Available online at http://scik.org
Adv. Fixed Point Theory, 6 (2016), No. 4, 352-386
ISSN: 1927-6303

FIXED POINT RESULTS IN MULTIPLICATIVE GENERALIZED METRIC SPACES

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Abstract. In this paper, first we introduce the notion of multiplicative generalized metric spaces and then prove the Banach contraction principle in setting of newly defined spaces. We also introduce the notion of weakly commuting, compatible maps and its variants, weakly compatible, weakly compatible with properties (E.A) and CLR in this space. Further, we prove some fixed point theorems on multiplicative generalized metric spaces and provide some suitable examples in support of our results.

Keywords: multiplicative generalized metric spaces; weakly commuting; compatible and its variants; weakly compatible; property (E.A) and property (CLR).

2010 AMS Subject Classification: 47H10, 54H25.

1. Introduction

In 2007, Bashirov [1] defined multiplicative calculus and gave the notion of multiplicative metric spaces as follows.

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Received May 4, 2016

Let X be a non empty set. A multiplicative metric is a mapping $d : X \times X \to R^+$ satisfying the following conditions:

(m1) $d(x,y) > 1$ for all $x,y \in X$ and $d(x,y) = 1$ if and only if $x = y$,

(m2) $d(x,y) = d(y,x)$ for all $x,y \in X$,

(m3) $d(x,z) \leq d(x,y) \cdot d(y,z)$ for all $x,y,z \in X$ (multiplicative triangle inequality).

The pair (X,d) is called a multiplicative metric space.

For more detail on multiplicative metric spaces, one can refer to [7].

In 2006, Mustafa and Sims [6] introduce the notion of G-metric space as follows.

Let X be a non empty set and let $G : X \times X \times X \to R^+$ be a function satisfying the following:

(G1) $G(x,y,z) = 0$ if $x = y = z$,

(G2) $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq ?y$,

(G3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z \neq ?y$,

(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetry in all variables),

(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangular inequality),

Then the function G is called a generalized metric and the pair (X,G) is known as generalized metric spaces.

We note that the set of positive real numbers $\mathbb{R}_+$ is not complete according to the generalized metric G defined as

$$G(x,y,z) = (|x-y| + |y-z| + |z-x|) \quad \text{for all } x,y,z \in X.$$

Let $X = \mathbb{R}_+$. Consider the sequence $x_n = \left\{ \frac{1}{n} \right\}$. It is obvious $\{x_n\}$ is a Cauchy sequence in X with respect to generalized metric and X is not complete G-metric space, since $0 \notin R_+$.

Now to overcome this problem we introduce the notion of multiplicative generalized metric spaces similar to notion of multiplicative metric spaces defined by Bashirov [1] as follows:

Definition 1.1. Let X be a non empty set. Let $G : X \times X \times X \to \mathbb{R}_+$ be a function satisfying:

(GM1) $G(x,y,z) = 1$ if $x = y = z$,

(GM2) $1 < G(x,x,y)$ for all $x,y \in X$ with $x \neq ?y$,

(GM3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$,

(GM4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetry in all variables), (GM5) $G(x,y,z) \leq G(x,a,a) \cdot G(a,y,z)$ for all $x,y,z,a \in X$ (rectangular multiplicative inequality),

Then the function G is called a multiplicative generalized metric and the pair (X,G) is called multiplicative generalized metric spaces. We shall denote it for briefly multiplicative G-metric spaces.

We note that if (X,d) be a multiplicative space then we can define multiplicative G-metric on X by setting $G(x,y,z) = \{d(x,y,)d(y,z) \cdot d(z,x)\}^{\frac{1}{3}}$.
Now

(GM1) $G(x,y,z) = 1$ if $d(x,y) = d(y,z) = d(z,x) = 1$, implies $x = y = z$,

(GM2) $G(x,x,y) = \{d(x,x) \cdot d(x,y) \cdot d(y,x)\}^{\frac{1}{3}} = \{d(x,y,) \cdot d(y,x)\}^{\frac{1}{3}} > 1$ for $x \neq y$.

(GM3) Since $d(x,y) \leq d(x,z) \cdot d(z,y)$, implies $(d(,x,y) \cdot d(y,x))^{\frac{1}{3}} \leq (d(x,z) \cdot d(z,y) \cdot d(y,x))^{\frac{1}{3}}$ i.e., $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$.

(GM4) Symmetry is obvious.

(GM5) $G(x,a,a) \cdot G(a,y,z) = (d(x,a) \cdot d(a,a) \cdot d(a,x))^{\frac{1}{3}} \cdot (d(a,y) \cdot d(y,z) \cdot d(z,a))^{\frac{1}{3}}$
$\geq ((x,y) \cdot d(y,z) \cdot d(z,x)^{\frac{1}{3}} = G(x,y,z)$.
Hence G is multiplicative generalized metric on X.

Remark 1.2. We note that $\mathbb{R}_+$ is complete in case of multiplicative generalized metric spaces. For this we take sequence $x_n = \{a^{\frac{1}{n}}\}$, where $a > 1$. Then $\{x_n\}$ is a Cauchy sequence since for $n > m > l$,

$$\begin{aligned}
G(x_n,x_m,x_l) &= \left|\frac{x_n}{x_m}\right|^* \left|\frac{x_m}{x_l}\right|^* \left|\frac{x_l}{x_n}\right|^* \\
&= \left|\frac{a^{\frac{1}{n}}}{a^{\frac{1}{m}}}\right|^* \left|\frac{a^{\frac{1}{m}}}{a^{\frac{1}{l}}}\right|^* \left|\frac{a^{\frac{1}{l}}}{a^{\frac{1}{n}}}\right|^* \\
&= |a^{\frac{1}{n}-\frac{1}{m}}|^* |a^{\frac{1}{m}-\frac{1}{l}}|^* |a^{\frac{1}{l}-\frac{1}{n}}|^*| \\
&= a^{2(\frac{1}{l}-\frac{1}{n})} \\
&< a^{2(\frac{1}{n}+\frac{1}{n})} < \varepsilon \text{ if } n,m,l > 4\frac{\log a}{\log \varepsilon},
\end{aligned}$$

where $|a|^* = \begin{cases} a & \text{if } a \geq 1; \\ \dfrac{1}{a} & \text{if } a < 1. \end{cases}$

Since $\{x_n\} \to 1$ as $n \to \infty$ and $1 \in \mathbb{R}_+$. Hence (X,d) is complete multiplicative metric space

Example 1.3. Let $X = R^+$. Define $G^* : X \times X \times X \to [1,\infty)$ as

$$G^*(x,y,z) = \left(\left|\frac{x}{y}\right|^* \cdot \left|\frac{y}{z}\right|^* \left|\frac{z}{x}\right|^* \right),$$

where $|x|^* = \begin{cases} x & \text{if } fx \geq 1 \\ \dfrac{1}{x} & \text{if } x < 1. \end{cases}$

Obviously the properties (GM1), (GM2), (GM3) and (GM4) hold clearly.

For (GM 5), we have

$$\begin{aligned} G^*(x,a,a).G^*(a,y,z) &= \left|\frac{x}{a}\right|^* \cdot \left|\frac{a}{a}\right|^* \cdot \left|\frac{a}{x}\right|^* \cdot \left|\frac{a}{x}\right|^* \cdot \left|\frac{x}{y}\right|^* \cdot \left|\frac{y}{a}\right|^* \\ &= \left|\frac{x}{a}\right|^* \cdot \left|\frac{a}{x}\right|^* \cdot \left|\frac{a}{y}\right|^* \cdot \left|\frac{y}{z}\right|^* \cdot \left|\frac{z}{a}\right|^* \\ &> \left|\frac{x}{y}\right|^* \cdot \left|\frac{y}{z}\right|^* \cdot \left|\frac{z}{x}\right|^* \\ &= G^*(x,y,z) \quad \text{forall } x,y,z \in X. \end{aligned}$$

Hence G^* is multiplicative generalized metric on X and (X,G^*) is a multiplicative G-metric space.

Example 1.4. Let $X = R$. Define $G^* : X \times X \times X \to [1,\infty)$ as $G^*(x,y,z) = a^{(|x-y|+|y-z|+|z-x|)}$, where $x,y,z \in X$ and $a > 1$. Then clearly G^* is a multiplicative generalized metric space.

Remark 1.5. We note that Example 1.3 is valid for positive real numbers and Example 1.4 is valid for all real numbers.

Now we state some useful properties of multiplicative G-metric spaces.

Proposition 1.6. *Let (X,G) be a multiplicative G-metric space, then for any $x,y,z,a \in X$, we have*

(i) $G(x,y,z) = 1$ *then* $x = y = z$

(ii) $G(x,y,z) \leq G(x,a,a) \cdot G(y,a,a) \cdot G(z,a,a)$

(iii) $G(x,y,z) \leq G(x,x,y) \cdot G(x,x,z)$

(iv) $G(x,y,y) \leq G^2(y,x,x)$.

It is easy to prove (i) to (iv).

Proposition 1.7. *Let (X,G) be a multiplicative G-metric space and let $k > 1$ then G_1 is also a multiplicative G-metric on X, where $G_1(x,y,z) = \min\{k, G(x,y,z)\}$.*

Remark 1.8. For any non empty set X, we can define a multiplicative G-metric on X and conversely.

For any multiplicative G-metric on X, we define $d_G(x,y) = G(x,y,y) \cdot G(x,x,y)$

(i) Clearly $d_G(x,y) \geq 1$.

