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CONVERGENCE RESULTS FOR TWO ASYMPTOTICALLY QUASI-I-NONEXPANSIVE MAPPINGS AND EQUILIBRIUM PROBLEM IN BANACH SPACES

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Abstract. In this article, we consider an implicit iterative scheme for two asymptotically quasi-I-nonexpansive mappings $\mathscr{T}_1,\ \mathscr{T}_2$ and two asymptotically quasi-nonexpansive mapping $I_1,\ I_2$ in Banach spaces. We prove weak and strong convergence results for considered iteration to common fixed point of such mappings. Our main results improve and compliment some known results.

Keywords: asymptotically quasi-I-nonexpansive; common fixed point; implicit iteration; uniformly convex Banach space.

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1. Introduction

Let D be a nonempty subset of a real normed linear space E and let $\mathscr{T} : D \to D$ be a mapping. Throughout this article, we assume that $\mathbb{N}$ is the set of natural numbers, we consider that E is real Banach space and $F(\mathscr{T})$ is nonempty. Suppose $F(\mathscr{T})$ denote the set of fixed points of $\mathscr{T}$ i. e., $F(\mathscr{T}) = \{x \in D : \mathscr{T}x = x\}$. Now, let us recall some known definitions.

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Definition 1.1. *Let D be a nonempty closed convex subset of real Banach space E. A mapping $\mathscr{T}: D \to D$ is said to be:*

(i) *nonexpansive* [4] if $\|\mathscr{T}x - \mathscr{T}y\| \leq \|x-y\|$, for all $x,\ y \in D$,

(ii) *quasi-nonexpansive* [18] if $\|\mathscr{T}x - q\| \leq \|x-q\|$, for all $x \in D$ and $q \in F(\mathscr{T})$,

(iii) *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that,

$$\|\mathscr{T}^n x - \mathscr{T}^n y\| \leq L\|x-y\|,\ \forall\ x,\ y \in D,\ \forall\ n \in \mathbb{N},$$

(iv) *asymptotically nonexpansive* [6] with a sequence $\{k_n\} \subset [1,\infty)$ and $\lim_{n\to\infty} k_n = 1$ such that,

$$\|\mathscr{T}^n x - \mathscr{T}^n y\| \leq k_n\|x-y\|,\ \forall\ x,\ y \in D,\ \forall\ n \in \mathbb{N},$$

(v) *asymptotically quasi-nonexpansive* [10] with a sequence $\{k_n\} \subset [1,\infty)$ and $\lim_{n\to\infty} k_n = 1$ such that

$$\|\mathscr{T}^n x - q\| \leq k_n\|x-q\|\ \forall\ x \in D,\ \forall\ n \in \mathbb{N} \textit{ and } q \in F(\mathscr{T}).$$

In 1916, Tricomi [18] introduced quasi-nonexpansive for real functions and later studied by Diaz and Metcalf [2] for mappings in Banach spaces. Ghosh and Debnath [5] established a necessary and sufficient condition for convergence of Ishikawa iterates of a quasi-nonexpansive mapping on a closed convex subset of a Banach space. In 1972, the class of asymptotically nonexpansive mappings was introduced as a generalization of the class of nonexpansive mappings by Goebel and Kirk [6]. In 2001, the class of asymptotically quasi-nonexpansive mapping was introduced as a generalization of the class of asymptotically nonexpansive mappings by Qihou [10]. Furthermore, it is easy to observe that, if $F(T) \neq \emptyset$, then a nonexpansive mapping must be quasi-nonexpansive and an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive mapping. But the converse implications need not be true.

There are many methods for approximating fixed point of a nonexpansive mapping. Xu and ori [19] introduced implicit iteration process to approximate a common fixed point of a finite family of nonexpansive mappings in a Hilbert space. After two years later, Sun [15] has extended an implicit iteration process for a finite of nonexpansive mappings, due to Xu and ori

[19], to the case of asymptotically quasi-nonexpansive mappings in a setting of Banach spaces.

In 2006, Rhodes and Temir [12] are proved strong convergence result of Mann iteration for I-nonexpansive mapping. Temir and Gul [17] are proved a weakly convergence result for asymptotically I-nonexpansive mapping in Hilbert space. In [8] weak and strong convergence of an implicit iteration process for asymptotically quasi I- nonexpansive mapping in Banach space has been proved. Recently, in [20] implicit iteration process for approximating the common fixed points of two asymptotically quasi I- nonexpansive mappings were studied.

There are many concepts which generalize a notion of nonexpansive mapping. One of such is I-nonexpansivity of a mapping $\mathscr{T}$ [14]. Let us recall some notions.

Definition 1.2. *Let D be a nonempty closed convex subset of real Banach space E. A mapping $\mathscr{T}$, $I : D \to D$ be two mappings of nonempty subset D of a real normed linear space E. Then $\mathscr{T}$ is said to be:*

(i) *I-nonexpansive if for all $x, y \in D$ and $F(\mathscr{T}) = \{x \in D : \mathscr{T}x = x\}$, the set of fixed points of $\mathscr{T}$,*

$$\|\mathscr{T}x - \mathscr{T}y\| \leq \|Ix - Iy\|,$$

(ii) *asymptotically-I-nonexpansive with a sequence $\{k_n\} \subset [1, \infty)$ and $\lim_{n\to\infty} k_n = 1$ such that, for all $x,\ y \in D$,*

$$\|\mathscr{T}^n x - \mathscr{T}^n y\| \leq k_n \|I^n x - I^n y\|,\ \forall\ n \in \mathbb{N},$$

(iii) *asymptotically quasi-I-nonexpansive with a sequence $\{k_n\} \subset [1, \infty)$ and $\lim_{n\to\infty} k_n = 1$ if, for all $x \in D$ and $q \in F(\mathscr{T}) = \{x \in D : \mathscr{T}x = x\}$, the set of fixed points of $\mathscr{T}$,*

$$\|\mathscr{T}^n x - q\| \leq k_n \|I^n x - q\|\ \forall\ n \in \mathbb{N}.$$

Remark 1.1. *If $F(\mathscr{T}) \cap F(I)$ is nonempty then an asymptotically I-nonexpansive mapping is a asymptotically quasi-I-nonexpansive. But, there exists a nonlinear continuous asymptotically quasi I-nonexpansive mappings which is asymptotically I-nonexpansive.*

Let ϕ be a bifunction of $D \times D$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $\phi : D \times D \to \mathbb{R}$ is to find $x \in D$ such that

$$\phi(x,\ y) \geq 0,\ \forall\, y \in D. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(\phi)$. Given a mapping $\mathscr{T} : D \to D$, let $\phi(x,\ y) = (\mathscr{T}x,\ y-x)$ for all $x,\ y \in D$. For solving the equilibrium problem for a bifunction $\phi : D \times D \to \mathbb{R}$, let us assume that ϕ satisfies the following conditions :

(C1) $\phi(x,\ x) = 0$ for all $x \in D$,

(C2) ϕ is monotone, that is, $\phi(x,\ y) + \phi(y,\ x) \leq 0$ forall $x,\ y \in D$,

(C3) for each $x,\ y,\ z \in D$,

$$\lim_{t \downarrow 0} \phi(tz + (1-t)x,\ y) \leq \phi(x,\ y),$$

(C4) for each $x \in D,\ y \mapsto \phi(x,\ y)$ is convex and lower semicontinuous.

