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GENERALIZED CONTRACTION RESULTING TRIPLED FIXED POINT THEOREMS IN COMPLEX VALUED METRIC SPACES

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Abstract. Owning the concept of complex valued metric spaces introduced by Azam et al.[1] many authors prove several fixed point results for mappings satisfying certain contraction conditions. Coupled and tripled fixed point problems have attracted much attention in recent times. In this note, common tripled fixed point theorems for a pairs of mappings satisfying certain rational contraction in complex valued metric spaces are proved. Some illustrative examples are also given which demonstrate the validity of the hypotheses of our results.

Keywords: common fixed point; complex valued metric space; complete complex valued metric spaces; Cauchy sequence; tripled fixed point.

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1. Introduction and Preliminaries

Fixed point theory in nonlinear analysis has a broad set of applications. In this theory Banach contraction is one of the main tools to prove the existence and uniqueness of a fixed point. Large

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number of generalizations has been made on this principle.

In 2011, Azam et al.[1] introduced the notion of complex valued metric spaces which is more general than the classical metric spaces. In 2006 T. Gnana Bhaskar and V. Lakshmikantham [5] introduced the notion of coupled fixed point and proved some fixed point theorems in this context, thereafter S.M. Kang et al.[7] introduced the notion of coupled fixed point for a mapping in complex valued metric spaces. Recently, Berinde and Borcut [3,4] introduced the notion of tripled fixed point for nonlinear contractive mappings in partially order complete metric spaces and obtained tripled coincidence and fixed point theorems for commuting mappings. Very recently Roldan et al. [11] introduced the tripled fixed point in fuzzy metric spaces and proved existence and uniqueness theorem for contractive type mappings in fuzzy metric spaces. In this manner many researchers have contributed with their works in coupled and tripled fixed point results. For detailed development one can see in ([2],[6],[7],[9]). In order to that we consider a slight modification of the concept of tripled fixed point for a mapping in complex valued metric spaces as follows. In what follows, we recall some definitions and notations that will be used in our note.

Let C be the set of complex numbers and $z_1, z_2 \in C$. Define a partial order $\precsim$ on C as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$and $Im(z_1) \leq Im(z_2)$.

It follows that $\precsim$ exists if one of the followings conditions is satisfied:

(C1): $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(C2): $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(C3): $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,

(C4): $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$;

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (C2), (C3) and (C4) is satisfied and we will write $z_1 \prec z_2$ if only (C4) is satisfied. Note that

The following definition is due to Azam et al.[1].

Definition 1.1. Let X be a non empty set. A mapping $d : X \times X \to C$ is called a complex valued metric on X if d satisfies the following conditions :

(CM1): $0 \precsim d(x,y)$ for all $x, y \in X$ and $d(x,y) \Leftrightarrow x = y$,

(CM2): $d(x,y) = d(y,x)$ for all $x, y \in X$,

(CM3): $d(x,y) \precsim d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Then d is called a complex valued metric on X, and (X,d) is called complex valued metric space.

Example 1.1. Let $X = C$ be a set of complex number. Define $d : C \times C \to C$. By

$$d(z_1,z_2) = |x_1 - x_2| + i|y_1 - y_2|$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then (C,d) is a complex valued metric space.

Definition 1.2. Let (X,d) is a complex valued metric space and $/x_n/$ be a sequence in X.

(1) for every $c \in C$ with $0 \prec c$ there exists $N \in \mathbb{N}$ such that $d(x_n,x)$ for all $n \geq N$, then x_n is said to be convergent to $x \in X$, and we denote this by $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.

(2) If for every $c \in C$ with $0 \prec c$ there exist $N \in \mathbb{N}$ such that $d(x_n,x_{n+m}) \prec c$ for all $n \geq N$, where $m \in \mathbb{N}$, then $/x_n/$ is said to be Cauchy sequence.

(3) If every Cauchy sequence in X is convergent, then (X,d) is said to be a complete complex valued metric space.

Lemma 1.1. Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.

Lemma 1.2. Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n,x_{n+m})| \to 0$ as $n \to \infty$ where $m \in \mathbb{N}$.

