Available online at http://scik.org

Adv. Fixed Point Theory, 6 (2016), No. 3, 308-329

ISSN: 1927-6303

KKM-PRINCIPLE AND FIXED POINT THEOREMS IN METRIC TYPE SPACE

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Abstract. The aim of this paper is to give some fixed point theorems using the KKM-Type mappings, some well-known results are extended to the case of metric type spaces. We also introduced a new concepts namely, multivalued co-set contraction, and quasi-subadmissibility in order to generalize concidence Fan's Theorem in metric type spaces.

Keywords: KKM property; metric type space; fixed point; nearly-subadmissible; quasi-subadmissible; quasi-upadmissible; co-set contraction.

2010 AMS Subject Classification: 47H09, 47H10, 54H25.

1. Introduction

In [3] I.A. Bakhtin was introduce the metric type space in order to generalize the Banach contraction principle in such spaces. M.A. Khamsi in [15] introduce this space which is generated by a cone metric space over a normal cone P and proved some coupled fixed point theorems.

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Received March 18, 2016

KKM-type mapping in metric space was introduced by A. Amini, M. Fakhar and J. Zafarni [1] and they established some fixed point theorems, Khamsi and Hussain in [14] extend this work on KKM-type compact mappings on sub-admissible subset of a metric type space. Chi-Ming Chen [6] generalize the A. Amini results [1] in the setting of nearly-sub-admissible subset in complete metric space for a k-set contraction and he establish some fixed point theorems where the compactness is not needed.

In this work, we generalize KKM-type result in the context of metric type space, and derive some related fixed point theorems, our work include the results of Chi-Ming [6] and M.A. Khamsi and Hussain [14]. As an application we deduce a Ky Fan-type coincidence point theorem, for this we introduce the notion of metric-type quasi-subadmissible and quasi-upadmissible sets, and we derive a Minimax-type result in such spaces.

2. Preliminaries

For the convenience of the reader we repeat the relevant material from [14] without proofs, thus making our exposition self-contained.

Definition 2.1. *Let X be a set. Let $D : X \times X \longrightarrow [0,\infty)$ be a function which satisfies*

(i) $D(x,y) = 0$ if and only if $x = y$;

(ii) $D(x,y) = D(y,x)$, for any $x,y \in X$;

(iii) $D(x,z) \leqslant K[D(x,y) + D(y,z)]$, for any points $x,y,z \in X$, for some constant $K > 0$.

The pair (X,D) is called a metric type space. This metric generalization is worthy of consideration only when $K > 1$, and throughout this paper we consider that $K \geqslant 1$.

We note that closed and open balls of (X,D) are defined respectively as follows :

$$B(x,r) = \{y \in X : D(x,y) \leqslant r\} \quad \text{and} \quad B_0(x,r) = \{y \in X : D(x,y) < r\} \ .$$

We define a natural topology in metric type space (X,D), this topology has most same properties of the metric space. A subset $A \subset X$ is said to be open if and only if for any $a \in A$, there exists $\varepsilon > 0$ such that the open ball $B_0(a,\varepsilon) \subset A$.

Now we recall some properties of metric type space; most of this results can be found in [14],

Definition 2.2. *Let (X,D) be a metric type space.*

(i) The sequence $\{x_n\}$ converges to $x \in X$ if and only if $\lim\limits_{n\to\infty} D(x_n,x) = 0$.

(ii) The sequence $\{x_n\}$ is Cauchy if and only if $\lim\limits_{n,m\to\infty} D(x_n,x_m) = 0$.

(iii) (X,D) is complete if and only if any Cauchy sequence in X is convergent.

The next result characterized the closure of metric type space.

Proposition 2.3. *Let (X,D) be a metric type space , then for any nonempty subset $A \subset X$ we have*

(i) A is closed if and only if for any sequence $\{x_n\}$ in A which converges to x, we have $x \in A$;

(ii) if we define $\overline{A}$ to be the intersection of all closed subsets of X which contains A, then for any $x \in \overline{A}$ and for any $\varepsilon > 0$, we have $B_0(x,\varepsilon) \cap A \neq \emptyset$.

Definition 2.4. *The subset A is called sequentially compact if and only if for any sequence $\{x_n\}$ in A, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges in A. Also A is called totally bounded if for any $\varepsilon > 0$, there exist $x_1, x_2, \ldots, x_n \in A$ such that $A \subset B_0(x_1,\varepsilon) \cup \cdots \cup B_0(x_n,\varepsilon)$.*

Proposition 2.5 *Let (X,D) be a metric type space and A a nonempty subset of X. The following properties are equivalent :*

(i) A is compact,

(ii) A is sequentially compact.

Proposition 2.6. *Let (X,D) be a metric type space. X is compact if and only if X is complete and totally bounded.*

Proof. Since sequential compactness and compactness are equivalent, we must show that every sequence in totally bounded set has a Cauchy subsequence. Let $\{x_n\}_n$ be a sequence in X. For each $n \in \mathbb{N}$ let D_n be a finite subset of X such that the open balls of radius K^n centred at the points of D_n cover X (where K is the constant associated to the triangle inequality satisfied by D).

Since D_0 is finite, so there is a point $y_0 \in D_0$ such that infinitely many points of $\{x_n\}_n$ are in $B(y_0,1)$. Let

$$A_0 = \{n \in \mathbb{N} : x_n \in B(y_0,1)\}$$

so that A_0 is infinite, we will denote by $\left\{x_{\varphi(0,n)}\right\}_n$ the subsequence of $\{x_n\}_n$ satisfying $\forall n \in A_0$: $x_{\varphi(0,n)} \in B(y_0,1)$.

Now D_1 is finite, so there is a $y_1 \in D_1$ such that

$$A_1 = \left\{ n \in \mathbb{N} : x_n \in B(y_0, 1) \bigcap B\left(y_1, \frac{1}{K}\right) \right\}$$

A_1 is infinite, denote by $\left\{x_{\varphi(1,n)}\right\}_n$ the subsequence of $\left\{x_{\varphi(0,n)}\right\}_n$ satisfying $\forall n \in A_1, x_{\varphi(1,n)} \in B(y_0, 1) \bigcap B\left(y_1, \frac{1}{K}\right)$.

Continuing in this vein, if A_m is an infinite subset of $\mathbb{N}$, there must be a $y_{m+1} \in D_{m+1}$ such that

$$A_{m+1} = \left\{ n \in \mathbb{N} : x_n \in \bigcap_{i=0}^{m+1} B\left(y_i, \frac{1}{K^i}\right) \right\}$$

In what follows, $\left\{x_{\varphi(m+1,n)}\right\}_n$ denotes the subsequence of $\left\{x_{\varphi(m,n)}\right\}_n$. Put for each $n \in \mathbb{N}$, $z_n = x_{\varphi(n,n)}$, for n large enough, we will get

$$D(z_n, z_{n+1}) \leqslant K\left[D(z_n, y_n) + D(y_n, z_{n+1})\right] \leqslant \frac{K}{K^n} + \frac{K}{K^n} = \frac{2}{K^{n-1}}$$

sine $K > 1$, then $\{z_n\}_n$ is a Cauchy sequence. This complete the proof.