Motivated by above works, in this paper, we proposed a new implicit iteration scheme for approximating the common fixed points of asymptotically quasi I-nonexpansive mappings $\mathscr{T}_1,\ \mathscr{T}_2$, asymptotically quasi-nonexpansive mapping $I_1,\ I_2$ and equilibrium problem :

$$\begin{cases} x_0 \in D, \\ x_n = a_n x_{n-1} + b_n \mathscr{T}_1^n y_n + c_n I_1^n x_n, \\ y_n = \widehat{a_n} x_n + \widehat{b_n} \mathscr{T}_2^n x_n + \widehat{c_n} I_2^n x_n. \end{cases} \tag{1.2}$$

where $\{a_n\},\ \{b_n\},\ \{c_n\},\ \{\widehat{a_n}\},\ \{\widehat{b_n}\},\ \{\widehat{c_n}\}$ are six real sequences in $(0,\ 1)$ satisfying $a_n + b_n + c_n = 1 = \widehat{a_n} + \widehat{b_n} + \widehat{c_n}$.

2. **Preliminaries**

Recall that a Banach space E is said to satisfy Opial condition [9] if for each sequence $\{x_n\}$ in E such that $\{x_n\}$ converges weakly to x implies that

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\| \tag{2.1}$$

for all $y \in E$ with $y \neq x$. It is well know that [3] inequality (2.1) is equivalent to

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|. \tag{2.2}$$

Definition 2.1. *Let E be a closed subset of a real Banach space E and let $\mathscr{T} : D \to D$ be a mapping.*

(i) *A mapping $\mathscr{T}$ is said to be semi-closed(demi-closed) at zero, if for each bounded sequence $\{x_n\}$ in D, the conditions x_n converges weakly to $x \in D$ and $\mathscr{T}x_n$ converges strongly to zero imply $\mathscr{T}x = 0$.*

(ii) *A mapping $\mathscr{T}$ is said to be semicompact, if for any bounded sequence $\{x_n\}$ in D such that $\|x_n - \mathscr{T}x_n\| \to 0,\ n \to \infty$, then there exists a subsequence $\{x_{n_p}\} \subset \{x_n\}$ such that $x_{n_p} \to x^* \in D$ strongly.*

We restate the following lemmas which play key roles in our proofs.

Lemma 2.1. [1] *Let D be a closed convex subset of a smooth, strictly convex and reflexive Banach space E and ϕ be a bifunction of $D \times D$ into $\mathbb{R}$ satisfying (C1) - (C4), let $r > 0$ and $x \in E$. Then, there exists $z \in D$ such that*

$$\phi(z,\ y) + \frac{1}{r}\langle y - z,\ Jz - Jx\rangle \geq 0\ \forall\ y \in D.$$

Lemma 2.2. [11] *Let D be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E and ϕ be a bifunction of $D \times D$ into $\mathbb{R}$ satisfying (C1) - (C4). For $r > 0$ and $x \in E$, define a mapping $S_r : E \to D$ as follows:*

$$S_r(x) = \left\{ z \in D : \phi(z,\ y) + \frac{1}{r}\langle y - z,\ Jz - Jx\rangle,\ \forall\ y \in D \right\}$$

for all $x \in E$. Then, the following hold:

(1) S_r is single-valued,

(2) S_r is firmly nonexpansive-type mapping i. e., for all $x,\ y \in E$,

$$\langle S_r x - S_r y,\ JS_r x - JS_r y\rangle \leq \langle S_r x - S_r y,\ Jx - Jy\rangle,$$

(3) $\phi(S_r) = EP(\phi)$,

(4) $EP(\phi)$ is closed and convex.

Lemma 2.3. [7] *Let E be a uniformly convex Banach space satisfying the Opial's condition, D be a nonempty closed subset of E and $\mathscr{S} : D \to D$ an asymptotically nonexpansive mapping. If the sequence $\{x_n\} \subset D$ is a weakly convergent sequence with the weak limit p and if* $\lim_{n\to\infty} \|x_n - \mathscr{S} x_n\| = 0$, *then $\mathscr{S} p = p$.*

Lemma 2.4. [13] *Let D be a uniformly convex Banach space and let* $0 < \beta < \gamma < 1$. *Suppose that $\{t_n\}$ is a sequence in $[\beta,\ \gamma]$ and $\{x_n\},\ \{y_n\}$ are two sequence in D such that*

$$\lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = d,\ \limsup_{n\to\infty} \|x_n\| \le d,\ \limsup_{n\to\infty} \|y_n\| \le d, \tag{2.3}$$

holds some $d \ge 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.5. [16] *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of nonnegative real numbers with* $\sum_{n=1}^{\infty} \beta_n < \infty$. *If one of the following conditions is satisfied:*

(1) $\alpha_{n+1} \le \alpha_n + \beta_n,\ n \ge 1,$

(2) $\alpha_{n+1} \le (1+\beta_n)\alpha_n,\ n \ge 1,$

then the limit $\lim_{n\to\infty} \alpha_n$ *exists.*

3. Main Results

Lemma 3.1. *Let E be a real Banach space and let D be a nonempty closed convex subset of E. Let $\mathscr{T}_1,\ \mathscr{T}_2 : D \to D$ be two asymptotically quasi-I-nonexpansive mapping with a sequences $\{k_n\},\ \{h_n\} \subset [1,\ \infty)$ and $I_1,\ I_2$ be two asymptotically quasi-nonexpansive self mapping of D with a sequence $\{g_n\},\ \{t_n\} \subset [1,\ \infty)$. Let $\mu_n = \max_{n\in\mathbb{N}} \{k_n,\ h_n,\ g_n,\ t_n\}$ and we assume that $R = \sup_n (1-a_n),\ \mathscr{M} = \sup_n \mu_n^2 \ge 1$ such that $F = F(\mathscr{T}_1) \cap F(\mathscr{T}_2) \cap F(I_1) \cap F(I_2)$ is nonempty and $q^* \in F$. And $\{a_n\},\ \{b_n\},\ \{c_n\},\ \{\widehat{a_n}\},\ \{\widehat{b_n}\},\ \{\widehat{c_n}\}$ are six real sequences in $(0,\ 1)$ which satisfy the following conditions :*

(i) $R(\mathscr{M} + \mathscr{M}^2) < 1,$

(ii) $\sum_{n=1}^{\infty}(1-a_n)(\mu_n^4+\mu_n^2-1)<\infty$.