(ii) $d_G(x,y) = 1$ iff $G(x,y,y) \cdot G(x,x,y) = 1$

iff $G(x,y,y) = G(x,x,y) = 1$

iff $x = y$.

(iii) $d_G(y,x) = G(y,x,x) \cdot G(y,y,x) = G(x,y,y) \cdot G(x,x,y) = d_G(x,y)$.

(iv) $d_G(x,z) \cdot d_G(z,y) = G(x,z,z) \cdot G(x,x,z) \cdot G(z,y,y) \cdot G(z,z,y)$

$\geq G(x,y,y) \cdot G(x,x,y) = d_G(x,y)$.

Hence d_G is multiplicative metric on X.

Remark 1.9. It can easily be shown that every generalized metric need not be multiplicative generalized metric and conversely i.e., generalized metric spaces and multiplicative generalized metric spaces are independent.

Consider $X = \mathbb{R}_+$ and generalized metric be defined as $G(x,y,z) = (|x-y| + |y-z| + |z-x|)$ for all $x,y,z \in X$.

Then (X,G) is a generalized metric space. But it is not multiplicative generalized metric space. For this take $x = 2$, $y = 4$, $z = 6$ and $a = 2$. Then $G(x,y,z) = 8$ and $G(x,a,a) \cdot G(a,y,z) = 0$ implies

$$G(x,y,z) > G(x,a,a) \cdot G(a,y,z).$$

This implies that (X,G) is not a multiplicative generalized metric space.

Now consider $X = \mathbb{R}_+$ and define $G^* : X \times X \times X \times X \to [1, \infty)$ as

$$G^*(x,y,z) = \left(\left|\frac{x}{y}\right|^* \cdot \left|\frac{y}{z}\right|^* \cdot \left|\frac{z}{x}\right|^* \right),$$

where $|x|^* = \begin{cases} x & \text{if } x \geq 1 \\ \frac{1}{x} & \text{if } x < 1. \end{cases}$

Then (X,G) be a multiplicative generalized metric space.

Now for $x = 2$, $y = 4$, $z = 10$ and $a = 6$, $G^*(x,y,z) = 25$ and $G^*(x,a,a) + G^*(a,y,z) = 9 + 6.25 = 15.25$.

Then clearly $G^*(x,y,z) > G^*(x,a,a) + G^*(a,y,z)$ implies that G is not a generalized metric on X.

2. The multiplicative G-metric topology

Definition 2.1. Let (X,G) be a multiplicative G-metric space then for $x_0 \in X$, $r > 1$, the multiplicative G-open ball with centre x_0 and radius r is defined as

$$B_G(x_0,r) = B_r(x_0) = \{y \in X, G(x_0,x,x) < r\}.$$

Similarly, we can define multiplicative G-closed ball as $B_G(x_0,r) = \{y \in X, G(x_0,x,x) \leq r\}$.

Proposition 2.2. *Let (X,G) be a multiplicative G-metric space. Then for any $x_0 \in X$ and $r > 1$, we have*

(i) *if $G(G_0,x,y) < r$, then $x,y \in B_G(x_0,r)$.*

(ii) *If $y \in B_G(x_0,r)$, then there exists $\delta > 1$ such that $B_G(y,\delta) \subseteq B_G(x_0,r)$.*

Proof. (i) The proof follows obviously from (GM3) since

$$G(x_0,x,x) \leq G(x_0,x,y) < r \quad \text{implies} \quad x \in B_G(x_0,r).$$

Also, $G(x_0,y,y) \leq G(x_0,x,y) < r$ implies $x \in B_G(x_0,r)$.

(ii) The proof follows from (GM5) with $\delta = r/G(x_0,y,y)$.

For $y \in B_G(x_0,r)$ implies $G(x_0,y,y) < r$ with $\delta > 1$.

Let $z \in B_G(y,\delta)$ implies $G(y,x,x) < \delta = \dfrac{r}{G(x_0,y,y)}$ i.e., $G(x_0,y,y) \cdot G(y,z,z) < r$.

Therefore, by rectangular inequality, we have $G(x_0,z,z) < r$ i.e., $z \in B_G(x_0,r)$.

Hence $B_G(y,\delta) \subseteq B_G(x_0,r)$. $\square$

Remark 2.3. It follows from Proposition 2.2(ii) that the family of G-balls $B = \{B_G(x,r) : x \in X, r > 1\}$ is the base for the multiplicative G-metric topology.

Proposition 2.4. *Let (X,G) be a multiplicative G-metric space.*
For all $x_0 \in X$ and $r > 1$, we have

$$B_G\left(x_0, \frac{r}{3}\right) \subseteq B_G(x_0,r) \subseteq B_G(x_0,r).$$

Consequently, the multiplicative G-topology $T(G)$ coincides with the multiplicative topology arising from d_G. Thus every multiplicative G-metric space is topologically equivalent to a multiplicative metric space.

Definition 2.5. Let (X,G) be a multiplicative G-metric space and $A \subset X$. We call $x \in A$ is a multiplicative G-interior point of A if there exists $\varepsilon > 1$ such that $B_\varepsilon(x) \subset A$.
The collection of all the interior points of A is called multiplicative G-interior of A and denoted by $int(A)$.

Definition 2.6. Let (X,G) be a multiplicative G-metric space and $A \subset X$. If every point of A is a multiplicative G-interior point of A, then A is called a multiplicative G-open set.

Lemma 2.7. *Let (X,G) be a multiplicative G-metric space then each multiplicative G-open ball of X is a multiplicative G-open set.*

Proof. Let $x \in X$ and $B_\varepsilon(x)$ be a multiplicative G-open ball. For $y \in B_\varepsilon(x)$, consider $\delta = \dfrac{\varepsilon}{G(x,y,y)}$ and $z \in B_\delta(y)$ then $G(y,z,z) < \delta$, i.e., $G(y,z,z) < \dfrac{\varepsilon}{G(x,y,y}$.
This implies $G(y,z,z) \cdot G(x,y,y) < \varepsilon$.
Now $G(x,z,z) < G(y,z,z) \cdot G(x,y,y) < \varepsilon$ implies GB_ε i.e., $B_\delta(y) \subset B_\in(x)$.
Thus $B_\varepsilon(x)$ is multiplicative G set. $\square$

Lemma 2.8. *Let (X,G) be a multiplicative G-metric space then X and ϕ are multiplicative G-open sets.*

Lemma 2.9. *The union of any countable or uncountable family of multiplicative G-open sets is also a multiplicative G-open set.*

Lemma 2.10. *The intersection of any finite family of multiplicative G-open sets is also a multiplicative G-open set.*

Proof. Let B_1 and B_2 be two multiplicative G-open sets and $y \in B_1 \cap B_2$ then there exists δ_1 and $\delta_2 > 1$ such that $B_{\delta_1} \subset B_1$ and $B_{\delta_2} \subset B_2$. Letting $\delta = \min\{\delta_1, \delta_2\}$, we conclude that $B_\delta \subset B_1$. Hence the intersection of any finite family of multiplicative G-open sets is a multiplicative G-open set. □

Definition 2.11. Let (X,G) be a multiplicative G-metric space. A point $x \in X$ is said to be a multiplicative G-limit point of $S \subset X$ iff $(B_\delta)\{x\} \setminus \{x\}) \cap S \neq \phi$ for every $\varepsilon > 1$. The set of all multiplicative G-limit points of the set S is denoted by S'.

Definition 2.12. Let (X,G) be a multiplicative G-metric space. We call a set $S \subset X$ multiplicative G-closed in (X,G) if S contains all of its multiplicative G-limit points.

Proposition 2.13. *Let (X,G) be a multiplicative G-metric space and $S \subset X$. Then $S \cup S'$ is a multiplicative G-closed set. This set is called multiplicative G-closure of the set S, which is denoted by $\overline{S}$.*

Proposition 2.14. *Let (X,G) be a multiplicative G-metric space and $S \subset X$. S is multiplicative G-closed iff $X \setminus S$, the complement of S, is multiplicative G-open.*

Definition 2.15. Let (X,G) be a multiplicative G-metric space. The sequence $\{x_n\}$ in X is said to be

(i) multiplicative G-convergent to $x \in X$ if for every multiplicative G-open ball $B_\varepsilon(x)$, there exists a natural number N such that $x_n \in B_\varepsilon(x)$ for all $n \geq N$. In brief denoted by $x_n \to x$ as $n \to \infty$.

(ii) multiplicative Cauchy if for all $\varepsilon > 1$, there exists a natural number N such that $G(x_m, x_l, x_n) > -\varepsilon$ for all $m,n,l \geq N$.

Lemma 2.16. *Let (X,G) be a multiplicative G-metric space and $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ in X is multiplicative G-convergent to $x \in X$ iff $G(x_n, x, x) \to 1$ as $n \to \infty$.*

Lemma 2.17. *Let (X,G) be a multiplicative G-metric pace and $\{x_n\}$ be a sequence in X. If the sequence is multiplicative G-convergent then it is multiplicative G-Cauchy sequence.*

Proof. Let $x \in X$ such that $x_n \to x$. So we have for any $\varepsilon > 1$, there exists a natural number N such that

$$G(x_n,x,x) < \varepsilon^{\frac{1}{3}} \text{ and } G(x_m,x,x) < \varepsilon^{\frac{1}{3}} \text{ and } G(x_l,x,x) < \varepsilon^{\frac{1}{3}} \text{ for all } n,m,1 \geq N.$$

By multiplicative rectangular inequality, we get

$$G(x_m,x_n,x_l) \leq G(x_n,x,x), G(x_m,x,x) \cdot G(x_m,x,x) < \varepsilon^{\frac{1}{3}} \cdot \varepsilon^{\frac{1}{3}} \cdot \varepsilon^{\frac{1}{3}} \cdot \varepsilon^{\frac{1}{3}} = \varepsilon,$$

which implies that $\{x_n\}$ is a multiplicative G-Cauchy sequence. □

Definition 2.18. Let (X,G) be a multiplicative G-metric space and $A \subset X$. The set A is called multiplicative G-bounded if there exists $x \in X$ and $M > 1$ such that $A \subseteq B_M(x)$.