Let X and Y be two topological spaces and $T : X \longrightarrow 2^Y$ be a set valued map with nonempty values. T is said to be :

(1) closed if its graph $G_T = \{(x,y) \in X \times Y \, , y \in T(x)\}$ is closed;
(2) compact if the closure $\overline{T(X)}$ is a compact subset of Y.
(3) lower semicontinuous if for each open set $B \subset Y$, the set $T^{-1}(B) = \{x \in X \, : \, T(x) \cap B \neq \emptyset\}$ is open in X.

For a set A, we denote the set of all nonempty finite subsets of A by $\langle A \rangle$. Let A be a bounded subset of a metric type space (X, D). Then

(1) $co(A) = \bigcap \{B \subset X \, , B \text{is a closed ball in } X \text{ such that} A \subset B\}$
(2) $\Lambda(X) = \{A \subset X \, , A = co(A)\}$, i.e. $A \in \Lambda(X)$ if and only if A is an intersection of all closed balls containing A. In this case,we say that A is an admissible set in X. (the empty set $\emptyset$ is assumed to be admissible).
(3) A is called subadmissible, if for each $B \in \langle A \rangle$, $co(B) \subset A$. Obviously, if A is an admissible subset of X, then A must be subadmissible.

(4) A is said to be nearly-subadmissible if for each compact subset A_1 of A and for each $\varepsilon > 0$, there exists a continuous mapping $f_{A_1,\varepsilon} : A_1 \longrightarrow A$ such that $x \in B_0\left(f_{A_1,\varepsilon}(x), \varepsilon\right)$ for each $x \in A_1$, and $co\left(f_{A_1,\varepsilon}(A_1)\right) \subset A$.

Remark 2.7. *If A is subadmissible set then A is a nearly-subadmissible*

Proof. Obviously each finite subset of A is compact , said $B = \{x_1, \ldots, x_n\} \subset A$, since the identity mapping $i_d : B \longrightarrow A$ is continuous we have $co(i_d(B)) = co(B) \subset A$, and for every $i \in \{1, 2, \ldots, n\}$ and for each $\varepsilon > 0$, $x_i \in B_0(i_d(x_i), \varepsilon)$. That implies A is nearly-subadmissible set.

Let X be a metric type space and A a nonempty subadmissible subset of X.

Definition 2.8. *A set valued mapping $T : A \longrightarrow 2^X$ is called a KKM mapping, if for each $B \in \langle A\rangle$, we have $co(B) \subset T(B) = \bigcup_{x\in B} T(x)$. More generally, if Y is a topological space and $T, S : A \longrightarrow 2^Y$ are two set valued mappings such that for any $B \in \langle A\rangle$, we have $S(co(B)) \subset T(B)$, then T is called a generalized KKM mapping with respect to S.*

If the set valued mapping $S : A \longrightarrow 2^Y$ satisfies the requirement that for any generalized KKM mapping $T : A \longrightarrow 2^Y$ with respect to S the family $\left\{\overline{T(x)}, x \in A\right\}$ has the finite intersection property, then S is said to have the KKM property. We define

$$\mathrm{KKM}(A,Y) = \left\{S : A \longrightarrow 2^Y, S \text{ has the KKM property}\right\}.$$

Let A be a nonempty subset of a metric type space X. Then $S : A \longrightarrow 2^X$ is said to have the approximate fixed point property if for any $\varepsilon > 0$, there exists an $x_\varepsilon \in A$ such that $S(x_\varepsilon) \bigcap B_0(x_\varepsilon, \varepsilon) \neq \emptyset$.

We conclude this section by giving an extension of Kuratowski measure of non-compactness and some related results in a complete metric type space.

Definition 2.9. *Let A be a bounded subset of a complete metric type space (X, D), the Kuratowski type-measure of noncompactness $\alpha(A)$ of A is defined by*

$$\alpha(A) = \inf\{\varepsilon > 0 : A \textit{ can be covered by finitely many sets with diameter} \leqslant \varepsilon\}.$$

The diameter $\delta(A)$ of A is defined by $\delta(A) = \sup\{D(x,y) : x, y \in A\}$ with $\delta(\emptyset) = 0$.

The next properties are satisfied in any complete metric type space.

Proposition 2.10. *For each nonempty bounded subsets A_1 and A_2 of (X,D) we have*

(i) : $\alpha(A_1) = 0$ *if and only if A_1 is totally bounded,*

(ii) : $A_1 \subset A_2 \Longrightarrow \alpha(A_1) \leqslant \alpha(A_2)$

(iii) : *If A_2 is finite subset of X, then $\alpha(A_1 \cup A_2) = \alpha(A_1)$,*

(iv) : $\alpha(A_1) \leqslant \alpha\left(\overline{A_1}\right) \leqslant K^2 \alpha(A_1)$.

Proof. (i) and (ii) are obvious by definition of the Kuratowski type-measure of noncompactness.

(iii) Put

$$\mathscr{D}_1 = \{\varepsilon > 0;\, A_1 \text{ can be covered by finitely many sets with diameter } \leqslant \varepsilon\}$$

,

$$\mathscr{D}_2 = \{\varepsilon > 0;\, A_1 \cup A_2 \text{ can be covered by finitely many sets with diameter } \leqslant \varepsilon\}$$

Let $\varepsilon \in \mathscr{D}_1$, then there exist a family of sets $M_1, M_2, \ldots, M_n$ of X such that, for each $i \in \{1, 2 \ldots, n\}$, $\delta(M_i) \leqslant \varepsilon$ and $A_1 \subset \bigcup_{i=1}^n M_i$.

Since A_2 is finite subset set $A_2 = \{a_1, a_2, \ldots, a_m\}$ such that for each $j \in \{1, 2 \ldots, m\}$:

$$A_1 \cup A_2 \subset \left(\bigcup_{i=1}^n M_i\right) \cup \left(\bigcup_{j=1}^m B_0\left(a_j, \frac{\varepsilon}{2K}\right)\right)$$

this implies that $\varepsilon \in \mathscr{D}_2$, in consequence, $\alpha(A_1 \cup A_2) \leqslant \alpha(A_1)$, and since $A_1 \subset A_1 \cup A_2$ this clearly forces $\alpha(A_1 \cup A_2) = \alpha(A_1)$.