If $\{x_n\}$ *is the implicit iterative sequence defined by (1.2), then*

(1) $\lim_{n\to\infty}\|x_n-q^*\|$ *exists for each* $q^*\in F$.

(2) The sequence $\{x_n\}$ *generated by (1.2) converges strongly to common fixed point in* F *if and only if* $\liminf_{n\to\infty} d(x_n,\ F)=0$.

Proof. As $q^*\in F$, it follows from (1.2) that

$$\begin{aligned}\|x_n-q^*\| &= \|a_nx_{n-1}+b_n\mathscr{T}_1^ny_n+c_nI_1^nx_n-q^*\|\\ &\le a_n\|x_{n-1}-q^*\|+b_n\|\mathscr{T}_1^ny_n-q^*\|+c_n\|I_1^nx_n-q^*\|\\ &\le a_n\|x_{n-1}-q^*\|+b_nk_n\|I_1^ny_n-q^*\|+c_ng_n\|x_n-q^*\|\\ &\le a_n\|x_{n-1}-q^*\|+(1-a_n-c_n)k_ng_n\|y_n-q^*\|+(1-a_n-b_n)g_n\|x_n-q^*\|\\ &\le a_n\|x_{n-1}-q^*\|+(1-a_n)\mu_n^2\|y_n-q^*\|+(1-a_n)\mu_n^2\|x_n-q^*\| \qquad (3.1)\end{aligned}$$

Again from (1.2), we obtain

$$\begin{aligned}\|y_n-q^*\| &= \|\widehat{a_n}x_n+\widehat{b_n}\mathscr{T}_2^nx_n+\widehat{c_n}I_2^nx_n-q^*\|\\ &\le \widehat{a_n}\|x_n-q^*\|+\widehat{b_n}h_n\|I_2^nx_n-q^*\|+\widehat{c_n}t_n\|x_n-q^*\|\\ &\le \widehat{a_n}\|x_n-q^*\|+\widehat{b_n}h_nt_n\|x_n-q^*\|+\widehat{c_n}t_n\|x_n-q^*\|\\ &\le \widehat{a_n}\mu_n^2\|x_n-q^*\|+\widehat{b_n}\mu_n^2\|x_n-q^*\|+\widehat{c_n}\mu_n^2\|x_n-q^*\|\\ &\le \mu_n^2\|x_n-q^*\| \qquad (3.2)\end{aligned}$$

Then from (3.1), we get

$$\begin{aligned}\|x_n-q^*\| &\le a_n\|x_{n-1}-q^*\|+(1-a_n)\mu_n^4\|x_n-q^*\|+(1-a_n)\mu_n^2\|x_n-q^*\|\\ &\le a_n\|x_{n-1}-q^*\|+(1-a_n)(\mu_n^4+\mu_n^2)\|x_n-q^*\|\end{aligned}$$

This implies that

$$[1-(1-a_n)(\mu_n^4+\mu_n^2)]\|x_n-q^*\| \leq a_n\|x_{n-1}-q^*\| \tag{3.3}$$

By condition (i) we obtain $(1-a_n)(\mu_n^4+\mu_n^2) \leq \sup(1-a_n)\sup(\mu_n^4+\mu_n^2) = R(\mathscr{M}^2+\mathscr{M}) < 1$, and therefore

$$1-(1-a_n)(\mu_n^4+\mu_n^2) \geq 1-R(\mathscr{M}^2+\mathscr{M}) > 0.$$

Therefore (3.3) we take

$$\begin{aligned}
\|x_n-q^*\| &\leq \frac{a_n}{1-(1-a_n)(\mu_n^4+\mu_n^2)}\|x_{n-1}-q^*\| \\
&\leq \left[1+\frac{a_n+(1-a_n)(\mu_n^4+\mu_n^2)-1}{1-(1-a_n)(\mu_n^4+\mu_n^2)}\right]\|x_{n-1}-q^*\| \\
&\leq \left[1+\frac{(1-a_n)(\mu_n^4+\mu_n^2-1)}{1-(1-a_n)(\mu_n^4+\mu_n^2)}\right]\|x_{n-1}-q^*\| \\
&\leq \left[1+\frac{(1-a_n)(\mu_n^4+\mu_n^2-1)}{1-R(\mathscr{M}+\mathscr{M}^2)}\right]\|x_{n-1}-q^*\|
\end{aligned}$$

Let $\beta_n = \frac{(1-a_n)(\mu_n^4+\mu_n^2-1)}{1-R(\mathscr{M}+\mathscr{M}^2)}$. Then the last inequality can be written as follows :

$$\|x_n-q^*\| \leq (1+\beta_n)\|x_{n-1}-q^*\|. \tag{3.4}$$

From condition (ii) we find

$$\sum_{n=1}^{\infty}\beta_n = \frac{1}{1-R(\mathscr{M}+\mathscr{M}^2)}\sum_{n=1}^{\infty}(1-a_n)(\mu_n^4+\mu_n^2-1) < \infty.$$

Now taking $\alpha_n = \|x_{n-1}-q^*\|$ in (3.4) we obtain

$$\alpha_{n+1} \leq (1+\beta_n)\alpha_n$$

and according to Lemma 2.5 the limit $\lim_{n\to\infty}\alpha_n$ exists. This means the limit

$$\lim_{n\to\infty}\|x_n-q^*\| = d \tag{3.5}$$

exists, where $d \geq 0$ is a constant.