Lemma 2.19. *A G-multiplicative Cauchy sequence is multiplicative G-bounded.*

Proof. Let (X,G) be multiplicative G-metric space and $\{x_n\}$ be a multiplicatve G-Cauchy sequence in X. From definition of Cauchy sequence for $\varepsilon = 2 > 1$, there exists a natural number n_0 such that $G(x_n,x_m,x_l) < 2$ for all $n,m,l \geq n_0$. Hence we have

$$M = \max\{x, G(x_n,x_{n_0},x_{n_0})\} < m \quad \text{for all } n \in N.$$

Thus we have

$$G(x_m,x_n,x_1) \leq G(x_n,x_{n_0},x_{n_0}) \cdot G(x_m,x_{n_0},x_{n_0}) \cdot G(x_l,x_{n_0},x_{n_0}) < M^3 \text{ for all } m,n,l \in N.$$

Thus the sequence is multiplicative G-bounded. □

Lemma 2.20. *Let $\{x_n\}$ be a multiplicative G-Cauchy sequence in a multiplicative G-metric space (X,G). If the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x \in X$ as $n_k \to \infty$, then $x_n \to X$ as $n \to \infty$.*

Lemma 2.21. *Let (X,G) be a multiplicative G-metric space and $\{x_n\}$ be a sequence in X. If the sequence $\{x_n\}$ is multiplicative G-convergent then the multiplicative G-limit point is unique.*

Proof. Let $x,y \in X$ such that $x_n \to X$ and $y_n \to y$ as $n \to \infty$ i.e., for every $\varepsilon > 1$, there exists $N \in N$ such that for all $n \geq N$, we have $G(x_n,x,x) < \sqrt{\varepsilon}$ and $G(x_n,y,y) < \sqrt{\varepsilon}$. Then we have $G(x,y,y) \leq G(x_n,x,x) \cdot G(x_n,y,y) < \varepsilon$. Since ε is arbitrary, therefore $G(x,y,y) = 1$ i.e., $x = y$. $\square$

Proposition 2.22. *Let (X,G) be a multiplicative G-metric space and $\{x_n\}$ be sequence in X and a point $x \in X$. The following are equivalent:*

(i) *$\{x_n\}$ is multiplicative G-convergent to x.*

(ii) *$d_G(x_n,x,x) \to 1$ as $n \to \infty$ i.e., $\{x_n\}$ converges to x relative to metric d_G.*

(iii) *$G(x_n,x_n,x) \to 1$ as $n \to \infty$.*

(iv) *$G(x_m,x_n,x) \to 1$ as $n \to \infty$.*

Definition 2.23. Let (X,G) and (Y,G') be two multiplicative G-metric spaces and $f : X \to Y$ is said to be continuous if given an $\varepsilon > 1$, there exists $\delta > 1$ such that $f(B_\delta(x)) \subset B \in (f(x))$, for all x in X.

Theorem 2.24. *Let (X,G) and (Y,G') be two multiplicative G-metric spaces and $f : X \to Y$ be a mapping. Then f is multiplicative G-continuous at a point $x \in X$ iff $f\{x_n\} \to f(x)$ for every sequence $\{x_n\}$ with $x_n \to x$ as $n \to \infty$.*

Proof. Suppose that f is multiplicative G-continuous at x and $x_n \to x$. From the definition of G-multiplicative continuity of f, we have, for every $\varepsilon > 1$, there exists$\delta > 1$ such that $f(B_\delta(x)) \subset B_\varepsilon(f(x))$.

Since $x_n \to x$ as $n \to \infty$, there exists N such that $n \geq N$ implies $x_n \in B_\delta(x)$. Then $f(x_n) \in B_\varepsilon(f(x))$ and hence $f(x_n) \to f(x)$ $(n \to \infty)$.

Conversely, assume that f is not G-multiplicative continuous at x, i.e., there exists an $\varepsilon > 1$ such that for each $\delta > 1$, we have $x' \in X$ with $G(x',x,x) < \delta$ but

$$G'(f(x'),f(x),f(x)) \geq \varepsilon. \tag{2.1}$$

Let us take any sequence of real numbers $\{\delta_n\}$ such that $\delta_n \to 1$ and $\delta_n > 1$ for each n.

For each n choose x that satisfy (2.1) and call this x'_n.

It is clear that $x_n' \to_* x$. but $f(x_n)$ is not multiplicative G-convergent to $f(x)$ implies f is not G-multiplicative continuous then not every sequence $\{x_n\}$ with $x_n \to x$ will yield a sequence $f(x_n) \to f(x)$, a contradiction. Hence the result. $\square$

3. Fixed point theorems

Now we introduce the multiplicative G-contraction map as follows.

Definition 3.1. Let (X,G) be a multiplicative G-metric space. A mapping $f : X \to X$ is called multiplicative G-contraction if there exists a real constant $\lambda \in [0,1)$ such that

$$G(fx, fy, fz) \leq G^{\lambda}(x,y,z) \quad \text{for all } x,y,z \in X. \tag{3.2}$$

Now in the setting of multiplicative G-metric spaces, we state Banach contraction principle in the setting of multiplicative G-metric space as follows:

Theorem 3.2. *Let T be a multiplicative G-contraction mapping from complete multiplicative G-metric space (X,G) into itself. Then f has a unique fixed point or every multiplicative G-contraction mapping on a complete multiplicative G-metric space has a fixed point.*

Proof. Consider the point $x_0 \in X$ then there exist $X \in X$ such that $fx_0 = x_1$, for this $x_1 \in X$ there exists $x_2 \in X$ such that $fx_1 = x_2$.
Continuing like this, we get a sequence $\{x_n\}$ in X such that $fx_{n-1} = x_n$.
Now consider

$$\begin{aligned} G(x_{n+1},x_n,x_n) &= G(fx_n, fx_{n-1}, fx_{n-1}) \\ &\leq G^{\lambda}(x_n,x_{n-1},x_{n-1}). \end{aligned}$$

Similarly, we get $G(x_n,x_{n-1},x_{n-1}) \leq G^{\lambda^n}(x_1,x_0,x_0)$.
Let $m,n,l \in N$ such that $m > n > l$, then we have

$$\begin{aligned} G(x_m,x_n,x_1) &< G(x_m,x_{m-1},x_{m-1}) \cdot G(x_{m-1},x_n,x_1) \\ &< G(x_m,x_{m-1},x_{m-1}) \cdot G(x_{m-1},x_{m-2},x_{m-2}) \cdot G(G(x_{m-2},x_n,x_1) \end{aligned}$$

$$< G(x_m,x_{m-1},x_{m-1})\cdots G(x_{n-1},x_{n-2},x_{n-2})\cdots G(G(x_{l+1},x_l,x_l)$$

$$\leq G^{\lambda^{m-1}+\cdots+\lambda^l}(x_1,x_0,x_0)$$

$$\leq G^{\frac{\lambda^l}{1-\lambda}}(x_1,x_0,x_0).$$

This implies that $G(x_m,x_n,x_1) \to 1$ as $m,n,l \to \infty$. Hence the sequence $\{x_n\}$ is a multiplicative G-Cauchy sequence in X. Since X is complete, so there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Also

$$G(fz,z,z) \leq (fz,fx_n,fx_n),G(fx_n,z,z)$$

$$\leq G^{\lambda}(z,x_n,x_n)\cdot G(x_{n+1},z,z) \to 1 \text{ as } n \to \infty.$$

i.e., $G(fz,z,z) = 1$. Therefore, $fz = z$.

Hence z is a fixed point of f i.e., $fz = z$.

Now if there is another point $y(\neq z)$ such that $fy = y$ then from (3.1)

$$G(z,y,y) = G(fz,fy,fy) \leq G^{\lambda}(z,y,y).$$

Hence $G(z,y,y) = 1$ and $y = z$, a contradiction.

This implies that z is the unique fixed point of f. □

Corollary 3.3. *Let (X,G) be a complete multiplicative G-metric space. For ε with $\varepsilon > 1$ and $x_0 \in X$, consider multiplicative G-closed ball $B_\varepsilon(x_0)$. Suppose the mapping $f : X \to X$ satisfies the multiplicative G-contraction condition*

$$G(fx,fy,fz) \leq G^{\lambda}(x,y,z) \quad \textit{for all } x,y \in B_\varepsilon,$$

where $\lambda \in [0,1)$ is a constant and $G(x_0,fx_0,fx_0) \leq \varepsilon^{1-\lambda}$.
Then f has a unique fixed point $\overline{B}_\varepsilon(x_0)$.