(iv) $A_1 \subset \overline{A_1}$ then $\alpha(A_1) \leqslant \alpha\left(\overline{A_1}\right)$. Let $\varepsilon \in \mathscr{D}_1$, there exist a family of sets $M_1, M_2, \ldots, M_n$ of X such that, for each $i \in \{1, 2 \ldots, n\}$

$$\delta(M_i) \leqslant \varepsilon \text{ and } A_1 \subset \bigcup_{i=1}^n M_i$$

then $\overline{A_1} \subset \overline{\bigcup_{i=1}^n M_i} = \bigcup_{i=1}^n \overline{M_i}$ and $\delta\left(\overline{M_i}\right) \leqslant K^2 \varepsilon$. Indeed, let $(x, y) \in \left(\overline{M_i}\right) \times \left(\overline{M_i}\right)$, there exists two sequences $\{x_n\}_n$ and $\{y_n\}_n$ such that $\lim D(x_n, x) = 0$ and $\lim D(y_n, y) = 0$, for each $n \in \mathbb{N}$ we get :

$$\begin{aligned} D(x, y) &\leqslant K\left(D(x, x_n) + K\left(D(x_n, y_n) + D(y_n, y)\right)\right) \\ &\leqslant K^2 \varepsilon + K\left(D(x_n, x) + K D(y_n, y)\right) \end{aligned}$$

take the limit with respect to n, we get $\delta\left(\overline{M_i}\right) \leqslant K^2 \varepsilon$. This gives $K^2 \varepsilon \in \overline{\mathscr{D}_1}$ with

$$\overline{\mathscr{D}_1} = \left\{\varepsilon > 0;\, \overline{A_1} \text{ can be covered by finitely many sets with diameter } \leqslant \varepsilon\right\}$$

which implies that $\alpha(A_1) \leqslant \alpha\left(\overline{A_1}\right) \leqslant K^2\alpha(A_1)$.

In the course of this paper, we will need the following generalized Cantor intersection,

Lemma 2.11. *Let (X,D) be a complete metric type space and $\{A_n\}$ be a decreasing sequence of nonempty, closed and bounded subsets of X and $\lim_{n\to\infty} \alpha(A_n) = 0$. Then the intersection A_∞ of all A_n is nonempty and compact.*

Proof. the proof is similar to the metric space, since every complete and totally bounded subset is compact by proposition2.6

The following lemma will be useful in the sequel, the proof of (i) will be find in [[14], Lemma 4.1.], and the proof of (ii) its similar to (ii) [[6], Lemma 2]

Lemma 2.12. *Let (X,D) be a metric type space and A a nonempty subset of X, and suppose that Y is topological space,*

(i) If $T \in KKM(A,Y)$ and $f \in C(Y,X)$ then $f \circ T \in KKM(A,X)$

(ii) If B is nonempty subset of A and $T \in KKM(A,Y)$ then $T|_B \in KKM(B,Y)$

Proposition 2.13. *Let (X,D) be a metric type space, then for each $x \in X$ and $\varepsilon > 0$ we have $B_0(x,\varepsilon) \subset \overline{B_0(x,K\varepsilon)^{c}}^{c} \subset B_0(x,K\varepsilon)$, for some $K \geqslant 1$.*

Proof. See the proof M.A. Khamsi and N. Hussain [[14], Theorem 4.1].

3. Main results

We introduce a slight different definition of the well known set valued $k-$set contraction, which is convenient with the structure of the space.

Definition 3.1. *Let A be a nonempty subset of a metric type space (X,D). A set valued mapping $T : A \longrightarrow 2^X$ is said to be a co-set contraction if, for all $B \subset A$ with B bounded, $T(B)$ is bounded and $\alpha(co(T(B))) \leqslant k\alpha(B)$, $0 < k < 1$.*

In this section we make the assumption that the set-valued map $T : A \to 2^X$ satisfies the condition : there exists $w \in A$ such that $Tw \neq \emptyset$, $T(Tw) \neq \emptyset$. Now we are able to give our main result.

Theorem 3.2. *Let (X,D) be a complete metric type space and A is a nonempty bounded nearly-subadmissible subset of X. If $T \in KKM(A,A)$ is a co-set contraction, and closed with $\overline{T(A)} \subset A$, then T has a fixed point in A.*

Proof. Let $\varepsilon > 0$ be given. Since T is co-set contraction we have $\alpha(co(T(B))) \leqslant k\alpha(B)$ for each nonempty bounded subset of A. Since $T(A) \neq \emptyset$, let $w \in A$ as above, and define a nonempty sequence $\{A_n\}$ of decreasing subsets by :

$$\begin{aligned} A_0 &= T(A) \\ A_1 &= co(T(A_0 \cup \{w\})) \cap T(A) \\ A_{n+1} &= co(T(A_n \cup \{w\})) \cap T(A) \end{aligned}$$

for each $n \in \mathbb{N}$.

Then $A_{n+1} \subset A_n$ for each $n \in \mathbb{N}$, indeed, by induction : obviously we have $A_1 \subset A_0$ suppose that $A_{i+1} \subset A_i$ for every $i \in \{0, 1, \cdots, n-1\}$ and let show that $A_{n+1} \subset A_n$, by hypothesis we have $A_n \subset A_{n-1}$ implies that

$$co(T(A_n \cup \{w\})) \subset co(T(A_{n-1} \cup \{w\})) \Longrightarrow A_{n+1} \subset A_n$$

Then by induction we have for each $n \in \mathbb{N}$

$$A_{n+1} \subset A_n \tag{1}$$

Now lets prove that for every n in $\mathbb{N}$ we have :

$$T(A_n) \subset A_{n+1} \tag{2}$$

suppose $x \in T(A_n)$ then $x \in co(T(A_n \cup \{w\}))$ and we know that $T(A_n) \subset T(A_0) \subset T(A)$ so

$$x \in co(T(A_n \cup \{w\})) \cap T(A) = A_{n+1}.$$

From the proprieties of measure of noncompactness and by definition of co-set contraction we get

$$\begin{aligned} \alpha(A_{n+1}) &= \alpha(co(T(A_n \cup \{w\})) \cap T(A)) \leqslant \alpha(co(T(A_n \cup \{w\}))) \\ &\leqslant k\alpha(A_n \cup \{w\}) \\ &\leqslant k\alpha(A_n) \end{aligned} \tag{3}$$

from (3) we get for each $n \in \mathbb{N}$

$$\alpha(A_n) \leqslant k^n \alpha(A_0)$$

Thus $\alpha(A_n) \underset{n\to\infty}{\longrightarrow} 0$, and hence $A_\infty = \bigcap\limits_{n\geqslant 0} A_n$ is a nonempty totally bounded set. Remark that $A_\infty \subset A_n \subset T(A)$ for each $n \in \mathbb{N}$, then

$$\overline{A_\infty} \subset \overline{T(A)} \subset A$$

Since $\overline{T(A)} \subset A$ we have $\overline{T(A_{n+1})} \subset \overline{T(A_n)} \subset A$ for each n in $\mathbb{N}$. That is $\left\{\overline{T(A_n)}\right\}_n$ is nonincreasing sequence of nonempty closed subsets and $\alpha\left(\overline{T(A_n)}\right) \underset{n\to\infty}{\longrightarrow} 0$, indeed, for each $n \in \mathbb{N}$, $T(A_n) \subset A_{n+1}$ then $\alpha(T(A_n)) \leqslant \alpha(A_{n+1})$ which implies $\lim\limits_{n\to\infty} \alpha(T(A_n)) = 0$, by closeness property of Kuratowski type-measure,

$$\alpha(T(A_n)) \leqslant \alpha\left(\overline{T(A_n)}\right) \leqslant K^2 \alpha(T(A_n))$$

hence $\lim \alpha\left(\overline{T(A_n)}\right) = 0$, then by Lemma 2.11, we get $\bigcap\limits_{n\geqslant 0} \overline{T(A_n)}$ is nonempty compact subset.