It follows from (3.4) that

$$d(x_n,\ F) \leq (1+\beta_n)d(x_{n-1},\ F).$$

So from Lemma 2.5, we obtain $\lim_{n\to\infty} d(x_n,\ F)$ exists. Furthermore, since $\liminf_{n\to\infty} d(x_n,\ F) = 0$, then $\lim_{n\to\infty} d(x_n,\ F) = 0$. Next we show that $\{x_n\}$ is a Cauchy sequence in D. Let $\varepsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n,\ F) = 0$, there exists a positive integer m_0 such that

$$d(x_n,\ F) < \frac{\varepsilon}{2},\ \forall\, n \geq m_0.$$

In particular, $\inf\{\|x_{m_0} - q\| :\ q \in F\} < \frac{\varepsilon}{2}$. Thus there must exists $q \in F$ such that

$$\|x_{m_0} - q\| < \frac{\varepsilon}{2}.$$

Now, for all $m,\ n \geq m_0$, we have

$$\begin{aligned} \|x_{m+n} - x_n\| &\leq \|x_{m+n} - q\| + \|x_n - q\| \\ &\leq 2\|x_{m_0} - q\| \\ &\leq 2\Big(\frac{\varepsilon}{2}\Big) = \varepsilon. \end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence in a closed subset D of a Banach space E and so it must converge to a point q^* in D. And $\lim_{n\to\infty} d(x_n,\ F) = 0$ gives that $d(q^*,\ F) = 0$. By the routine proof, we know F is a closed subset of D. Thus $q^* \in F$. This completes the proof.

□

Theorem 3.2. *Let E be a real uniformly convex Banach space and let D be a nonempty closed convex subset of E. Let $\phi : D \times D \to \mathbb{R}$ be a bifunction which satisfy the conditions (C1)-(C4).Let $\mathscr{T}_1,\ \mathscr{T}_2 : D \to D$ be two uniformly L_1 and L_2- Lipschitizian asymptotically quasi-I-nonexpansive mapping with a sequences $\{k_n\},\ \{h_n\} \subset [1,\ \infty)$ and $I_1,\ I_2$ be two uniformly L_3 and L_4- Lipschitizian asymptotically quasi-nonexpansive self mapping of D with a sequence $\{g_n\},\ \{t_n\} \subset [1,\ \infty)$. Let $\mu_n = \max_{n\in\mathbb{N}}\ \{k_n,\ h_n,\ g_n,\ t_n\}$ and we assume that $R = \sup_n(1 - a_n)$, $\mathscr{M} = \sup_n \mu_n^2 \geq 1$ such that $F = F(\mathscr{T}_1) \cap F(\mathscr{T}_2) \cap F(I_1) \cap F(I_2) \cap EP(\phi)$ is nonempty and $q^* \in F$. For an initial point $x_0 \in D$, generate a sequence $\{x_n\}$ by $v_n \in D$ such that*

$$
\begin{cases}
\phi(v_n,\, y) + \frac{1}{r_n}\langle y - v_n,\, Jv_n - Jx_n\rangle \geq 0,\; \forall\, y \in D, \\
x_n = a_n x_{n-1} + b_n \mathscr{T}_1^n y_n + c_n I_1^n x_n, \\
y_n = \widehat{a_n} x_n + \widehat{b_n} \mathscr{T}_2^n x_n + \widehat{c_n} I_2^n x_n,\; \forall\, n \geq 1,
\end{cases} \tag{3.6}
$$

where $\{a_n\},\ \{b_n\},\ \{c_n\},\ \{\widehat{a_n}\},\ \{\widehat{b_n}\},\ \{\widehat{c_n}\}$ *are six real sequences in* $(0,\ 1)$ *satisfying* $a_n + b_n + c_n = 1 = \widehat{a_n} + \widehat{b_n} + \widehat{c_n}$ *and* $\{r_n\} \subset [\rho,\ \infty)$ *for* $\rho > 0$, *which satisfy the following conditions :*

(i) $R(\mathscr{M} + \mathscr{M}^2) < 1$,

(ii) $\sum_{n=1}^{\infty}(1 - a_n)(\mu_n^4 + \mu_n^2 - 1) < \infty$,

(iii) $\liminf_{n\to\infty} r_n > 0$.

Then the implicit iterative sequence $\{x_n\}$ *defined by (3.6), satisfies the following :*

$$
\lim_{n\to\infty} \|x_n - \mathscr{T}_1 x_n\| = \lim_{n\to\infty} \|x_n - \mathscr{T}_2 x_n\| = \lim_{n\to\infty} \|x_n - I_1 x_n\| = \lim_{n\to\infty} \|x_n - I_2 x_n\| = 0.
$$

Proof. We divide the proof into two steps.

Step 1. First, we will prove that

$$
\lim_{n\to\infty} \|x_n - \mathscr{T}_1^n x_n\| = \lim_{n\to\infty} \|x_n - \mathscr{T}_2^n x_n\| = \lim_{n\to\infty} \|x_n - I_1^n x_n\| = \lim_{n\to\infty} \|x_n - I_2^n x_n\| = 0.
$$

According to Lemma 3.1, we know that $\lim_{n\to\infty} \|x_n - q^*\|$ exists for any $q^* \in F$. We have suppose that $\lim_{n\to\infty} \|x_n - q^*\| = d$. It follows from (3.6) that

$$
\begin{aligned}
\lim_{n\to\infty} \|x_n - q^*\| &= \lim_{n\to\infty} \|a_n x_{n-1} + b_n \mathscr{T}_1^n y_n + c_n I_1^n x_n - q^*\| \\
&= \lim_{n\to\infty} \|a_n(x_{n-1} - q^*) + b_n(\mathscr{T}_1^n y_n - q^*) + c_n(I_1^n x_n - q^*)\| \\
&= \lim_{n\to\infty} \Big\| a_n(x_{n-1} - q^*) + (1 - a_n)\Big[\frac{b_n}{1 - a_n}(\mathscr{T}_1^n y_n - q^*) \\
&\qquad + \frac{c_n}{1 - a_n}(I_1^n x_n - q^*)\Big]\Big\| = d.
\end{aligned} \tag{3.7}
$$

It follows from $\lim_{n\to\infty}\|x_n-q^*\|=d$ that $\lim_{n\to\infty}\|x_{n-1}-q^*\|=d$. Taking $\limsup$ on both sides, we have

$$\limsup_{n\to\infty}\|x_{n-1}-q^*\|=0. \tag{3.8}$$

In addition, from (3.7), we have

$$\begin{aligned}
&\limsup_{n\to\infty}\left\|\frac{b_n}{1-a_n}(\mathscr{T}_1^n y_n-q^*)+\frac{c_n}{1-a_n}(I_1^n x_n-q^*)\right\| \\
&\le \limsup_{n\to\infty}\left[\frac{b_n}{1-a_n}\|\mathscr{T}_1^n y_n-q^*\|+\frac{c_n}{1-a_n}\|I_1^n x_n-q^*\|\right] \\
&\le \limsup_{n\to\infty}\left[\frac{b_n}{1-a_n}\mu_n^2\|y_n-q^*\|+\frac{c_n}{1-a_n}\mu_n^2\|nx_n-q^*\|\right] \\
&\le \limsup_{n\to\infty}\left[\frac{b_n}{1-a_n}\mu_n^4\|x_n-q^*\|+\frac{c_n}{1-a_n}\mu_n^4\|nx_n-q^*\|\right] \\
&\le \limsup_{n\to\infty}\left[\frac{\mu_n^4}{1-a_n}(b_n+c_n)\|x_n-q^*\|\right] \\
&= \limsup_{n\to\infty}\mu_n^4\|x_n-q^*\|=d.
\end{aligned} \tag{3.9}$$