Proof. We only need to prove $B_\varepsilon(x_0)$ is complete and $fx \in B_\varepsilon(x_0)$ for all $x \in B_\varepsilon(x_0)$.
Suppose $\{x_n\}$ be a multiplicative Cauchy sequence in $B_\varepsilon(x_0)$. Then $\{x_0\}$ is also a multiplicative Cauchy sequence in X. By completeness of X, there exists $x \in X$ such that $x_n \to X$, therefore,

$$G(x_0,x,x) \leq G(x_0,x_n,x_n)\cdot G(x_n,x,x) < G(x_n,x,x)\cdot \varepsilon.$$

Since $x_n \to x$, therefore, $G(x_n,x,x) \to 1$. Hence $G(x_0,x,x) < \varepsilon$ and $x \in \overline{B}_\varepsilon$. It follows that $\overline{B}_\varepsilon(x_0)$ is complete. For every $x \in B_\varepsilon(x_0)$,

$$G(x_0,fx,fx) \leq G(x_0,fx_0,fx_0) \cdot G(fx_0,fx,fx) \leq \varepsilon^{1-\lambda} \cdot G^\lambda(x_0,x,x) \leq \varepsilon^{1-\lambda} \cdot \varepsilon^\lambda = \varepsilon$$

Thus $fx \in B_\varepsilon(x_0)$. □

Corollary 3.4. *Let (X,G) be a complete multiplicative G-metric space. If a mapping $f : X \to X$ satisfies for some positive integer n,*

$$G(f^n x, f^n y, f^n z) \leq G^\lambda(x,y,z) \quad \textit{for all } x,y,z \in X,$$

where $\lambda \in [0,1)$ is a constant then f has a unique fixed point in X.

Proof. From Theorem 3.2, f^n has a unique fixed point say $z \in X$. Also $f^n(fz) = f(f^n z) = fz$, implies fz is also fixed point of f^n. But from uniqueness of fixed point, we have $fz = z$, implies z is fixed point of f. Since the fixed point of f is also fixed point of f^n, so the fixed point of f is unique. □

Example 3.5. Let $X = R^+ = (0,\infty)$ be given space.
Define $G^* : X \times X \times X \times \to [1,\infty)$ as

$$G^*(x,y,z) = \left|\frac{x}{y}\right|^* \cdot \left|\frac{y}{z}\right|^* \cdot \left|\frac{z}{x}\right|^*.$$

Then (X,G) is complete multiplicative metric space let $f : X \to X$ be a map defined as $fx = \sqrt{x}$. Then by Banach contraction principle unique fixed point 1.

Theorem 3.6. *Let (X,G) be a complete multiplicative G-metric space. Suppose the mapping $f : X \to X$ satisfying the following condition:*

$$G(fx,fy,fz) \leq (G(fx,x,x).g(fy,y,y).G(fz,z,z))^\lambda \quad \textit{for all } x,y,z \in X, \tag{3.2}$$

where $\lambda \in \left[0,\frac{1}{2}\right)$.
Then f has a unique fixed point in X, and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to a fixed point.

Proof. Choose $x_0 \in X$, set $x_1 = fx_0$, $x_2 = fx_1 = f^2x_0, \ldots, x_n = f^{n+1}x_0, \ldots$.

Consider

$$\begin{aligned} G(x_{n+1},x_n,x_n) &= G(fx_n, fx_{n-1}, fx_{n-1}) \\ &\leq (G(fx_n,x_n,x_n)\cdot G(fx_{n-1},x_{n-1},x_{n-1})\cdot G(fx_{n-1},x_{n-1},x_{n-1}))^{\lambda} \\ &\leq (G(x_{n+1},x_n,x_n)\cdot G(x_n,x_{n-1},x_{n-1})\cdot G(x_n,x_{n-1},x_{n-1}))^{\lambda}. \end{aligned}$$

Thus we have

$$G(x_{n+1},x_n,x_n) \leq G^{\frac{\lambda^2}{1-\lambda}}(x_n,x_{n-1},x_{n-1}) = G^h(x_n,x_{n-1},x_{n-1}),$$

where $h = \dfrac{\lambda^2}{1-\lambda}$.

For $m > n > l$, consider

$$\begin{aligned} G(x_m,x_n,x_l) &\leq G(x_m,x_{m-1},x_{m-1})\cdot G(x_{m-1},x_{m-2},x_{m-2})\cdots G(x_{1+l},x_l,x_l) \\ &= G^{h^{m-1}+\ldots+h^l}(x_1,x_0,x_0) \leq G^{\frac{h^1}{1-h}}(x_1,x_0,x_0) \end{aligned}$$

which implies that

$$G(x_m,x_n,x_l) \to 1 \text{ as } n,m,1 \to \infty.$$

Hence $\{x_n\}$ is a multiplicative G-Cauchy sequence. Since X is complete, so there exists a $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Consider

$$\begin{aligned} G(fz,z,z) &\leq G(fz,fx_n,fx_n)\cdot G(fx_n,z,z) \\ &\leq (G(fz,z,z)\cdot G(fx_n,x_n,x_n)\cdot G(x_n,x_n,x_n))^{\lambda}\cdot G(x_{n+1},z,z). \end{aligned}$$

i.e., $G(fz,z,z) \leq (G^{\lambda^2}(fx_n,x_n,x_n)\cdot G(x_{n+1},z,z))^{\frac{1}{1-\lambda}} \to 1$ as $n \to \infty$.

Hence $G(fz,z,z) = 1$, implies $fz = z$. The uniqueness follows easily. □

Example 3.7. Let $X = R^+ = (0,\infty)$. Define $G^* : X \times X \times X \to [1,\infty)$ as

$$G^*(x,y,z) = \left|\frac{x}{y}\right|^* \cdot \left|\frac{y}{z}\right|^* \left|\frac{z}{x}\right|^*.$$

Then (X,G^*) is complete multiplication metric space.

Let $f : X \to X_0$ be a map defined as

$$fx = \begin{cases} 1 & \text{if } x \le 1 \\ x^{\frac{1}{4}} & \text{if } x > 1 \end{cases}$$

We find that $G^*(fx, fy, fz) \le (G^*(fx,x,x) \cdot G^*(fy,y,y) \cdot G^*(fz,z,z))^{\lambda}$ holds for all $x,y,z \in X$ and for some $\lambda = \frac{1}{3} \in \left[0, \frac{1}{2}\right)$.

Also for any $x \in X$, iterative sequence $\{f^n x\}$ converges to fixed point. Since all conditions of Theorem 3.6 is satisfied. Hence f has a unique fixed point 1.

Theorem 3.8. *Let (X,G) be a complete multiplicative G-metric space. Suppose the mapping $f : X \to X$ satisfied the following*

$$G(fx, fy, fz) \le (G(fx,y,z) \cdot G(fy,x,z) \cdot G(fz,x,y))^{\lambda} \tag{3.3}$$

for all $x,y,z \in X$, where $\lambda \in \left[0, \frac{1}{2}\right)$.

Then f has a unique fixed point in X and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to fixed point.

Proof. Choose $x_0 \in X$ and define $x_1 = fx_0,\ x_2 = fx_0,\ x_2 = fx_1 = f^2 x_0, \ldots, x_{n+1} = fx_n = f^{n+1}x_0, \ldots$.

Consider

$$\begin{aligned} G(x_{n+1}, x_n, x_n) &= G(fx_n, fx_{n-1}, fx_{n-1}) \\ &\le (G(fx_n, x_{n-1}, x_{n-1}) \cdot G(fx_{n-1}, x_n, x_{n-1}) \cdot G^{\lambda}(fx_{n-1}, x_n, x_{n-1})) \\ &\le (G(x_{n+1}, x_{n-1}, x_{n-1}) \cdot G(x_n, x_n, x_{n-1}) \cdot G(x_n, x_n, x_{n-1})^{\lambda} \\ &\le (G(x_{n+1}, x_n, x_n) \cdot G(x_n, x_{n-1}, x_{n-1}) \cdot G(x_n, x_{n-1}, x_{n-1}) \cdot G(x_{n-1}, x_n, x_{n-1}) \\ &\qquad \cdot G(x_n, x_{n-1}, x_{n-1}) \cdot G(x_{n-1}, x_n, x_{n-1}))^{\lambda} \end{aligned}$$

Thus we have

$$G(x_{n+1}, x_n, x_n) \le G^{\frac{\lambda^5}{1-\lambda}}(x_n, x_{n-1}, x_{n-1}) = G^h(x_n, x_{n-1}, x_{n-1}),$$

where $h = \dfrac{\lambda^5}{1-\lambda}$.

For $m > n > l$, we can consider

$$\begin{aligned} G(x_m, x_n, x_l) &\leq G(x_m, x_{m-1}, x_{m-1}) \cdot G(x_{m-1}, x_{m-2}, x_{m-2}) \cdots G(x_{l+1}, x_l, x_l) \\ &= G^{h^{m-1}+\cdots+h^1}(x_1, x_0, x_0) \\ &\leq G^{\frac{h^l}{1-h}}(x_1, x_0, x_0), \end{aligned}$$

which implies that $G(x_m, , x_n, x_l) \to 1$ as $n, m, l \to \infty$.

Hence $\{x_n\}$ is a multiplicative G-Cauchy sequence. Since X is complete, therefore, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Consider

$$\begin{aligned} G(fz, z, z) &\leq G(fz, fx_n, fx_n) \cdot G(fx_n, z, z) \\ &\leq (G(fz, x_n, x_n) \cdot G(fx_n, z, x_n) \cdot G^{\lambda}(fx_n, z, x_n) \cdot G(fx_n, z, z) \\ &\leq (G(fz, z, z) \cdot G(z, x_n, x_n) \cdot G(x_{n+1}, z, x_n) \cdot G^{\lambda}(x_{n+1}, z, x_n)) \cdot G(x_{n+1}, z, z). \\ G(fz, z, z) &\leq G^{\lambda}(z, x_n, x_n) \cdot G^{\frac{\lambda^2}{1-\lambda}}(x_{n+1}, z, x_n) \cdot G^{\frac{1}{1-\lambda}} G(x_{n+1}, z, z) \to 1 \text{ as } n \to \infty. \end{aligned}$$

Hence $G(fz, z, z) = 1$, implies $fz = z$. The uniqueness follows easily. □

Example 3.9. Let $X = R^+ = (0, \infty)$ be given space. Define $G^* : X \times X \times X \times \to [1, \infty)$ as

$$G^*(x, y, z) = \left|\frac{x}{y}\right|^* \cdot \left|\frac{y}{z}\right|^* \cdot \left|\frac{z}{x}\right|^*.$$

Then (X, G^*) is complete multiplicative metric space.