Since $\overline{T(A_\infty)}$ is closed and $\overline{T(A_\infty)} \subset \bigcap\limits_{n\geqslant 0} \overline{T(A_n)}$, then $\overline{T(A_\infty)}$ is compact. A is nearly-subadmissible for any $\varepsilon > 0$ there exists a continous mapping $f_{\overline{T(A_\infty)},\varepsilon} : \overline{T(A_\infty)} \longmapsto A$ such that $x \in B_0\left(f_{\overline{T(A_\infty)},\varepsilon}(x), \dfrac{\varepsilon}{K^2}\right)$ for each $x \in \overline{T(A_\infty)}$ and $Z = co\left(f_{\overline{T(A_\infty)},\varepsilon}\left(\overline{T(A_\infty)}\right)\right) \subset A$. From $T \in \mathrm{KKM}(A,A)$ and A_∞ is nonempty subset of A by Lemma 2.12 (ii)

$$T|_{A_\infty} \in \mathrm{KKM}(A_\infty, A)$$

Put $L(x) := f_{\overline{T(A_\infty)},\varepsilon}(T|_{A_\infty}(x))$ for each $x \in A_\infty$ and by Lemma 2.12(i) we have $L \in \mathrm{KKM}(A_\infty, Z)$.

$\overline{L(A_\infty)} = \overline{f_{\overline{T(A_\infty)},\varepsilon}(T(A_\infty))}$ then $\overline{L(A_\infty)} \subset \overline{f_{\overline{T(A_\infty)},\varepsilon}\left(\overline{T(A_\infty)}\right)} = f_{\overline{T(A_\infty)},\varepsilon}\left(\overline{T(A_\infty)}\right) \subset Z$ and $\overline{L(A_\infty)}$ is compact in Z.

We have to claim that for each $\lambda > 0$, there exists an $x_{\lambda,\varepsilon} \in A_\infty$ such that $B\left(x_{\lambda,\varepsilon}, \dfrac{\lambda+\varepsilon}{K}\right) \cap L\left(x_{\lambda,\varepsilon}\right) \neq \emptyset$. For that suppose the contrary, then there exists $\lambda > 0$ such that for each $x \in A_\infty$

$$B_0\left(x, \frac{\lambda+\varepsilon}{K}\right) \cap L\left(x_{\lambda,\varepsilon}\right) = \emptyset.$$

Define a set valued map $S : A_\infty \to 2^Z$ by $S(x) = \overline{L(A_\infty)} \cap \overline{B_0\left(x, \dfrac{\lambda+\varepsilon}{K}\right)}^c$, then

(1) $S(x)$ is closed for each $x \in A_\infty$

(2) S is a generalized KKM map with respect to L,

Indeed, its obvious too see that $S(x)$ is compact since $\overline{B_0\left(x, \dfrac{\lambda+\varepsilon}{K}\right)^c}$ is closed for each $x \in A_\infty$. For (2) we prove it by contradiction, assume that there exists $\{x_1, \ldots, x_n\} \subset A_\infty$ such that

$$L(co(x_1, \ldots, x_n) \cap A_\infty) \nsubseteq \bigcup_{i=1}^{n} S(x_i),$$

then there exists $u \in co(x_1, \ldots, x_n) \cap A_\infty$ and $v \in L(u) \subset \overline{L(A_\infty)}$ such that $v \notin \bigcup_{i=1}^{n} S(x_i)$ by proposition 2.13, we get $v \in \overline{B_0\left(x_i, \dfrac{\lambda+\varepsilon}{K}\right)^c}^c \subset B_0\left(x_i, \dfrac{\lambda+\varepsilon}{K}\right)$ for each $i = 1, 2, \ldots, n$, implies that $x_i \in B_0\left(v, \dfrac{\lambda+\varepsilon}{K}\right)$ for each $i = 1, 2, \ldots, n$, therefore $\{x_1, \ldots, x_n\} \subset B_0\left(v, \dfrac{\lambda+\varepsilon}{K}\right)$ and since

$$u \in co(x_1, \ldots, x_n) \quad \text{and} \quad v \in L(u)$$

we have $u \in co(x_1, \ldots, x_n) \subset B_0\left(v, \dfrac{\lambda+\varepsilon}{K}\right)$ then $v \in B_0\left(u, \dfrac{\lambda+\varepsilon}{K}\right)$ which implies that $v \in B_0\left(u, \dfrac{\lambda+\varepsilon}{K}\right) \cap L(u)$ this contradict the fact that $B\left(x, \dfrac{\lambda+\varepsilon}{K}\right) \cap L(x) = \emptyset$ for all $x \in A_\infty$, then S is a generalized KKM map with respect to L.

And since $L \in \mathrm{KKM}(A_\infty, Z)$ the family $\{Sx, x \in A_\infty\}$ has the finite intersection property, and so we have

$$\bigcap_{x \in A_\infty} S(x) \neq \emptyset.$$

Let choose $u \in \bigcap_{x \in A_\infty} S(x)$, then $u \in \overline{L(A_\infty)} \cap \overline{B_0\left(x, \dfrac{\lambda+\varepsilon}{K}\right)^c}$ for each $x \in A_\infty$.

First we have $\bigcap_{x \in A_\infty} S(x) \subset \overline{L(A_\infty)} = f_{\overline{T(A_\infty)},\varepsilon}\left(\overline{T(A_\infty)}\right)$. Since $f_{\overline{T(A_\infty)},\varepsilon} : \overline{T(A_\infty)} \longmapsto A$ is continuous we have $f_{\overline{T(A_\infty)},\varepsilon}\left(\overline{T(A_\infty)}\right)$ is compact in nearly-subadmissible subset A, and by definition of nearly subadmissible subset $x \in \overline{T(A_\infty)}$ implies that :

$$x \in B_0\left(f_{\overline{T(A_\infty)},\varepsilon}(x), \frac{\varepsilon}{K^2}\right) \subset \overline{B_0\left(f_{\overline{T(A_\infty)},\varepsilon}(x), K \times \frac{\varepsilon}{K^2}\right)^{c}}^{c}$$

which gives $f_{\overline{T(A_\infty)},\varepsilon}(x) \in \overline{B_0\left(x, \frac{\varepsilon}{K}\right)^{c}}^{c}$ for each $x \in \overline{T(A_\infty)}$, i.e.

$$f_{\overline{T(A_\infty)},\varepsilon}\left(\overline{T(A_\infty)}\right) \subset \bigcup_{x \in \overline{T(A_\infty)}} \overline{B_0\left(x, \frac{\varepsilon}{K}\right)^{c}}^{c}.$$

Therefore, $T(A_n) \subset A_n \Rightarrow \overline{T(A_\infty)} \subset \overline{A_\infty}$ implies that

$$\bigcup_{x \in \overline{T(A_\infty)}} \overline{B_0\left(x, \frac{\varepsilon}{K}\right)^{c}}^{c} \subset \bigcup_{x \in \overline{A_\infty}} \overline{B_0\left(x, \frac{\varepsilon}{K}\right)^{c}}^{c}.$$