From (3.7), (3.8), (3.9) and Lemma 2.4, we get

$$\begin{aligned}
&\lim_{n\to\infty}\left\|(x_{n-1}-q^*)-\left[\frac{b_n}{1-a_n}(\mathscr{T}_1^n y_n-q^*)+\frac{c_n}{1-a_n}(I_1^n x_n-q^*)\right]\right\| \\
&=\lim_{n\to\infty}\left(\frac{1}{1-a_n}\right)\|(1-a_n)(x_{n-1}-q^*)-b_n(\mathscr{T}_1^n y_n-q^*)-c_n(I_1^n x_n-q^*)\| \\
&=\lim_{n\to\infty}\left(\frac{1}{1-a_n}\right)\|x_n-x_{n-1}\|=0.
\end{aligned}$$

Since the sequence $\{a_n\}$ in $(0,\ 1)$, there are some constants $a,\ b\in(0,\ 1)$ such that $0<a\le a_n<b<1$. Therefore, we have

$$\lim_{n\to\infty}\|x_n-x_{n-1}\|=0. \tag{3.10}$$

On the other hand, from (3.7), we have

$$\begin{aligned}
\lim_{n\to\infty}\|x_n-q^*\|=\lim_{n\to\infty}\Big\|b_n(\mathscr{T}_1^n y_n-q^*)+(1-b_n)\Big[&\frac{a_n}{1-b_n}(x_{n-1}-q^*) \\
&+\frac{c_n}{1-b_n}(I_1^n x_n-q^*)\Big]\Big\|=d.
\end{aligned} \tag{3.11}$$

Since $\mathscr{T}_1$ is uniformly L_1-Lipschitizian asymptotically quasi-nonexpansive mapping, we have

$$\lim_{n\to\infty}\|\mathscr{T}_1^n y_n - q^*\| \le \mu_n^2\|y_n - q^*\| \le \mu_n^4\|x_n - q^*\|.$$

Taking $\limsup$ on both sides, we take

$$\limsup_{n\to\infty}\|\mathscr{T}_1^n y_n - q^*\| \le d, \tag{3.12}$$

and

$$\begin{aligned}
&\limsup_{n\to\infty}\left\|\frac{a_n}{1-b_n}(x_n-q^*)+\frac{c_n}{1-b_n}(I_1^n x_n - q^*)\right\| \\
&\le \limsup_{n\to\infty}\left[\frac{a_n}{1-b_n}(x_n-q^*)+\frac{c_n}{1-b_n}\mu_n^2\|x_n-q^*\|\right] \\
&\le \limsup_{n\to\infty}\left[\frac{a_n}{1-b_n}\mu_n^2\|x_n-q^*\|+\frac{c_n}{1-b_n}\mu_n^2\|x_n-q^*\|\right] \\
&\le \limsup_{n\to\infty}\left[\frac{\mu_n^2}{1-b_n}(a_n+c_n)\|x_n-q^*\|\right] \\
&= \limsup_{n\to\infty}\mu_n^2\|x_n-q^*\| = d. \qquad (3.13)
\end{aligned}$$

From (3.11), (3.12), (3.13) and Lemma 2.4, we have

$$\lim_{n\to\infty}\|x_n - \mathscr{T}_1^n y_n\| = 0. \tag{3.14}$$

Similarly, we obtain

$$\lim_{n\to\infty}\|x_n - I_1^n x_n\| = 0. \tag{3.15}$$

From (3.10) and (3.14), we take

$$\lim_{n\to\infty}\|x_{n-1} - \mathscr{T}_1^n y_n\| \le \lim_{n\to\infty}\|x_{n-1}-x_n\| + \lim_{n\to\infty}\|x_n - \mathscr{T}_1^n y_n\| = 0. \tag{3.16}$$

Consider

$$\begin{aligned}
\|x_{n-1}-q^*\| &\le \|x_{n-1}-\mathscr{T}_1^n y_n\| + \|\mathscr{T}_1^n y_n - q^*\| \\
&\le \|x_{n-1}-\mathscr{T}_1^n y_n\| + k_n g_n\|y_n - q^*\| \\
&\le \|x_{n-1}-\mathscr{T}_1^n y_n\| + \mu_n^2\|y_n - q^*\|,
\end{aligned}$$

this implies that, we have

$$\|x_{n-1}-q^*\|-\|x_{n-1}-\mathscr{T}_1^n y_n\| \leq \mu_n^4\|x_n-q^*\|.$$

From (3.7) and (3.16) with Squeeze theorem yield

$$\lim_{n\to\infty}\|y_n-q^*\|=d. \tag{3.17}$$

Again from (3.6), we can see that

$$\begin{aligned}
\lim_{n\to\infty}\|y_n-q^*\| &= \lim_{n\to\infty}\|\widehat{a}_n x_n+\widehat{b}_n\mathscr{T}_2^n x_n+\widehat{c}_n I_2^n x_n-q^*\| \\
&= \lim_{n\to\infty}\|\widehat{a}_n(x_n-q^*)+\widehat{b}_n(\mathscr{T}_2^n x_n-q^*)+\widehat{c}_n(I_2^n x_n-q^*)\| \\
&= \lim_{n\to\infty}\Big\|\widehat{a}_n(x_n-q^*)+(1-\widehat{a}_n)\Big[\frac{\widehat{b}_n}{1-\widehat{a}_n}(\mathscr{T}_2^n x_n-q^*) \\
&\qquad +\frac{\widehat{c}_n}{1-\widehat{a}_n}(I_2^n x_n-q^*)\Big]\Big\| = d.
\end{aligned} \tag{3.18}$$