Let $f : X \to X$ be a map defined as

$$fx = \begin{cases} 1 & \text{if } x \leq 1 \\ \sqrt{x} & \text{if } x > 1. \end{cases}$$

Then $G^*(fx, fy, fz) \leq (G^*(fx, y, z) \cdot G^*(fy, x, z) \cdot G^*(fz, x, y))^{\lambda}$ holds for all $x, y, z \in X$ and for some $\lambda = \dfrac{1}{3} \in \left[0, \dfrac{1}{2}\right)$.

Also for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. Since all conditions of Theorem 3.8 are satisfied. Hence f has a unique fixed point 1.

4. Weakly commuting

In 1982, Sessa [11] introduced weakly commuting mappings in metric space.

In the similar mode, we state this in the setting of multiplicative G-metric spaces as follows:

Definition 4.1. Two self mappings f and g of a multiplicative G-metric space (X,G) are called weakly commuting iff $G(fgx,gfx,gfx) \leq G(fx,gx,gx)$ and $G(fgx,fgx,gfx) \leq G(fx,fx,gx)$ for all $x \in X$.

Example 4.2. Let $X=[0,1]$. Define $G: X\times X\times X \to [0,\infty)$ as $G(x,y,z) = \left|\frac{x}{y}\right|^* \left|\frac{y}{z}\right|^* \left|\frac{z}{x}\right|^*$ where $|x|^* = \begin{cases} x & \text{if } x>1 \\ \frac{1}{x} & \text{if } x \leq 1 \end{cases}$. i.e., absolute multiplicative.

Then (X,G) be multiplicative generalised metric space.

Define constant mappings $f,g: X \to X$ by

$$fx = a \text{ and } gx = b,\ a \neq ?b, \quad \text{where } a,b \in [0,1].$$

Then $G(fgx,gfx,gfx) = G(fb,ga,ga) = G(a,b,b)$ and $G(fx,gx,gx) = G(a,b,b)$, implies that, f and g are weakly commuting maps.

Remark 4.3. Every weakly commuting mappings need not be commuting.

Example 4.4. Let $X=[0,1]$ with multiplicative generalized metric G defined as $G(x,y,z) = \left|\frac{x}{y}\right|^*, \left|\frac{y}{z}\right|^*, \left|\frac{z}{x}\right|^*$ where $|x|^* = \begin{cases} x & \text{if } x>1 \\ \frac{1}{x} & \text{if } x \leq 1 \end{cases}$, absolute multiplicative value.

Define $f,g: X \to X$ as

$$f(x) = \frac{x}{2-x} \quad \text{and} \quad g(x) = x \quad \text{for all } x \in X.$$

Then f and g are weakly commuting maps but not commuting.

Theorem 4.5. *Let (X,G) be a complete G-multiplicative metric space. Let f and g be self mapping of X satisfying the following conditions*

(4.1) $f(X) \subseteq g(X)$,

(4.2) *f or g is continuous,*

(4.3) *$G(fx, fy, fz) \leq m^k(X,Y,Z)$, where*

$M(x,y,z)) = \max\{G(Gx,gy,gz), G(gx,fy,gz), G(gy,fx,gz), G(gx,fx,gz), G(gy,fy,gz),$
for all $x,y,z \in X$, where $k \in \left(0, \frac{1}{2}\right)$.

(4.4) *f and g are weakly commuting maps,*

Then f and g have a unique fixed point in X.

Proof. Let x_0 be an arbitrary point in X. From hypothesis there exists a point $x_1 \in X$ such that $fx_0 = gx_1$. In general there exists x_{n+1} such that $y_n = fx_n = gx_{n+1}$, $n = 0,1,2,\ldots$.
We may assume that $gx_n \neq gx_{n+1}$, for each n. Since if there exists n such that $gx_n = gx_{n+1}$, then $gx_n = gx_{n+1} = fx_n$, yields f and g have a coincidence point.
From hypothesis,

$$\begin{aligned}
G(gx_n, gx_n, gx_{n+1}) &= G(fx_{n-1}, fx_{n-1}, fx_n) \\
&\leq \max\{G(gx_{n-1}, gx_{n-1}, gx_n), G(gx_{n-1}, fx_{n-1}, gx_n), (G(gx_{n-1}, fx_{n-1}, gx_n), \\
&\qquad G(gx_{n-1}, fx_{n-1}, gx_n), G(gx_{n-1}, fx_{n-1}, gx_n)\}^k \\
&\leq \max\{G(gx_{n-1}, gx_{n-1}, gx_n), G(gx_{n-1}, gx_n, gx_n), G(gx_{n-1}, gx_n, gx_n), \\
&\qquad G(gx_{n-1}, gx_n, gx_n), G(gx_{n-1}, gx_n, gx_n)\}^k \\
&\leq \max\{G(gx_{n-1}, gx_{n-1}, gx_n), G(gx_{n-1}, gx_n, gx_n)\}^k
\end{aligned}$$

But $G(gx_{n-1}, gx_n, gx_n) \leq G^2(gx_{n-1}, gx_{n-1}, gx_n)$.
Hence $G(gx_n, gx_n, gx_{n+1}) \leq G^{k^2}(gx_{n-1}, gx_{n-1}, gx_n)$.
Let $k^2 = q$ then $0 \leq q \leq 1$, continuing the above process, we obtain

$$G(gx_n, gx_n, gx_{n+1}) \leq G^{q^n}(gx_0, gx_0, gx_1).$$

For every $m,n \in N$, $m > n$, and by repeated use of inequality, we have

$$\begin{aligned}
G(gx_n, gx_n, gx_m) &\leq \prod_{j=n}^{m-1} G(gx_j, gx_j, gx_{j+1}) \\
&\leq \prod_{j=n}^{m-1} G^{q^j}(gx_0, gx_0, gx_1)
\end{aligned}$$

$$\leq G^{\frac{q^n}{1-q}}(gx_0, gx_0, gx_1).$$

Therefore, $G(gx_n, gx_n, gx_m) \to 1$ as $m, n \to \infty$. Hence $\{gx_n\}$ is a G-multiplicative Cauchy sequence. Since (X, G) be complete, therefore, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$.

Since f and g are weakly commuting therefore,

$$G(fgx_n, gfx_n, gfx_n) \leq G(fx_n, gx_n, gx_n)).$$

Letting $n \to \infty$, and using continuity of g, we have

$$\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} ggx_n = \lim_{n\to\infty} ggx_n = gz.$$

Consider

$$G(fgx_n fx_n, fx_n) \leq \max\{G(ggx_n, gx_n, gx_n), G(ggx_n, fx_n, gx_n), G(gx_n, fgx_n, gx_n),$$
$$G(ggx_n, fgx_n, gx_n), G(gx_n, fgx_n, gx_n)\}^k$$

Proceeding limit $n \to \infty$, we have

$$G(gz,z,z) \leq \max\{G(gz,z,z), G(gz,z,z), G(z,gz,z), G(gz,gz,z), G(z,gz,z)\}^k$$
$$\leq \max\{G(gz,z,z), G(gz,z,z), G(z,gz,z), G(gz,z,z), G(z,gz,z), G(z,gz,z)\}^k$$
$$G(gz,z,z) \leq G^{k^2}(gz,z,z), \text{ which is contradiction since } k \in \left(0, \frac{1}{2}\right).$$

Therefore, $gz = z$.

Now we show that $fz = gz = z$.

Consider

$$G(fx_n, fz, fz)) \leq \max\{G(gx_n, gz, gz), G(gx_n, fz, gz), G(gz, fx_n, gz), G(gx_n, fx_n, gz), G(gz, fx_n, gz)\}^k$$

Proceeding limit $n \to \infty$, we have

$$G(z, fz, fz) \leq \max\{G(z, gz, gz), G(z, fz, gz), G(gz, z, gz), G(z, z, gz), G(gz, z, gz)\}^k$$
$$\leq G^k(z, fz, fz), \text{ which is contradiction since } k \in \left(0, \frac{1}{2}\right).$$

Therefore, $fz = z$.

Hence $z = gz$ implies z be common fixed point of f and g.

Assume that there exists another $p \in X_i$ such that $fv = gv = v$. Then if $gv \neq gz$.

From hypothesis

$$\begin{aligned} G(gv, gz, gz) &= G(fv, fz, fz) \\ &\leq \max\{G(gv, gz, gz), G(gv, fz, gz)G(gz, fv, gz), G(gv, fv, gz), G(gz, fz, gz)\}^k \\ &= \max\{G(gv, gz, gz), G(gv, gv, gz)\}^k \\ &\leq G^{k^2}(gv, gz, gz), \ \text{ a contradiction Hence } gv = gz. \end{aligned}$$

So g and f have unique common fixed point.

5. Compatible maps and its variants

In 1986, Jungck [3] introduced compatible mappings in metric spaces as follows:

Definition 5.1. Let f and g be two self mappings on a metric space (X, d). The mappings f and g are said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} fx_n = z$ for some $z \in X$.