Then,

$$u \in \bigcap_{x \in A_\infty} S(x) \subset f_{\overline{T(A_\infty)},\varepsilon}\left(\overline{T(A_\infty)}\right) \subset \bigcup_{x \in \overline{T(A_\infty)}} \overline{B_0\left(x, \frac{\varepsilon}{K}\right)^{c}}^{c} \subset \bigcup_{x \in \overline{A_\infty}} \overline{B_0\left(x, \frac{\varepsilon}{K}\right)^{c}}^{c}.$$

so there exists $x_0 \in \overline{A_\infty}$ such that :

$$\begin{cases} u & \in \overline{B_0\left(x_0, \frac{\varepsilon}{K}\right)^{c}}^{c} \subset \overline{B_0\left(x_0, \frac{\lambda+\varepsilon}{K}\right)^{c}}^{c} \\ u & \in \overline{L(A_\infty)} \cap \overline{B_0\left(x_0, \frac{\lambda+\varepsilon}{K}\right)}^{c} \end{cases}$$

this is a contradiction, then for any $\varepsilon > 0$ there exists $x_{\lambda,\varepsilon} \in A_\infty$ such $B\left(x_{\lambda,\varepsilon}, \frac{\lambda+\varepsilon}{K}\right) \cap L\left(x_{\lambda,\varepsilon}\right) \neq \emptyset$.

Hence there exists $y_{\lambda,\varepsilon} \in B\left(x_{\lambda,\varepsilon}, \dfrac{\lambda+\varepsilon}{K}\right) \cap L\left(x_{\lambda,\varepsilon}\right)$. $y_{\lambda,\varepsilon} \in L\left(x_{\lambda,\varepsilon}\right) = f_{\overline{T(A_\infty)},\varepsilon}\left(T\left(x_{\lambda,\varepsilon}\right)\right)$ and choose $z_{\lambda,\varepsilon} \in T\left(x_{\lambda,\varepsilon}\right)$ such that $y_{\lambda,\varepsilon} = f_{\overline{T(A_\infty)},\varepsilon}\left(z_{\lambda,\varepsilon}\right)$. Note that

$$z_{\lambda,\varepsilon} \in B_0\left(f_{\overline{T(A_\infty)},\varepsilon}\left(z_{\lambda,\varepsilon}\right), \frac{\varepsilon}{K^2}\right) = B_0\left(y_{\lambda,\varepsilon}, \frac{\varepsilon}{K^2}\right) \subset B_0\left(y_{\lambda,\varepsilon}, \frac{\lambda+\varepsilon}{K}\right)$$

since $K \geqslant 1$, therefore

$$z_{\lambda,\varepsilon} \in B_0\left(y_{\lambda,\varepsilon}, \frac{\lambda+\varepsilon}{K}\right) \cap T\left(x_{\lambda,\varepsilon}\right) \neq \emptyset \Longrightarrow z_{\lambda,\varepsilon} \in B_0\left(x_{\lambda,\varepsilon}, \lambda+\varepsilon\right) \cap T\left(x_{\lambda,\varepsilon}\right) \neq \emptyset$$

since $\left\{z_{\lambda,\varepsilon}\right\}_{\varepsilon} \subset \overline{T\left(A_\infty\right)}$ and $\left\{x_{\lambda,\varepsilon}\right\}_{\varepsilon} \subset \overline{A_\infty}$ compact subsets and without loss of generality we may assume that

$$z_{\lambda,\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} z_{0,\varepsilon}$$

$$x_{\lambda,\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} x_{0,\varepsilon}$$

Then $z_{0,\varepsilon} \in B\left(x_{0,\varepsilon}, K^2\varepsilon\right) \cap T\left(x_{0,\varepsilon}\right)$ because $\overline{A_\infty} \subset A$ and T is closed in A, since $\varepsilon > 0$ is choose arbitrary which just as before, implies T has a fixed point in A.

Example 3.3. *Let $X = \mathbb{R}$ equipped with metric type distance $D(x,y) = (x-y)^2$ it is obvious that for each $x,y,z \in X$ we get*

$$D(x,z) \leqslant 2\left[D(x,y) + D(y,z)\right].$$

Put $A = [0,1] = B\left(\frac{1}{2}, \frac{1}{4}\right)$, it is clear that A is bounded admissible subset of X then A is nearly-subadmissible. A is also closed, indeed, let $\{x_n\}_n$ be some sequence of A which converge to x, let claim that $x \in A$, by assumption for each $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that for each $n \geqslant N$, $(x_n - x)^2 \leqslant \dfrac{1}{2^n} \Longleftrightarrow |x_n - x| \leqslant \sqrt{\dfrac{1}{2^n}}$ and since

$$\left|x - \frac{1}{2}\right| \leqslant |x - x_n| + \left|x_n - \frac{1}{2}\right| \leqslant \sqrt{\frac{1}{2^n}} + \frac{1}{2}$$

so taking the limit with respect to n yields, we get

$$\left|x-\frac{1}{2}\right| \leqslant \frac{1}{2} \Longleftrightarrow \left(x-\frac{1}{2}\right)^2 \leqslant \frac{1}{4}$$

then $x \in B\left(\frac{1}{2},\frac{1}{4}\right)$. *Furthermore, closed balls are also closed for this topology.*

Define a set valued mapping $T : [0,1] \to 2^{[0,1]}$ *as follows :*

$$Tx = [0,x]$$

for each $x \in A$. *Since for any bounded subset* B *of* A *we have*

$$T(B) = \bigcup_{x\in B} Tx = \begin{cases} [0,\hat{x}] \\ or \\ [0,\hat{x}[\end{cases}$$

where $\hat{x} = \sup B$. *Otherwise, we have* $\alpha(co(T(B))) = \alpha\left(B\left(\frac{\hat{x}}{2},\frac{\hat{x}^2}{4}\right)\right) = 0$ *since* $T(B)$ *is totally bounded subset of* A *which implies for some fixed* $k \in]0,1[$

$$\alpha(co(T(B))) \leqslant k\alpha(B).$$

Also, for each convex subset $B \subset A$, *we have* $\overline{T(B)} = [0,\hat{x}]$ *is convex, by [[12], Theorem 2.2] we get* $T \in KKM(A,A)$.

T *is closed map, indeed, let* $\{x_n\}_n$ *be a sequence in* A *such that* $x_n \to x$*,and define* $\{y_n\}_n$ *by for each* $n \in \mathbb{N}$, $y_n \in Tx_n$ *with* $y_n \to y$, *observe that* $y_n \in [0,x_n]$ *for each* $n \in \mathbb{N}$, *then* $y \in \bigcap_{n\in\mathbb{N}}[0,x_n] = [0,x] = Tx$.

So all assumptions of Theorem 3.2 are satisfied, then T *has at least one fixed point in* A. *Note that* $0 \in T0$.