In addition, from (3.18), we get

$$\begin{aligned}
&\lim_{n\to\infty}\sup\Big\|\frac{\widehat{b}_n}{1-\widehat{a}_n}(\mathscr{T}_2^n x_n-q^*)+\frac{\widehat{c}_n}{1-\widehat{a}_n}(I_2^n x_n-q^*)\Big\| \\
&\leq \lim_{n\to\infty}\sup\Big[\frac{\widehat{b}_n}{1-\widehat{a}_n}\|\mathscr{T}_2^n x_n-q^*\|+\frac{\widehat{c}_n}{1-\widehat{a}_n}\|I_2^n x_n-q^*\|\Big] \\
&\leq \lim_{n\to\infty}\sup\Big[\frac{\widehat{b}_n}{1-\widehat{a}_n}\mu_n^2\|x_n-q^*\|+\frac{\widehat{c}_n}{1-\widehat{a}_n}\mu_n^2\|x_n-q^*\|\Big] \\
&\leq \lim_{n\to\infty}\sup\Big[\frac{\mu_n^2}{1-a_n}(\widehat{b}_n+\widehat{c}_n)\|x_n-q^*\|\Big] \\
&= \lim_{n\to\infty}\sup\mu_n^2\|x_n-q^*\| = d.
\end{aligned} \tag{3.19}$$

From (3.5), (3.18), (3.19) and Lemma 2.4, we get

$$\begin{aligned}
&\lim_{n\to\infty}\Big\|(x_n-q^*)-\Big[\frac{\widehat{b}_n}{1-\widehat{a}_n}(\mathscr{T}_2^n x_n-q^*)+\frac{\widehat{c}_n}{1-\widehat{a}_n}(I_2^n x_n-q^*)\Big]\Big\| \\
&= \lim_{n\to\infty}\Big(\frac{1}{1-\widehat{a}_n}\Big)\|(1-\widehat{a}_n)(x_n-q^*)-\widehat{b}_n(\mathscr{T}_2^n x_n-q^*)-\widehat{c}_n(I_2^n x_n-q^*)\| \\
&= \lim_{n\to\infty}\Big(\frac{1}{1-\widehat{a}_n}\Big)\|x_n-y_n\| = 0.
\end{aligned}$$

Since the sequence $\{\widehat{a_n}\}$ in $(0,\ 1)$, there are some constants $a,\ b \in (0,\ 1)$ such that $0 < a \leq \widehat{a_n} < b < 1$. Therefore, we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.20}$$

In a similar way, we obtain

$$\lim_{n\to\infty} \|x_n - \mathscr{T}_2^n x_n\| = 0. \tag{3.21}$$

and

$$\lim_{n\to\infty} \|x_n - I_2^n x_n\| = 0. \tag{3.22}$$

From (3.10), (3.16) and (3.20), we take

$$\begin{aligned}\lim_{n\to\infty} \|x_n - \mathscr{T}_1^n x_n\| &\leq \lim_{n\to\infty} [\|x_n - x_{n-1}\| + \|x_{n-1} - \mathscr{T}_1^n y_n\| + \|\mathscr{T}_1^n y_n - \mathscr{T}_1^n x_n\|] \\ &\leq \lim_{n\to\infty} [\|x_n - x_{n-1}\| + \|x_{n-1} - \mathscr{T}_1^n y_n\| + L_1\|y_n - x_n\|] = 0.\end{aligned} \tag{3.23}$$

Consider, from (3.10) and (3.21), we take

$$\lim_{n\to\infty} \|x_{n-1} - \mathscr{T}_2^n x_n\| \leq \lim_{n\to\infty} \|x_{n-1} - x_n\| + \lim_{n\to\infty} \|x_n - \mathscr{T}_2^n x_n\| = 0. \tag{3.24}$$

From (3.10) and (3.15), we take

$$\lim_{n\to\infty} \|x_{n-1} - I_1^n x_n\| \leq \lim_{n\to\infty} \|x_{n-1} - x_n\| + \lim_{n\to\infty} \|x_n - I_1^n x_n\| = 0. \tag{3.25}$$

and by (3.10) and (3.22), we take

$$\lim_{n\to\infty} \|x_{n-1} - I_2^n x_n\| \leq \lim_{n\to\infty} \|x_{n-1} - x_n\| + \lim_{n\to\infty} \|x_n - I_2^n x_n\| = 0. \tag{3.26}$$

Finally, we get

$$\begin{aligned}\|x_n - \mathscr{T}_1 x_n\| &\leq \|x_n - \mathscr{T}_1^n x_n\| + \|\mathscr{T}_1^n x_n - \mathscr{T}_1 x_n\| \\ &\leq \|x_n - \mathscr{T}_1^n x_n\| + L_1\|\mathscr{T}_1^{n-1} x_n - x_n\| \\ &\leq \|x_n - \mathscr{T}_1^n x_n\| + L_1\left[\|\mathscr{T}_1^{n-1} x_n - \mathscr{T}_1^{n-1} x_{n-1}\| + \|\mathscr{T}_1^{n-1} x_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\|\right] \\ &\leq \|x_n - \mathscr{T}_1^n x_n\| + L_1\left[L_1\|x_n - x_{n-1}\| + \|\mathscr{T}_1^{n-1} x_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\|\right] \\ &\leq \|x_n - \mathscr{T}_1^n x_n\| + L_1(L_1+1)\|x_n - x_{n-1}\| + L_1\|\mathscr{T}_1^{n-1} x_{n-1} - x_{n-1}\|\end{aligned}$$

with (3.10) and (3.23), we obtain

$$\lim_{n\to\infty}\|x_n - \mathscr{T}_1 x_n\| = 0. \tag{3.27}$$

Analogously, one has

$$\|x_n - \mathscr{T}_2 x_n\| \le \|x_n - \mathscr{T}_2^n x_n\| + L_2(L_2+1)\|x_n - x_{n-1}\| + L_2\|\mathscr{T}_2^{n-1}x_{n-1} - x_{n-1}\|,$$

$$\|x_n - I_1 x_n\| \le \|x_n - I_1^n x_n\| + L_3(L_3+1)\|x_n - x_{n-1}\| + L_3\|I_1^{n-1}x_{n-1} - x_{n-1}\|,$$

and

$$\|x_n - I_2 x_n\| \le \|x_n - I_2^n x_n\| + L_4(L_2+1)\|x_n - x_{n-1}\| + L_4\|I_2^{n-1}x_{n-1} - x_{n-1}\|,$$