In 2012, Choudhury et al. [2] introduced the notion of compatible maps in G-metric space as follows:

Definition 5.2. Let f and g be self maps of a G-metric space as (X, G). The maps f and g are said to be compatible map if

$$\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0 \text{ or } \lim_{n\to\infty} G(gfx_n, fgx_n, fgx_n) = 0,$$

whenever $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

In 1993, Jungck et al. [4] introduced compatible mappings of type (A) in metric spaces as follows:

Definition 5.3. Let f and g be self maps of a metric space (X, d). The maps f and g are said to be compatible maps of type (A) if

$$\lim_{n\to\infty} d(ffx_n, gfx_n) = 0 \ \text{ and } \ \lim_{n\to\infty} d(ggx_n, fgx_n) = 0.$$

whenever $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} dfx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

In 1995, Pathak and Khan [8] introduced compatible mappings of type (B) as follows:

Definition 5.4. Let f and g be self maps of a metric space (X,d). The maps f and g are said to be compatible maps of type (B) if

$$\lim_{n\to\infty} d(fgx_n, ggx_n) \leq \frac{1}{2}(\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, ffx_n))$$

and

$$\lim_{n\to\infty} d(gfx_n, ggx_n) \leq \frac{1}{2}(\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, ggx_n)),$$

whenever $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

In 1995, Pathak et al. [9] introduced compatible mappings of type (P) as follows:

Definition 5.5. [[9]] Let f and g be self maps of a metric space (X,d). The maps f and g are said to be compatible maps of type (P) if

$$\lim_{n\to\infty} (ffx_n, ggx_n) = 0 \text{ or } \lim_{n\to\infty} d(ggx_n, ffx_n) = 0,$$

whenever $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

In 1998, Pathak et al. [10] introduced compatible mappings of type (C) as follows:

Definition 5.6. [[10]] Let f and g be self maps of a metric space (X,d). The maps f and g are said to be compatible maps of type (C) if

$$\lim_{n\to\infty} d(fgx_n, ggx_n) \leq (\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, ffx_n) + \lim_{n\to\infty} d(ft, ggx_n)\}$$

and

$$\lim_{n\to\infty} d(gfx_n, ffx_n) \leq \frac{1}{3}(\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, ggx_n)) + \lim_{n\to\infty} d(gt, ffx_n)).$$

whenever $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Recently Kang et al. [5] introduced the notion of compatible maps and its variants as follows:

Definition 5.7. Let (X,d) be a multiplicative metric space and $f,g:X\to X$ be mappings. The mappings f and g are called

(i) compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n)=1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t\in X$.

(ii) compatible of type (A) if $\lim_{n\to\infty} d(fgx_n, g^2fx_n)=1$ and $\lim_{n\to\infty} d(gfx_n, f^2x_n)=1$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = 1 \lim_{n\to\infty} gx_n = t$ for some $t\in X$.

(iii) compatible of type (B) if

$$\lim_{n\to\infty} d(fgx_n, g^2x_n)\le [\lim_{n\to\infty} d(fgx_n, ft), \lim_{n\to\infty} d(ft, f^2x_n)]^{1/2}$$

and

$$\lim_{n\to\infty} d(gfx_n, g^2x_n)\le [\lim_{n\to\infty} d(gfx_n, gt), \lim_{n\to\infty} d(gt, g^2x_n)]^{1/2}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t\in X$.

(iv) compatible of type (C) if

$$\lim_{n\to\infty} d(fgx_n, g^2x_n)\le (\lim_{n\to\infty} d(fgx_n, ft)\cdot \lim_{n\to\infty} d(ft, f^2x_n))\cdot \lim_{n\to\infty} d(ft, g^2x_n))^{1/3}$$

and

$$\lim_{n\to\infty} d(gfx_n, f^2x_n)\le (\lim_{n\to\infty} d(fgx_n, ft)\cdot \lim_{n\to\infty} d(ft, f^2x_n) \lim_{n\to\infty} d(ft, g^2x_n))^{1/3}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t\in X$.

(v) compatible of type (P) if

$\lim_{n\to\infty} d(f^2x_n, g^2x_n)=1$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n \lim_{n\to\infty} gx_n = t$ for some $t\in X$.

Now we state compatible maps and its variants in setting of multiplicative G-metric space as follows:

Definition 5.8. Let f and g be self maps of a multiplicative G-metric space (X,G). The maps f and g are said to be

(i) compatible map iff

$$\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n)=1 \text{ or } \lim_{n\to\infty} G(gfx_n, fgx_n, fgx_n)=1,$$

(ii) compatible of type(A) iff

$$\lim_{n\to\infty} G(ffx_n, gfx_n, gfx_n) = 1 \text{ or } \lim_{n\to\infty} G(ggx_n, fgx_n, fgx_n) = 1,$$

(iii) compatible of type (B) iff

$$\lim_{n\to\infty} G(fgx_n, fgx_n, ggx_n) \leq (\lim_{n\to\infty} G(fgx_n, fgx_n, ft) \lim_{n\to\infty} G(ft, ffx_n, ffx_n))^{\frac{1}{2}}$$

and

$$\lim_{n\to\infty} G(gfx_n, gfx_n, ffx_n) \leq (\lim_{n\to\infty} G(gfx_n, gfx_n, gt) \lim_{n\to\infty} G(gt, ggx_n, ggx_n)^{\frac{1}{2}},$$

(iv) compatible of type (C) iff

$$\lim_{n\to\infty} G(fgx_n, fgx_n, ggx_n) \leq (\lim_{n\to\infty} G(fgx_n, fgx_n, ft) \lim_{n\to\infty} G(ft, ff, x_n, ffx_n) \lim_{n\to\infty} G(ft, ggx_n, ggx_n))^{\frac{1}{3}}$$

and

$$\lim_{n\to\infty} G(gfx_n, gfx_n, ffx_n) \leq (\lim_{n\to\infty} G(gfx_n, gfx_n, gt) \lim_{n\to\infty} G(gt, ggx_n, ggx_n) \lim_{n\to\infty} G(gt, ffx_n, ffx_n))^{\frac{1}{3}}$$

(v) compatible of type (P) iff

$$\lim_{n\to\infty} G(ffx_n, ffx_n, ggx_n) = 1 \text{ or } \lim_{n\to\infty} G(ggx_n, ggx_n, ffx_n) = 1$$

whenever $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Now we highlight relationship between compatible mappings and its variants.

Proposition 5.9. *Let f and g be compatible of type (A) if one of f and g is continuous then f and g are compatible.*

Proof. Since f and g are compatible of type (A), therefore

$$\lim_{n\to\infty} G(ffx_n, gfx_n, gfx_n) = 1 \text{ or } \lim_{n\to\infty} G(ggx_n, fgx_n, fgx_n) = 1,$$

whenever $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.
Suppose f is continuous. Then $\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fgx_n = ft$ for some $t \in X$ and $\lim_{n\to\infty} ggx_n = \lim_{n\to\infty} fgx_n = ft$ for some $t \in X$. Now we get $\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} fgx_n = 1$, i.e., f and g are compatible maps. Similarly if g is continuous then we also have f and g are compatible maps. □

Proposition 5.10. *Let f and g be compatible of type (A) then f and g are compatible of type (B).*

Proof. Since f and g are compatible of type (A) therefore

$$\lim_{n\to\infty} G(ffx_n, gfx_n, gfx_n) = 1 \text{ or } \lim_{n\to\infty} G(ggx_n, fgx_n, fgx_n) = 1,$$

whenever $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.
Now

$$\begin{aligned} 1 &= \lim_{n\to\infty} G(ggx_n, fgx_n, fgx_n) \\ &\leq (\lim_{n\to\infty} G(gfx_n, gfx_n, gt) \lim_{n\to\infty} G(gt, ggx_n, ggx_n))^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} 1 &= \lim_{n\to\infty} G(gfx_n, gfx_n, ffx_n) \\ &\leq (\lim_{n\to\infty} G(gfx_n, gfx_n, gt) \lim_{n\to\infty} G(gt, ggx_n, ggx_n))^{\frac{1}{2}} \end{aligned}$$

Implies f and g are compatible of type (B). ☐

Proposition 5.11. *Let f and g be continuous mappings from a multiplicative G-metric space (X,G) into itself. Then the following are equivalent:*

(i) *f and g are compatible of type (A),*

(ii) *f and g are compatible of type (B),*

(iii) *f and g are compatible.*

Proof. Suppose f and g are compatible of type (B) therefore

$$\lim_{n\to\infty} G(fgx_n, fgx_n, ggx_n) \leq (\lim_{n\to\infty} G(fgx_n, fgx_n, ft) \lim_{n\to\infty} G(ft, ffx_n, ffx_n))^{\frac{1}{2}}$$

and

$$\lim_{n\to\infty} G(gfx_n, gfx_n, ffx_n) \leq (\lim_{n\to\infty} G(gfx_n, gfx_n, gt) \lim_{n\to\infty} G(gt, ggx_n, ggx_n))^{\frac{1}{2}},$$

whenever $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$ for some $t \in X$.

Since f and g are continuous therefore,

$$\lim_{n\to\infty} G(fg, x_n, fgx_n, ggx_n) \leq (\lim_{n\to\infty} G(fg, fgx_n, ft) \lim_{n\to\infty} G(ft, ffx_n, ffx_n))^{\frac{1}{2}} = 1$$

and

$$\lim_{n\to\infty} G(gfx_n, gfx_n, ffx_n) \leq (\lim_{n\to\infty} G(gfx_n, gfx_n, gt) \lim_{n\to\infty} G(gt, ggx_n, ggx_n)^{\frac{1}{2}} = 1.$$

Implies f and g are compatible of type (A).