Remark 3.4. *We replace the assumption the set valued map is not a k-set contraction but a co-set contraction, since the property* $\alpha(co(A)) = \alpha(A)$ *in complete metric space is not verified, for that we give the following example*

Example 3.5. *Let* $X = l^\infty(\mathbb{N},\mathbb{R})$ *be the space of bounded real sequences* $u = (u_n)_{n\in\mathbb{N}}$ *endowed by the usual norm*

$$\|u\|_\infty = \sup_{n\in\mathbb{N}} |u_n|$$

it is well known that (X, d_∞) *is a complete metric space. Let* $A = \{a, b\}$ *which* $a = (0, 0, \ldots)$ *and* $b = (1, 1, \ldots)$*, since* A *is compact we get* $\alpha(A) = 0$.

For $co(A)$ *note that if* $A \subset B(x, r)$ *for some* $x \in X$ *and* $r > 0$ *we must have,* $\|x\|_\infty \leqslant r$ *and* $|1 - \|x\|_\infty| \leqslant r$,

(i) if $1 - \|x\|_\infty \geqslant 0$, *then* $1 - r \leqslant 1 - \|x\|_\infty \leqslant r$, *hence* $r \geqslant \dfrac{1}{2}$

(ii) if $1 - \|x\|_\infty \leqslant 0$, *then* $r \geqslant \|x\|_\infty \geqslant 1$, *hence* $r > \dfrac{1}{2}$, *then any closed ball which contains* A, *its radius is greater than or equal to* $\dfrac{1}{2}$.

For any closed ball which contains A, *and centered at* $x = (x_n)_{n \in \mathbb{N}} \in X$ *with radius equal to* $\dfrac{1}{2}$ *we have,*

$$\begin{cases} \|a - x\|_\infty \leqslant & \dfrac{1}{2} \\ \|b - x\|_\infty \leqslant & \dfrac{1}{2} \end{cases}$$

then for each $n \in \mathbb{N}$, $|x_n| \leqslant \frac{1}{2}$ *and* $|1 - x_n| = 1 - x_n \leqslant \frac{1}{2}$, *hence* $x_n = \frac{1}{2}$, *which implies for any closed ball which contains* A, *with radius equal to* $\dfrac{1}{2}$ *is centered at* $x = \left(\dfrac{1}{2}, \dfrac{1}{2}, \ldots\right)$, *as* $co(A) = B\left(x, \dfrac{1}{2}\right)$ *is not a compact in* X *we get* $\alpha(co(A)) \neq \alpha(A)$.

As a direct consequence of Theorem3.2 , we have the next corollary.

Corollary 3.6. *Let* (X, D) *be a complete metric type space and* A *is a nonempty bounded subadmissible subset of* X. *If* $T \in KKM(A, A)$ *is a co-set contraction, and closed with* $\overline{T(A)} \subset A$, *then* T *has a fixed point in* A.

From Theorem 3.2 we have immediately the following [[6], Theorem 1]

Theorem 3.7. *Let* (X, d) *be a complete metric space and* A *is a nonempty bounded nearly-subadmissible subset of* X. *If* $T \in KKM(A, A)$ *is a co-set contraction , and closed with* $\overline{T(A)} \subset A$, *then* T *has a fixed point in* A.

By a similar proof as that given in [[6], Theorem 2] we can obtain the following result.

Corollary 3.8. *Let (X,D) be a metric type space and A is a nonempty nearly-subadmissible subset of X. If $T \in KKM(A,A)$ is compact and closed, then T has a fixed point in A.*

As a consequence,

Corollary 3.9. *Let (X,D) be a metric type space and A is a nonempty nearly-subadmissible subset of X. If $S,T \in KKM(A,A)$ are closed. We assume T or S is compact. Then there exists $x_0 \in A$ such that $x_0 \in T(x_0) \bigcap S(x_0)$.*

From Corollary 3.8, we have immediately the following [[14], Theorem 4.2].

Theorem 3.10. *Let (X,D) be a metric type space and A a nonempty subadmissible subset of X. Let $T \in KKM(A,A)$ be closed and compact. Then T has a fixed point, i.e. there exists $x \in A$ such that $x \in T(x)$.*

By a similar proof as that given in [[6], Theorem 4] we can obtain the following theorem.

Theorem 3.11. *Let A be a nonempty subadmissible subset of a metric type space (X,D), C a nonempty subset of A such that $co(C) \subset A$, and let $T : C \to 2^A$ be a closed-valued KKM mapping.*

Then $\{T(x)\}_{x\in C}$ has the finite intersection property.

The next result can be seen as a modification of condition $T \in \mathrm{KKM}(A,A)$ in Theorem by T is a lower semi-continuous.

Theorem 3.12. *Let (X,D) be a metric type space and A is a nonempty nearly-subadmissible subset of X.*

If $T : A \longrightarrow 2^A$ is lower semi-continuous, compact and closed, then T has a fixed point in A.

Proof. Let $\varepsilon > 0$ there exists a finite subset $C = \{x_1, \dots, x_n\}$ in A satisfying $\overline{T(A)} \subset \bigcup_{i=1}^n B_0(x_i,\varepsilon)$, since T is compact. A is nearly sub-admissible and $C = \{x_1, \dots, x_n\} \subset A$ then, there exists a continuous function $h : C \to A$ such that

$$x_i \in B_0(h(x_i),\varepsilon)\ \forall i \in \{1,2,\dots,n\} \quad \text{and} \quad Z = co(h(C)) \subset A.$$

Define $S : h(C) \to 2^Z$ by $S(h(x_i)) = \left\{z \in Z : T(z) \cap \overline{B_0(x_i,K\varepsilon)^{c}}^{c} = \emptyset\right\}$ for each $i \in \{1,\dots,n\}$.

T is lower semi-continuous, then $S(h(x_i))$ is closed in Z for all $i \in \{1,\dots,n\}$, and

$$\bigcap_{i=1}^n S(h(x_i)) = \left\{z \in Z : T(z) \cap \bigcup_{i=1}^n \overline{B_0(x_i,K\varepsilon)^{c}}^{c} = \emptyset\right\} = \emptyset$$

By Theorem 3.11, S is not a KKM map, and there exists a finite subset $h(C_1) = \{h(x_{i_1}), \ldots, h(x_{i_m})\} \in \langle h(C) \rangle$ such that

$$co(h(C_1)) \nsubseteq \bigcup_{j=1}^{m} S\left(h\left(x_{i_j}\right)\right)$$

thus there exists $y_\varepsilon \in co(h(C_1))$ such that $y_\varepsilon \notin \bigcup_{j=1}^{m} S\left(h\left(x_{i_j}\right)\right)$ wich implies by definition of S that for each $j \in \{1, \ldots, m\}$,

$$\emptyset \neq T(y_\varepsilon) \cap \overline{B_0\left(x_{i_j}, K\varepsilon\right)^{c}}^{c} \subset T(y_\varepsilon) \cap B_0\left(x_{i_j}, K\varepsilon\right)$$