which with (3.10), (3.15), (3.21) and (3.22) imply

$$\lim_{n\to\infty}\|x_n - \mathscr{T}_2 x_n\| = 0 \tag{3.28}$$

$$\lim_{n\to\infty}\|x_n - I_1 x_n\| = 0 \tag{3.29}$$

and

$$\lim_{n\to\infty}\|x_n - I_2 x_n\| = 0. \tag{3.30}$$

Step 2. Assume $z^* \in F = F(\mathscr{T}_1)\cap F(\mathscr{T}_2)\cap F(I_1)\cap F(I_2)\cap EP(\phi)$. From the definition of notation T_r in Lemma 2.2, we know that $v_n = T_{r_n}x_n$. So, it follows that

$$\|v_n - z^*\| = \|T_{r_n}x_n - T_{r_n}z^*\| \le \|x_n - z^*\|.$$

Since $\{v_n\}$ is bounded, there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that $\{v_{n_k}\}$ converges weakly to $z^* \in D$ when $z^* = J^{-1}w^*$ for some $w^* \in J(D)$. By (3.4), we have that $\{x_{n_k}\}$ converges weakly to $z^* \in D$ and from (3.20), we also have that $\{y_{n_k}\}$ converges weakly to $z^* \in D$. Also, from (3.15), (3.21), (3.22),(3.23) and Lemma 2.3, we obtain $z^* \in F(\mathscr{T}_1)\cap F(\mathscr{T}_2)\cap F(I_1)\cap F(I_2)$.

Next, we show that $z^* \in EP(\phi)$, that is $Jz^* = w^* \in J(EP(\phi))$. By $v_n = T_{r_n}x_n$, since J is uniformly norm-to-norm continuous on bounded subset of E, it follows that

$$\lim_{n\to\infty} \|Jx_n - Jv_n\| = 0. \tag{3.31}$$

From the assumption $r_n \in [\rho,\ \infty)$, one sees

$$\lim_{n\to\infty} \frac{\|Jx_n - Jv_n\|}{r_n} = 0. \tag{3.32}$$

Since $\{x_n\}$ is bounded and so is $\{Jx_n\}$, there exists a subsequence $\{Jx_{n_k}\}$ of $\{Jx_n\}$ such that $\{Jx_n \rightharpoonup\ w^*\}$. Since $\{v_n\}$ is bounded, by (3.40), we also obtain $\{Jv_n \rightharpoonup\ w^*\}$. Noticing that $v_n = T_{r_n}x_n$, we obtain

$$\phi(v_n,\ y) + \frac{1}{r_n}\langle y - v_n,\ Jv_n - Jx_n\rangle \geq 0,\ y \in D,$$

$$\phi(v_{n_k},\ y) + \langle y - v_{n_k},\ \frac{Jv_{n_k} - Jx_{n_k}}{r_{n_k}}\rangle \geq 0,\ y \in D, \tag{3.33}$$

According to (3.40), we obtain $\lim_{k\to\infty}\left[\frac{Jv_{n_k}-Jx_{n_k}}{r_{n_k}}\right] = 0$. Then, from (C2), we find

$$\frac{1}{r_n}\langle y - v_n,\ Jv_n - Jx_n\rangle \geq -\phi(v_n,\ y),$$

$$\langle y - v_n,\ \frac{Jv_n - Jx_n}{r_n}\rangle \geq \phi(y,\ v_n). \tag{3.34}$$

Since $\frac{\|Jx_n - Jv_n\|}{r_n} \to 0$ and $\{Jv_n \rightharpoonup\ w^*\}$, we obtain

$$\phi(y,\ w^*) \leq 0\ y \in D. \tag{3.35}$$

For t with $0 \leq t \leq 1$ and $y \in D$, let $y_t = ty + (1-t)w^*$. Since $y \in D$ and $w^* \in D$, we have $y_t \in D$ and hence $\phi(y_t,\ w^*) \leq 0$. So from condition (C1) and (C3), we have

$$0 = \phi(y_t,\ y_t) \leq t\phi(y_t,\ y) + (1-t)\phi(y_t,\ w^*) \leq t\phi(y_t,\ y), \tag{3.36}$$

and hence $0 \leq \phi(y_t,\ y)$. From (C3), we have

$$0 \leq \phi(w^*,\ y),\ \forall\ y \in D,$$

and hence $w^* \in EP(\phi)$. This completes the proof.

□

Theorem 3.3. *Let E be a real uniformly convex Banach space satisfying Opial condition and let D be a nonempty closed convex subset of E. Suppose $X : E \to E$ is an identity mapping, let $\mathcal{T}_1,\ \mathcal{T}_2 : D \to D$ be two uniformly L_1 and L_2-Lipschitizian asymptotically quasi-I-nonexpansive mapping with a sequences $\{k_n\},\ \{h_n\} \subset [1,\ \infty)$ and $I_1,\ I_2$ be two uniformly L_3 and L_4-Lipschitizian asymptotically quasi-nonexpansive self mapping of D with a sequence $\{g_n\},\ \{t_n\} \subset [1,\ \infty)$. Let $\mu_n = \max_{n\in\mathbb{N}}\ \{k_n,\ h_n,\ g_n,\ t_n\}$ and we assume that $R = \sup_n(1-a_n),\ \mathcal{M} = \sup_n \mu_n^2 \geq 1$ such that $F = F(\mathcal{T}_1)\cap F(\mathcal{T}_2)\cap F(I_1)\cap F(I_2)$ is nonempty and $q^* \in F$. And $\{a_n\},\ \{b_n\},\ \{c_n\},\ \{\widehat{a_n}\},\ \{\widehat{b_n}\},\ \{\widehat{c_n}\}$ are six real sequences in $(0,\ 1)$ which satisfy the following conditions :*

(i) $R(\mathcal{M}+\mathcal{M}^2) < 1,$

(ii) $\sum_{n=1}^{\infty}(1-a_n)(\mu_n^4+\mu_n^2-1) < \infty.$

If the mappings $X-\mathcal{T}_1,\ X-\mathcal{T}_2,\ X-I_1,\ X-I_2$ are semiclosed at zero, then the implicitly iterative sequence $\{x_n\}$ defined by (1.2) converges weakly to common fixed point of F.

