotonically to Bx pointwise on J. Next, we will show that $\{Bx_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in X. Let $t_1, t_2 \in I$ with $t_1 < t_2$. Then

$$\begin{aligned}
\left|Bx_n(t_2) - By_n(t_1)\right| &\leq |q(t_1) - q(t_2)| + \left|\int_0^{t_2} g(s, x(s))ds - \int_0^{t_1} g(s, x(s))ds\right| \\
&\leq |q(t_1) - q(t_2)| + \left|\int_{t_1}^{t_2} |g(s, x(s))|ds\right| \\
&\leq |q(t_1) - q(t_2)| + M_g|t_2 - t_1| \\
&\to 0 \qquad as \qquad t_2 - t_1 \to 0
\end{aligned}$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $Bx_n \to Bx$ is uniform and hence B is partially continuous on X.

Step IV: *B is uniformly partially compact operator on X.*

Let C be an arbitrary chain in X. We show that $B(C)$ is a uniformly bounded and equicontinuous set in X. First, we show that $B(C)$ is uniformly bounded. Let $y \in B(C)$ be any element. Then there is an element $x \in C$, such that $y = Bx$. Now, by hypothesis (C2),

$$\begin{aligned}\big|y(t)\big| &\leq |q(t)| + \int_0^t |g(s,x(s))|ds \\ &\leq \|q\| + M_g T \\ &\leq r\end{aligned}$$

for all $t \in J$. Taking supremum over t, we obtain $\|y\| = \|Bx\| \leq r$ for all $y \in B(C)$. Hence, B(C) is a uniformly bounded subset of X. Moreover, $\|B(C)\| \leq r$ for all chains C in X. Hence, B is a uniformly partially bounded operator on X.

Next, we will show that $B(C)$ is an equicontinuous set in X. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then, for any $y \in B(C)$, one has

$$\begin{aligned}\big|y(t_2) - y(t_1)\big| &= |Bx(t_2) - Bx(t_1)| \\ &\leq |q(t_1) - q(t_2)| + \left|\int_0^{t_2} g(s,x(s))ds - \int_0^{t_1} g(s,x(s))ds\right| \\ &\leq |q(t_1) - q(t_2)| + \left|\int_{t_1}^{t_2} |g(s,x(s))|ds\right| \\ &\leq |q(t_1) - q(t_2)| + M_g|t_2 - t_1| \\ &\to 0 \qquad as \qquad t_2 - t_1 \to 0\end{aligned}$$

uniformly for all $y \in B(C)$. Hence $B(C)$ is an equicontinuous subset of X. Now, $B(C)$ is a uniformly bounded and equicontinuous set of functions in X, so it is compact. Consequently, B is a uniformly partially compact operator on X into itself.

Step V: *u satisfies the operator inequality* $u \leq AuBu$.

By hypothesis (C7), the QFIE (1) has a lower solution u defined on J. Then, we have

$$u(t) \leq [f(t,u(t))]\left(q(t) + \int_0^t g(s,u(s))ds\right) \tag{15}$$

for all $t \in J$. From definitions of the operators A and B it follows that $u(t) \leq Au(t)Bu(t)$, for all $t \in J$. Hence $u \leq AuBu$.

Step VI: D-fuction ϕ *is satisfies the growth condition* $M\phi(r) < r, r > 0$

Finally, the D-function ϕ of the operator A satisfies the inequality given in hypothesis (d) of Theorem 2.1. Now from the estimate given in Step IV, it follows that

$$M\phi_A(r) \leq (\|q\| + M_g T\Big)\phi(r) < r,$$

for all $r > 0$.

Thus, A and B satisfy all the conditions of Theorem 2.1 and we apply it to conclude that the operator equation $AxBx = x$ has a solution. Consequently the integral Equation and the QFIE (1) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}_{n=1}^{\infty}$ of successive approximations defined by (11) converges monotonically to x^*. This completes the proof.

Example . Given a closed and bounded interval J = [0; 1], consider the QFIE,

$$x(t) = [2 + \tan^{-1} x(t)]\left(\frac{t}{t+1} + \int_0^t \frac{[1 + \tanh x(s)]}{4} ds\right) \tag{16}$$

for $t \in J$.

Here, $q(t) = \frac{t}{t+1}$ which is continuous and $\|q(t)\| = \frac{1}{2}$ Similarly, the functions f and g are defined by $f(t,x) = \tan^1 x + 2$ and $g(t,x) = \frac{[1+\tanh x]}{4}$.The function f satisfies the hypothesis (C3) with $\phi(r) = \frac{r}{1+\xi^2}$ for $0 < \xi < r$.To see this we have,

$$0 \leq f(t,x) - f(t,y) \leq \frac{1}{1+\xi^2}.(x-y)$$

for all x, $y \in \mathbb{R}$, $x \geq y$ and $x > \xi > y$. Moreover, the function f(t, x) is bounded on $J \times \mathbb{R}$ with bound $M_f = 3$ and so the hypothesis (C_2) is satisfied. Again, since g is bounded on $J \times \mathbb{R}$ by $M_g = \frac{1}{2}$, the hypothesis (C_6) holds. Furthermore, $g(t,x)$ is nondecreasing in x for all $t \in J$, and thus hypothesis (C_7) is satisfied. Also condition (10) of Theorem 3.1 is held. Finally, the QFIE (14) has a lower solution u(t) = 0 defined on J. Thus all hypotheses of Theorem 3.1 are satisfied. Hence we apply Theorem 3.1 and conclude that the QFIE (14) has a solution x^* defined on J and the sequence $\{x_n\}_{n\in N}$ defined by

$$x_{n+1}(t) = [2 + \tan^{-1} x_n(t)]\left(\frac{t}{t+1} + \int_0^t \frac{[1 + \tanh x_n(s)]}{4} ds\right) \tag{17}$$

for all t $\in J$, where $x_0 = 0$, converges monotonically to x^*

4. Conclusion

Finally, while concluding this paper we mention that the quadratic integral equation considered here is of very simple nature for which we have illustrated the Dhage iteration method to obtain the algorithms for the positive solutions under weaker partially Lipschitz and compactness conditions. However, an analogous study could also be made for other complex quadratic integral equations using similar method with appropriate modifications. Some of the results along this line will be reported elsewhere.

Conflict of Interests

The authors declare that there is no conflict of interests.

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