Let $z \in T(y_\varepsilon) \cap B_0\left(x_{i_j}, K\varepsilon\right)$ for each $j \in \{1, \ldots, m\}$, then $z \in T(y_\varepsilon)$ and $z \in B_0\left(x_{i_j}, K\varepsilon\right)$ for each $j = 1, 2, \ldots, m$. This implies that $x_{i_j} \in B_0(z, K\varepsilon)$ for each $j = 1, \ldots, m$, since $h\left(x_{i_j}\right) \in B_0\left(x_{i_j}, \varepsilon\right)$, we have $h\left(x_{i_j}\right) \in B_0(z, K(1+K)\varepsilon)$ imlpies that

$$y_\varepsilon \in co(h(C_1)) \subset B_0(z, K(1+K)\varepsilon)$$

so $z \in B_0(y_\varepsilon, K(1+K)\varepsilon)$. Therefore $T(y_\varepsilon) \cap B_0(y_\varepsilon, K(1+K)\varepsilon) \neq \emptyset$ for each $\varepsilon > 0$ and since ε is arbitrary, T has the finite intersection property, for each $\varepsilon > 0$ there exist $x_\varepsilon, y_\varepsilon$ such that

$$x_\varepsilon \in T(y_\varepsilon) \cap B_0(y_\varepsilon, \varepsilon).$$

Now since T is compact we may assume that x_ε converge to some x_0 as $\varepsilon \to 0$. Consequently, $y_\varepsilon \to x_0$ and T is closed then $x_0 \in T(x_0)$.

Remark 3.13. *As we can see, the continuity of the function $h : C \to A$ is not needed in the proof, thus the assumption "A is nearly-subadmissibile" can be weakened. For example C. M. Yen* [6] *makes the assumption A is an almost-subadmissible set where he removed the continuity of h.*

4. Application : KKM mappings and coincidence theorem.

We give a new concept of quasi-subadmissible and quasi-upadmissible subsets in metric type space as follows :

Definition 4.1. *A real-valued function φ defined on a nonempty subadmissible set A in metric type space (X,D) is said to be :*

(i) metric type quasi-subadmissible if the set $\{x \in A : \varphi(x) > r\}$ is subadmissible for each $r \in \mathbb{R}$;

(ii) metric type quasi-upadmissible if $-\varphi$ is quasi-subadmissible.

(iii) lower semicontinuous if the set $\{x \in A : \varphi(x) > \lambda\}$ is open for each $\lambda \in \mathbb{R}$.

(iv) upper semicontinuous if $-\varphi$ is lower semicontinuous.

Remark 4.2. [[20], *Definition 2.3.16], recall that a point $x \in X$ is said to be a maximal of the set valued map $T : X \longrightarrow 2^X$ provided $Tx = \emptyset$, we will denote by $Max(T)$ the set of all maximal points of T.*

The next theorem is a sharpened version of Ky Fan's coincidence theorem [[11], Theorem 4.1] in the context of subadimissible sets in metric type space.

Theorem 4.3. *Let $A, B \subset X$ be nonempty compact subadmissible sets in the metric type space X. Let $T, S : A \longrightarrow 2^B$ be two set valued maps such that*

(i) $\mathscr{M}ax(T)$ and $\mathscr{M}ax\left(T^{-1}\right)$, $\mathscr{M}ax(S)$ and $\mathscr{M}ax\left(S^{-1}\right)$ have finite cardinality

(ii) Tx and $S^{-1}y$ are open sets for each $x \in A$ and $y \in B$;

(iii) Sx and $T^{-1}y$ are subadmissible sets for each $x \in A$ and $y \in B$

(iv) And for each finite subset $\{(u_1,v_1),\dots,(u_n,v_n)\}$ in $A \times B$ we have

$$co\left(\{(u_1,v_1),\dots,(u_n,v_n)\}\right) \subseteq co\left(u_1,\dots,u_n\right) \times co\left(v_1,\dots,v_n\right).$$

Then there is an $x_0 \in A$ such that $Tx_0 \cap Sx_0 \neq \emptyset$.

Example 4.4. *Let $X = \mathbb{R}$ and $D(x,y) = (x-y)^2$ for each $x, y \in \mathbb{R}$, then obviously (X,D) is metric type space. Choose A and B as follows :*

$$\begin{cases} A &= B\left(\frac{1}{2},\frac{1}{4}\right) = [0,1] \\ B &= B(0,1) = [-1,1] \end{cases}$$

and define $T, S : A \longrightarrow 2^B$ by

$$Tx =]-1, 1-x[\text{ for each } x \in A. \quad \text{And,} \quad Sx = \begin{cases} \emptyset & \text{if } x = \{0,1\} \\ ]-1,1[& \text{if } x \in]0,1[\end{cases}$$

Hence,

$$T^{-1}y = \begin{cases} [0,1] & \text{if } y \in]-1,0[\\ [0,1-y[& \text{if } y \in [0,1[\\ \emptyset & \text{if } y = -1 = 1 \end{cases}$$

and

$$S^{-1}y = \begin{cases} \emptyset & \text{if } y = \{-1,1\} \\ ]0,1[& \text{if } y \in]-1,1[\end{cases}$$

Since the empty set $\emptyset$ is assumed to be admissible so it is subadmissible, and for each finite subset $\{(u_1,v_1),\dots,(u_n,v_n)\}$ in $A\times B$ we have $co(\{(u_1,v_1),\dots,(u_n,v_n)\}) = \bigcap B^{uv}$ where B^{uv} is a closed ball in $A\times B$ such that $\{(u_1,v_1,\dots,(u_n,v_n)\} \subset B^{uv}$ with $u=(u_1,\dots,u_n)$ and $v=(v_1,\dots,v_n)$, since $B^{uv}\subset B^u\times B^v$, where B^u and B^v are closed balls in A and B respectively, and

$$\bigcap B^{uv} \subset \bigcap B^u \times \bigcap B^v$$

we get,

$$co(\{(u_1,v_1),\dots,(u_n,v_n)\}) \subset co(u_1,\dots,u_n)\times co(v_1,\dots,v_n).$$

So all the assumptions of Theorem4.3 are satisfied then there is an $x_0\in A$ such that $Tx_0\cap Sx_0\neq\emptyset$, for that we can see that $x_0=\frac{1}{2}$ verify the desired requirement.