Suppose f and g are compatible of type (B). Let $\{x_n\}$ be sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$. Since f and g are continuous therefore $\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fgx_n = ft$ and $\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_n = gt$.

Now

$$\lim_{n\to\infty} (fgx_n fgx_n, ggx_n) \leq (\lim_{n\to\infty} G(fgx_n, fgx_n, ft) \lim_{n\to\infty} G(ft, ffx_n, ffx_n)^{\frac{1}{2}}$$

and

$$\lim_{n\to\infty} G(gf, x_n, gfx_n, ffx_n) \leq (\lim_{n\to\infty} G(gfx_n, gfx_n, gt) \lim_{n\to\infty} G(gt, ggx_n, ggx_n))^{\frac{1}{2}}$$

This implies that $\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} fgx_n$, hence f and g are compatible maps.

Now suppose f and g are compatible maps. Therefore,

$$\lim_{n\to\infty} G(fgx_n, fgx_n, gfx_n) = 1 \text{ and } \lim_{n\to\infty} G(gfx_n, gfx_n, fgx_n) = 1,$$

whenever $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gxn = t$ for some $t \in X$.

Since f and g are continuous maps therefore $\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} ffx_n = ft$ and $\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_n = gt$ implies $\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fgx_n = \lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_n$.

Now

$$\begin{aligned} 1 &= \lim_{n\to\infty} G(ggx_n, fgx_n, fgx_n) \\ &\leq (\lim_{n\to\infty} G(gfx_n, gfx_n, gt) \lim_{n\to\infty} G(gt, ggx_n, ggx_n))^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} 1 &= \lim_{n\to\infty} G(gf, x_n, gfx_n, ffx_n) \\ &\leq (\lim_{n\to\infty} G(gfx_n, gfx_n, gt) \lim_{n\to\infty} G(gt, ggx_n, ggx_n))^{\frac{1}{2}}. \end{aligned}$$

Implies f and g are compatible of type (B). □

Remark 5.12. Every weakly commuting mapping is compatible but converse need not be true.

Proof. Since f and g are weakly commuting mappings, therefore

$$G(fgx,gfx,gfx) \leq G(fx,gx,gx) \quad \text{for all } x \in X.$$

Let $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$, then

$$G(fgx_n,gfx_n,gfx_n) \leq G(\lim_{n\to\infty} fx_n,gx_n,gx_n) \text{ and } G(fgx_n,fgx_n,gfx_n) \leq G(fx)n,fx_n,gx_n).$$

Implies

$$\lim_{n\to\infty} G(fgx_n,fgx_n,gfx_n) = 1 and \lim_{n\to\infty} G(fgx_n,gfx_n,gfx_n) = 1.$$

Hence f and g are compatible. □

Remark 5.13. Compatible maps need not be weakly commuting.

Example 5.14. Let $X = [0,1]$ with multiplicative generalized metric G defined as $G(x,y,z) = \left|\frac{x}{y}\right|^* \left|\frac{y}{z}\right|^* \left|\frac{z}{x}\right|^*$ where $|x|^* = \begin{cases} x & \text{if } x > 1 \\ \frac{1}{x} & \text{if } x \leq 1 \end{cases}$ i.e., absolute multiplicative value.
Define $f,g : X \to X$ as

$$f(x) = x^3 \text{ and } g(x) = 2x^3 \quad \text{for all } x \in X.$$

Then clearly f and g are compatible maps but not weakly commuting.

Example 5.15. Let $X = [2,12]$ and let $f,g : X \to X$ be defined as

$$f(x) = \begin{cases} 3 & \text{if } x = 2 \text{ or } x > 5 \\ 12 & \text{if } 2 < x \leq 5 \end{cases} \text{ and } g(x) = \begin{cases} 3 & \text{if } x = 2 \\ 12 & \text{if } 2 < x \leq 5 \\ \frac{x+1}{2} & \text{if } x > 5 \end{cases}$$

Then clearly f and g are compatible of type (A), type (B), type (C) and also of type (P) but not compatible maps. We can see this by taking a sequence $\{x_n\}$ defined as $x_n = 5 + \frac{1}{n}$; $n > 0$.
Then clearly $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 3$.

$$\lim_{n\to\infty} G(fgx_n,gfx_n,gfx_n) \neq = 1.$$

But

$$\lim_{n\to\infty} G(ffx_n,gfx_n,gfx_n) = 1, \quad \lim_{n\to\infty} G(ggx_n,fgx_n,fgx_n) = 1,$$

$$\lim_{n\to\infty} G(ffx_n, ffx_n, ggx_n) = 1, \text{ and } \lim_{n} \to \infty G(ggx_n, ggx_n, ffx_n) = 1.$$

As well as condition of compatible of type (B) and type (C) are satisfied.

Here f and g are not continuous maps. This show that Proposition 5.11 is not valid if f and g are not continuous maps.

Proposition 5.16. *Let f and g be continuous self mappings of a multiplicative G-metric space (X,G). If $ft = gt$ for some $t \in X$, then $fgt = gft = fft = ggt$.*

Proof. Suppose that $\{x_n\}$ be a sequence in X defined by $x_n = t$, $n = 1,2,3,\dots$ for some $t \in X$ and $ft = gt$. Then $G(fx_n, gx_n, gx_n) \to 1$ as $n \to \infty$. Since f and g are compatible, therefore, we have

$$G(fgt, gft, gft) = \lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 1).$$

Hence we have $fgt = gft$. Since $ft = gt$, therefore, $fgt = fft = ggt = gft$. □

Proposition 5.17. *Let f and g be continuous self mappings of a multiplicative G-metric space (X,G). Suppose that $\lim_{n\to\infty} Gfx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$. Then*

(i) $\lim_{n\to\infty} gfx_n = ft$ *if f is continuous at t.*
(ii) $\lim_{n\to\infty} fgx_n = gt$ *if g is continuous at t.*
(iii) $fgt = gft$ *and* $ft = gt$ *if f and g are continuous at t.*

Proof. (i) Suppose that f be continuous at t. Since $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$, we have $fgx_n \to ft$ as $n \to \infty$. Since f and g are compatible, we have

$$\lim_{n\to\infty} G(gfx_n, ft, ft) \leq \lim_{n\to\infty} G(gfx_n, fgx_n, fgx_n) \cdot G(fgx_n, ft, ft)$$
$$\leq \lim_{n\to\infty} G(gfx_n, gfx_n, fgx_n) \cdot G(fgx_n, ft, ft) = 1 \text{ implies that } \lim_{n} \to gfx_n = ft$$

(ii) This proof follows similar to (i).

(iii) Suppose that f and g are continuous at t. Since f is continuous, therefore $\lim_{n\to\infty} gfx_n = ft$. Also, since g is continuous, therefore $\lim_{n\to\infty} fgx_n = gt$. Therefore $ft = gt$ by uniqueness of the limit. Using Proposition 5.15. $fgt = gft$. □

Now we prove some fixed point theorems for compatible maps and its variants as follows:

Theorem 5.18. *Let (X,G) be a complete G-multiplicative metric space. Let f and g be self mapping of X satisfying the following conditions (4.1), (4.2), (4.3) and the following*

(5.1) *f and g are compatible maps. Then f and g have a unique fixed point in X.*

Proof. Proceeding as in Theorem 4.5, we have Cauchy sequence $\{x_n\}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$.
Since f and g are compatible and suppose g is continuous, therefore,

$$\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} gxx_n = \lim_{n\to\infty} ggx_n = gz.$$

Rest part follows from Theorem 4.5. □

Theorem 5.19. *Let (X,G) be a complete G-multiplicative metric space. Let f and g be self mapping of X satisfying the following conditions (4.1), (4.2), (4.3) and the following*

(5.2) *maps f and g are compatible maps of type (A)*

then f and g have a unique common fixed point.

Proof. Since f and g are compatible of type (A). From the Proposition 5.9 we have f and g are compatible maps. Hence using Theorem 5.17 we have f and g have unique common fixed point. □

Theorem 5.20. *Let (X,G) be a complete G-multiplicative metric space. Let f and g be self mapping of X satisfying the following conditions (4.1), (4.2), (4.3) and the following*

(5.3) *maps f and g are compatible maps of type (B)*

then f and g have a unique common fixed point.

Proof. From Theorem 4.5 we get the sequence $\{x_n\}$ is such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u.$$

As g is continuous there $\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_n = gu$.
Since f and g are compatible of type (B), therefore

$$\lim_{n\to\infty} G(gfx_n, gfx_n, ffx_n) \leq \big(\lim_{n\to\infty} G(gfx_n, gfx_n, gu) \lim_{n\to\infty} G(gu, ggx_n, ggx_n)\big)^{\frac{1}{2}}$$

implies

$$\lim_{n\to\infty} G(gu,gu,ffx_n) \leq (\lim_{n\to\infty}(gu,gu,gu)\lim_{n\to\infty} G(gu,gu,gu))^{\frac{1}{2}} = 1$$

Thus, we have $\lim_{n\to\infty} ffx_n = gu$.

Consider

$$G(ffx_n, fx_n, fx_n) \leq \max\{G(gfx_n, gx_n, gx_n), G(gfx_n, fx_n, gx_n), G(gx_n, ffx_n, gx_n),$$
$$\cdot G(gfx_n, fgx_n, gx_n) \cdot G(gx_n, fx_n, gx_n)\}^k$$

Letting $n \to \infty$, we get

$$\begin{aligned} G(gu,u,u) &\leq \max(G(gu,u,u), G(gu,u,u), G(u,gu,u), G(gu,gu,u), G(u,u,u))^k \\ &\leq G^{k^2}(gu,u,u), \ \text{ a contradiction} \end{aligned}$$

Hence $gu = u$.