Proof. Let $q^* \in F$, then according to Lemma 3.1 the sequence $\{\|x_n-q^*\|\}$ converges. This provides that $\{x_n\}$ is bounded sequence. Since E be a uniformly convex, then every bounded subset of E is weakly compact. Since $\{x_n\}$ is bounded sequence in D, then there exists a subsequence $\{x_{n_r}\} \subset \{x_n\}$ such that $\{x_{n_r}\}$ converges weakly to $q \in D$. Therefore, from (3.27), (3.28), (3.29) and (3.30) it follows that

(3.37)
$$\lim_{n_r\to\infty}\|x_{n_r}-\mathcal{T}_1x_{n_r}\| = \lim_{n_r\to\infty}\|x_{n_r}-\mathcal{T}_2x_{n_r}\| = \lim_{n_r\to\infty}\|x_{n_r}-I_1x_{n_r}\| = \lim_{n_r\to\infty}\|x_{n_r}-I_2x_{n_r}\| = 0.$$

Since the mapping $X-\mathcal{T}_1,\ X-\mathcal{T}_2,\ X-I_1$ and $X-I_2$ are semiclosed at zero, hence, we find $\mathcal{T}_1q = q,\ \mathcal{T}_2q = q,\ I_1q = q$ and $I_2q = q$ which means $q \in F = F(\mathcal{T}_1)\cap F(\mathcal{T}_2)\cap F(I_1)\cap F(I_2)$.

Finally, let us prove that $\{x_n\}$ converges weakly to q. In fact, suppose the contrary, that is, there exists some subsequence $\{x_{n_r}\} \subset \{x_n\}$ such that $\{x_{n_r}\}$ converges weakly to $q_1 \in D$ and

$q_1 \neq q$. Then by the same method as given above, we can also prove that $q_1 \in F = F(\mathscr{T}_1) \cap F(\mathscr{T}_2) \cap F(I_1) \cap F(I_2)$.

From Lemma 3.1, we can prove that the $\lim_{n\to\infty} \|x_n - q\|$ and $\lim_{n\to\infty} \|x_n - q_1\|$ exist, and we have

$$\lim_{n\to\infty} \|x_n - q\| = d, \ \lim_{n\to\infty} \|x_n - q_1\| = d_1, \tag{3.38}$$

where d and d_1 are two nonnegative numbers. By asset of the Opial condition of E, we take

$$\begin{aligned} d &= \lim_{n_r\to\infty} \sup \|x_{n_r} - q\| < \lim_{n_r\to\infty} \sup \|x_{n_r} - q_1\| = d_1 \\ &= \lim_{n_k\to\infty} \sup \|x_{n_k} - q_1\| < \lim_{n_k\to\infty} \sup \|x_{n_k} - q\|. \end{aligned} \tag{3.39}$$

This is a contradiction. Therefore $q_1 = q$. This implies that $\{x_n\}$ converges weakly to q. This completes the proof.

□

Now we formulate next results concerning strong convergence of the sequence $\{x_n\}$.

Theorem 3.4. *Let E be a real uniformly convex Banach space and let D, $\mathscr{T}_1$, $\mathscr{T}_2$, I_1, I_2, $\{x_n\}$ be same as in Theorem 3.3. Suppose that the conditions in Theorem 3.3 is satisfied. If at least one mapping of the mappings $\mathscr{T}_1$, $\mathscr{T}_2$, I_1 and I_2 are semicompact, then an explicitly iterative sequence $\{x_n\}$ defined by (1.2) converges strongly to a common fixed point in F.*

Proof. Without any loss of generality, we may assume the $\mathscr{T}_1$, $\mathscr{T}_2$, I_1 and I_2 are semicompact. Then from (3.27), (3.28), (3.29) and (3.30), we have

$$\lim_{n\to\infty} \|x_n - \mathscr{T}_1 x_n\| = \lim_{n\to\infty} \|x_n - \mathscr{T}_2 x_n\| = \lim_{n\to\infty} \|x_n - I_1 x_n\| = \lim_{n\to\infty} \|x_n - I_2 x_n\| = 0.$$

From the semicompactness $\mathscr{T}_1$, $\mathscr{T}_2$, I_1 and I_2 there exists a subsequence $\{x_{n_r}\} \subset \{x_n\}$ such that $x_{n_r} \to q$ converges strongly to a $q \in D$. Again, using (3.27), (3.28), (3.29) and (3.30), we obtain

$$\lim_{n_r\to\infty} \|x_{n_r} - \mathscr{T}_1 x_{n_r}\| = \|q - \mathscr{T}_1 q\| = 0, \ \lim_{n_r\to\infty} \|x_{n_r} - \mathscr{T}_2 x_{n_r}\| = \|q - \mathscr{T}_2 q\| = 0,$$
$$\lim_{n_r\to\infty} \|x_{n_r} - I_1 x_{n_r}\| = \|q - I_1 q\| = 0, \ \lim_{n_r\to\infty} \|x_{n_r} - I_2 x_{n_r}\| = \|q - I_2 q\| = 0.$$

This shows that $q \in F = F(\mathscr{T}_1) \cap F(\mathscr{T}_2) \cap F(I_1) \cap F(I_2)$. According to Lemma 3.1 the limit $\lim_{n\to\infty} \|x_n - q\|$ exists. Then

$$\lim_{n\to\infty} \|x_n - q\| = \lim_{n_r\to\infty} \|x_{n_r} - q\| = 0,$$

which means that $\{x_n\}$ converges to $q \in F$. This completes the proof.

□

We give example for equilibrium problem as follows :

Example 3.5. *Let $E = \mathbb{R}$ and $D = [-100,\ 100]$. The equilibrium problem is to find $x \in D$ such that*

$$\phi(x,\ y) \geq 0,\ \forall\, y \in D, \tag{3.40}$$

where we define

$$\phi(x,\ y) = -2x^2 + xy + y^2.$$

Now, we can easily know that ϕ satisfy the conditions (C1)-(C4) as follows:

(C1) $\phi(x,\ x) = -2x^2 + x^2 + x^2 = 0,\ \forall\, x \in [-100,\ 100]$,

(C2) $\phi(x,\ y) + \phi(y,\ x) = -(x-y)^2 \leq 0,\ \forall\, x,\ y \in [-100,\ 100]$,

(C3) *for all* $x,\ y,\ z \in [-100,\ 100]$,

$$\begin{aligned}
\lim_{t\downarrow 0} \phi(tz + (1-t)x,\ y) &= \lim_{t\downarrow 0} \phi(x + t(z-x),\ y) \\
&= \lim_{t\downarrow 0} [-2(x + t(z-x))^2 + (x + t(z-x))y + y^2] \\
&= \lim_{t\downarrow 0} \phi(x + t(z-x),\ y) \\
&= \lim_{t\downarrow 0} [-2(x + t(z-x))^2 + (xy + t(z-x)y + y^2)] \\
&= -2x^2 + xy + y^2 \leq \phi(x,\ y),
\end{aligned}$$

(C4) *for each* $x \in [-100,\ 100]$, $\phi(x,\ y) = -2x^2 + xy + y^2$ *is convex and weakly lower semicontinuous.*

Conflict of Interests

The authors declare that there is no conflict of interests.

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