Proof of Theorem 4.3

Since $A\times B$ is compact, then for any $\varepsilon>0$ there exists a finite subset $\{x_1,x_2,\dots,x_n\}\times\{y_1,y_2,\dots,y_n\}\subset A\times B$, such that

$$A\times B\subset \bigcup_{i=1}^{n} B_0(x_i,\varepsilon)\times B_0(y_i,\varepsilon)$$

Define a set valued map L by : $(x,y)\longmapsto (A\times B)\cap\left(S^{-1}y\times Tx\right)^c$ for each $x\in A\setminus Max(T)$ and $y\in B\setminus Max\left(S^{-1}\right)$, each $L(x,y)$ is nonempty closed set in $A\times B$ then it is compact. Since for each $x\in A$ and $y\in B$, $S^{-1}y$ and Tx are open sets in A and B respectively, we have $S^{-1}y\times Tx\subset A\times B\subset\bigcup_{i=1}^{n}B_0(x_i,\varepsilon)\times B_0(y_i,\varepsilon)$. then

$$\bigcap_{i=1}^{n} L(x_i,y_i) = (A\times B)\cap\left\{\bigcup_{i=1}^{n}\left(S^{-1}y_i\times Tx_i\right)\right\}^c \subset (A\times B)\cap\left(\bigcup_{i=1}^{n}B_0(x_i,\varepsilon)\times B_0(y_i,\varepsilon)\right)^c = \emptyset$$

then by Theorem 3.11, L is not a KKM map. Therefore there are a finite elements $\{z_1, z_2, \ldots, z_m\}$ in $A \times B$ such that

$$co(z_1, z_2, \ldots, z_m) \nsubseteq \bigcup_{i=1}^{m} L(z_i)$$

with $z_i = (u_i, v_i)$ for each $i \in \{1, 2, \ldots, m\}$.

So there exists $\omega = (\mu, \nu) \in co(z_1, z_2, \ldots, z_m) \subset co(u_1, \ldots, u_m) \times co(v_1, \ldots, v_m)$ such that $\omega \notin L(z_i)$ for any $i \in \{1, 2, \ldots, m\}$. In other words, we have $\omega \in \bigcap_{i=1}^{m} \left(S^{-1} v_i \times T u_i\right)$ for $i \in \{1, 2, \ldots, m\}$. So $\mu \in S^{-1} v_i$ and $\nu \in T u_i$ for each $i \in \{1, 2, \ldots, m\}$, the first inclusion i.e. $\mu \in S^{-1} v_i$ shows that $v_i \in S\mu$ for each $i = 1, 2, \ldots, m$. And the second inclusion shows that $u_i \in T^{-1} \nu$ for each $i = 1, \ldots, m$. Since $S\mu$ and $T^{-1}\nu$ are both subadmissible sets we have for each $i \in \{1, 2, \ldots, m\}$,

$$\begin{aligned} co(u_1, \ldots, u_m) &\subset T^{-1}\nu \\ co(v_1, \ldots, v_m) &\subset S\mu \end{aligned}$$

and since $\mu \in co(u_1, \ldots, u_n)$ we have $\mu \in T^{-1}\nu \Leftrightarrow \nu \in T\mu$ and $\nu \in co(v_1, \ldots, v_n)$ then $\nu \in S\mu$ we conclude that $\nu \in T\mu \cap S\mu \neq \emptyset$, and the proof is complete.

Next theorem is an immediate application from the above coincidence result, is an analogous result of Minimax-type principle in metric type space.

Theorem 4.5. *Let $A, B \subset X$ be nonempty compact admissible sets in the metric type space X. Let $f : A \times B \longrightarrow \mathbb{R}$ satisfying :*

(i) $y \mapsto f(x, y)$ is lower semicontinous and metric type quasi-subadmissible for each fixed $x \in A$;

(ii) $x \mapsto f(x, y)$ is upper semicontinous and metric type quasi-upadmissible for each fixed $y \in B$.

(iii) And for each finite subset $\{(u_1, v_1), \ldots, (u_n, v_n)\}$ in $A \times B$ we have

$$co(\{(u_1, v_1), \ldots, (u_n, v_n)\}) \subseteq co(u_1, \ldots, u_n) \times co(v_1, \ldots, v_n).$$

Then

$$\sup_{x \in A} \inf_{y \in B} f(x, y) = \inf_{y \in B} \sup_{x \in A} f(x, y).$$

Proof. Because of upper semicontinuity, $\sup\limits_{x\in A} f(x,y)$ exists for each y and is a lower semicontinuous function of y, so $\inf\limits_{y\in B}\sup\limits_{x\in A} f(x,y)$ exists; similarly, $\sup\limits_{x\in A}\inf\limits_{y\in B} f(x,y)$ exists.

Since $f(x,y) \leqslant \sup\limits_{x\in A} f(x,y)$ we have $\inf\limits_{y\in B} f(x,y) \leqslant \inf\limits_{y\in B}\sup\limits_{x\in A} f(x,y)$; therefore

$$\sup_{x\in A}\inf_{y\in B} f(x,y) \leqslant \inf_{y\in B}\sup_{x\in A} f(x,y).$$

Next we prove that

$$\inf_{y\in B}\sup_{x\in A} f(x,y) \leqslant \sup_{x\in A}\inf_{y\in B} f(x,y). \tag{4}$$

Suppose the contrary, that inequality (4) hold, then by density of $\mathbb{R}$ there would be some $r \in \mathbb{R}$ satisfying

$$\sup_{x\in A}\inf_{y\in B} f(x,y) < r < \inf_{y\in B}\sup_{x\in A} f(x,y)$$

Define two set valued maps $T, S : A \to 2^B$ by

$$Sx = \{y \in B : f(x,y) < r\}, \quad \text{and} \quad Tx = \{y \in B : f(x,y) > r\}$$

. Each Tx is open since $y \mapsto f(x,y)$ is lower semicontinous, Sx is subadmissible by the metric type quasi-upadmissibility of $y \mapsto f(x,y)$, and is nonempty because

$$\sup_{x\in A}\inf_{y\in B} f(x,y) < r.$$

Since $T^{-1}y = \{x \in A : f(x,y) > r\}$ and $S^{-1}y = \{x \in A : f(x,y) < r\}$, we find in the same way that each $T^{-1}y$ is nonempty subadmissible and $S^{-1}y$ is open. Then by the coincidence theorem , there would be some (x_0, y_0) with

$$y_0 \in T(x_0) \cap S(x_0)$$

which gives the contradiction $r < f(x_0, y_0) < r$. Thus, the inequality cannot hold, and the proof is complete.

We conclude with an example:

Example 4.6. *Let $A = B = [0,1]$ in $X = \mathbb{R}$ equipped with the usual distance, and let f defined on $A \times B$ by $f(x,y) = y$, for each $x \in \mathbb{R}$. Both of mappings $x \mapsto f(x,y)$ and $y \mapsto f(x,y)$ are continuous, then $x \mapsto f(x,y)$ is upper semi-continuous and $y \mapsto f(x,y)$ is lower semi-continuous.*

The set $\{y \in B : f(x,y) > r\}$ *is subadmissible for each* $r \in \mathbb{R}$*, indeed :*

$$\{y \in B : f(x,y) > r\} = \begin{cases} [0,1] & \text{if } r < 0 \\ \emptyset & \text{if } r \geqslant 1 \\ ]r,1] & \text{if } 0 \leqslant r < 1 \end{cases}$$

and $\{x \in A : f(x,y) < r\}$ *is also subadmissible for each* $r \in \mathbb{R}$ *since*

$$\{x \in A : f(x,y) < r\} = \begin{cases} [0,1] & \text{if } y < r \\ \emptyset & \text{if } y \geqslant r \end{cases}$$

Then all assumptions of Theorem4.5 are satisfied, hence

$$\sup_{x\in A} \inf_{y\in B} f(x,y) = \inf_{y\in B} \sup_{x\in A} f(x,y) = 0.$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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