Now to show $fu = gu = u$.

Consider

$$G(fx_n, fu, fu) \leq \max\{G(gx_n, gu, gu), G(gx_n, fu, gu), G(gu, fx_n, gu),$$
$$G(gx_n, fx_n, gu), g(gu, fu, gu)\}^k.$$

Letting $n \to \infty$, we have

$$\begin{aligned} G(u, fufu) &\leq \max\{(G(u,gu,gu), G(u,fu,gu), G(gu,u,gu), G(u,u,gu), G(gu,fu,gu))\}^k \\ &\leq \max\{(G(u,u,u), G(u,fu,u), G(u,u,u), G(u,u,u), G(u,fu,u))\}^k \\ &= G^k(u,fu,u) \leq G^{k^2}(u,fu,fu), \ \text{ a contradiction.} \end{aligned}$$

Hence $fu = gu = u$.

Implies u is fixed point of f and g.

Now to prove uniqueness. Let v be another common fixed point of f and g.

Consider

$$\begin{aligned} G(v,u,u) &= G(fv, fu, fu) \\ &\leq \max\{G(gv,gu,gu), G(gv,fu,gu), G(gu,fv,gu), G(gv,fv,gu), G(gu,fu,gu)\}^k \end{aligned}$$

$$\leq \max\{G(v,u,u),G(v,u,u),G(u,v,u),G(v,v,u),G(u,u,u)\}^k$$
$$\leq G^{k^2}(v,u,u), \text{ a contradiction.}$$

Hence $u = v$ shows uniqueness of common fixed point. □

Theorem 5.21. *Let (X,G) be a complete G-multiplicative metric space. Let f and g be self mapping of X satisfying the following conditions (4.1), (4.2), (4.3) and the following*

(5.4) *f and g are compatible maps of type (C)*

then f and g have a unique common fixed point.

Proof. From Theorem 4.5 we get the sequence $\{x_n\}$ is such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u.$$

As g is continuous therefore $\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_n = gu$.
Since f and g are compatible of type (C), therefore

$$\lim_{n\to\infty} G(gfx_n,gfx_n,ffx_n) \leq (\lim_{n\to\infty} G(gfx_n,gfx_n,gt) \lim_{n\to\infty} G(gt,ggx_n,ggx_n) \lim_{n\to\infty} G(gt,ffx_n,ffx_n))^{\frac{1}{3}}$$

Implies

$$\begin{aligned}\lim_{n\to\infty} G(gu,gu,ffx_n) &\leq (\lim_{n\to\infty} G(gu,gu,gu) \lim_{n\to\infty} G(gu,gu,gu) \lim_{n\to\infty} G(gu,ffx_n,ffx_n))^{\frac{1}{3}} \\ &\leq \lim_{n\to\infty} G^{\frac{2}{3}}(gu,gu,ffx_n).\end{aligned}$$

Thus, we have $\lim_{n\to\infty} ffx_n, = gu$.
Rest part of proof follows from Theorem 5.20. □

Theorem 5.22. *Let (X,G) be a complete G-multiplicative metric space. Let f and g be self mapping of X satisfying the following conditions (4.1), (4.2), (4.3) and the following*

(5.5) *f and g are compatible maps of type (P)*

then f and g have a unique common fixed point.

Proof. From Theorem 4.5 we get the sequence $\{x_n\}$ is such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u\,.$$

As g is continuous therefore $\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_n = gu$.
Since f and g are compatible of type (P), therefore

$$\lim_{n\to\infty} G(ggx_n, ggx_n, ffx_n) = 1, \text{ implies } \lim_{n\to\infty} ffx_n = gu.$$

Rest part of proof follows from Theorem 5.20. □

Example 5.23. Let $X = [1, \infty)$ be multiplicative G metric space. Let $f, g : X \to X$ be defined as $fx = x$ and $gx = x^3$. Then clearly $f(X) \subseteq g(X)$. f and g are continuous.
And also $G(fx, fy, fz) = \left|\frac{x}{y}\right|^* \left|\frac{y}{z}\right|^* \left|\frac{z}{x}\right|^* \leq \left(\left|\frac{x^3}{y^3}\right|^* \left|\frac{y^3}{z^3}\right|^* \left|\frac{z^3}{x^3}\right|^*\right)^{\frac{1}{3}} \leq G^{\frac{1}{3}}(gx, gy, gz) \leq M^{\frac{1}{3}}(x, y, z)$.
All the conditions of Theorem 5.17 are satisfied for $k = \frac{1}{3}$? Here we also find that f and g are compatible maps. So from Theorem 5.17 f and g have unique common fixed point 1. Also as f and g are continuous so from Proposition f and g are compatible of type (A), type (B), type (C) and also of type (P). Hence f and g have unique common fixed point.

6. Weakly compatible

In 1996, Jungck [3] introduce the notion of weakly compatible maps as follows:

Definition 6.1. Two maps f and g are said to be weakly compatible if they commute at coincidence points.

Proposition 6.2. *Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

Proof. Since $w = fx = gx$ and f and g are weakly compatible, we have $fw = fgx = gfx = gw$ implies w is point of coincidence of f and g. So $w = fw = gw$. Moreover if $z = fz = gz$, then z is the point of coincidence of f and g and therefore $z = w$. By uniqueness of coincidence point, w is the unique common fixed point of f and g. □

Theorem 6.3. *Let (X, G) be a complete G-multiplicative metric space. Let f and g be self mapping of X satisfying the following conditions (4.1), (4.3) and the following*

(6.1) *fX or gX be closed,*

(6.2) *f and g are weakly compatible maps.*

then f and g have a unique common fixed point.

Proof. Proceeding as in Theorem 4.5, we have Cauchy sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} mfx_n = \lim_{n\to\infty} gx_n = z.$$

Suppose gX be closed. Therefore, there exists $p \in X$ such that $gp = z$.

Now we claim that $fp = gp = z$.

Consider

$$G(fx_n, fp, fp) \leq \max\{G(x_n, gp, gp), G(gx_n, fp, gp), G(gp, fx_n, gp),$$
$$G(gx_n, fx_n, gp), G(gp, fp, gp)\}^k.$$

Proceeding limit $n \to \infty$, we have

$$\begin{aligned} G(z, fp, fp) &\leq \max\{G(z, gp, gp), G(z, z, fp, gp), G(gp, z, gp), G(z, z, gp), G(gp, fp, gp)\}^k \\ &= \max\{G(z, z, z), G(z, fp, z), G(z, z, z), G(z, z, z), G(z, fp, z)\}^k \\ &= G^k(z, fp, z) \leq G^{k^2}(z, fp, fp). \end{aligned}$$

Implies $fp = z = gp$. Thus p is coincidence point of f and g.

Since f and g are weakly compatible maps. Therefore $fgp = gfporfz = gz$.

Now we will show that $fz = z$.

Consider

$$G(fx_n, fx_n, fz) \leq \max\{G(gx_n, gx_n, gz), G(Gx_n, fx_n, gz), G(gx_n, fx_n, gz),$$
$$G(gx_n, fx_n, gz), G(gx_n, fx_n, gp))\}^k.$$

Proceeding limit $n \to \infty$, we have

$$\begin{aligned} G(z, z, fz) &\leq \max\{G(z, fz, fz), G(z, z, fz), G(z, z, fz), G(z, z, fz), G(z, z, fz)\}^k \\ &\leq G^{k^2}(z, z, fp). \end{aligned}$$

Implies that $fz = z$. Therefore we have $fz = gz = z$ implies z is fixed point of f and g.

Uniqueness follows easily.

7. Weakly compatible with (E.A) property

Amari and Moutawakil [12] introduced property (E.A) in metric spaces as follows:

Definition 7.1. Let f and g be two self maps of a metric space (X,d). The pair (f,g) is said to satisfy property (E.A), if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t.$$

Theorem 7.2. *Let (X,G) be a complete G-multiplicative metric space. Let f and g be self mapping of X satisfying the following conditions (4.3) and the following conditions:*

(7.1) *f and g satisfy the (E.A) property,*

(7.2) *$g(X)$ is closed subspace of X,*

(7.3) *f and g are weakly compatible maps,*

then f and g have a unique common fixed point.

Proof. Since f and g satisfy the Property (E.A), therefore there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $u \in X$.
Since $g(X)$ is closed subspace of X, therefore

$$\lim_{n\to\infty} fx_n = z = gp = \lim_{n\to\infty} gx_n \quad \text{for some } p \in X.$$

Rest proof follows from Theorem 6.3. □

8. Weakly compatible with (CLRg) property

Sintunavarat and Kumam [13] introduced a new property called Common Limit Range property (i.e.,(CLR) property) as follows:

Definition 8.1. Let (X,d) be a metric space and $f,g : X \to X$ two mappings. The maps f and g are said to be satisfy the common limit in the range of g property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$ for some $x \in X$.

The common limit in the range of g property will be denoted by (CLRg) property.

Theorem 8.2. *Let (X,G) be a complete G-multiplicative metric space. Let f and g be self mapping of X satisfying the following conditions (4.3) and the following conditions:*

(6.1) *f and g satisfy the (CLRg) property,*

(6.2) *f and g are weakly compatible maps,*

then f and g have a unique common fixed point.

Proof. Since f and g satisfy (CLRg) Property, so there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gp = u \quad \text{for some } u \in X.$$

Rest proof follows from Theorem 6.3. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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