Definition 1.3. (Bhaskar and Lakshmikantham [5]) Let (X,d) be a complex valued metric space. Then an element $(x,y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if $x = F(x,y)$ and $y = F(y,x)$.

Definition 1.4. (Bhaskar and Ciric[10]) Let (X,d) be a complex valued metric space. Then an element $(x,y) \in X \times X$ is called

(i): a coupled coincidence point of mapping $F : X \times X \to X$ and $g : X \to X$ if $gx = F(x,y)$ and $gy = F(y,x)$ and (gx,gy) is called a coupled point of coincidence;

(ii): a common coupled fixed point of mapping $F : X \times X \to X$ and $g : X \to X$ if $x = gx = F(x,y)$ and $y = gy = F(y,x)$.

Note that if g is the identity mapping then Definition 1.4 reduced to Definition 1.3.

In 2010 Samet and Vetro [12] introduced the fixed point of order $N \geq 3$. In particular for $N = 3$, we have the following definition.

Definition 1.5. Let (X,d) be a complex valued metric space. Then an element $(x,y,z) \in X \times X \times X$ is said to be tripled fixed point of the mapping $F : X \times X \times X \to X$ if $x = F(x,y,z), y = F(y,z,x)$ and $z = F(z,x,y)$.

Example 1.2. Let $X = C$ and define $d : X \times X \times X \to C$ by $d(x,y) = i|x-y|$ then (X,d) is a complex valued metric space. Consider the mapping $F : X \times X \times X \to X$ with $i\frac{(x+y+z)}{i}$. For all $x,y,z \in X$, then $(0,0,0)$ is tripled fixed point of F.

2. Main Result

In this section, we prove some common fixed point theorems for contraction conditions described by rational expressions.

Theorem 2.1. Let (X,d) be a complete complex valued metric space and let the mappings $S,T : X \times X \times X \to X$, satisfy

$$\begin{aligned} d(S(x,y,z),T(u,v,w)) \precsim & \frac{\alpha[d(x,u)+d(y,w)+d(z,w)]}{3} + \\ & \frac{\beta[d(x,S(x,y,z)).d(x,T(u,v,w)+d(u,T(u,v,w)).d(u,S(x,y,z))]}{1+d(x,T(u,v,w))+d(u,S(x,y,z))}, \end{aligned} \tag{1}$$

for all $x,y,z,u,v,w \in X$ and α, β non negative reals with $\alpha + \beta < 1$. Then S and T have a unique common tripled fixed point.

Proof. Let x_0, y_0 *and* z_0 be arbitrary points in X.

Define

$x_{2n+1} = S(x_{2n}, y_{2n}, z_{2n}),\ y_{2n+1} = S(y_{2n}, z_{2n}, x_{2n})$ and $z_{2n+1} = S(z_{2n}, x_{2n}, y_{2n})$

$x_{2n+2} = T(x_{2n+1},\ y_{2n+1}, z_{2n+1}),\ y_{2n+2} = T(y_{2n+1}, z_{2n+1}, x_{2n+1}),$

$z_{2n+2} = T(z_{2n+1}, x_{2n+1}, y_{2n+1})$ for all $n = 0,1,2,3...$.

Then

$$\begin{aligned}
d(x_{2n+1},x_{2n+2}) &= d(S(x_{2n},y_{2n},z_{2n}),T(x_{2n+1},y_{2n+1},z_{2n+1}))\\
&\precsim \frac{\alpha[d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})+d(z_{2n},z_{2n+1})]}{3}\\
&+\frac{\beta\left\{\begin{array}{l} d(x_{2n},S(x_{2n},y_{2n},z_{2n}))d(x_{2n},T(x_{2n+1},y_{2n+1},z_{2n+1}))+\\ \qquad d(x_{2n+1},T(x_{2n+1},y_{2n+1},z_{2n+1}))d(x_{2n+1},S(x_{2n},y_{2n},z_{2n}))\end{array}\right\}}{1+d(x_{2n},T(x_{2n+1},y_{2n+1},z_{2n+1}))+d(x_{2n+1},S(x_{2n},y_{2n},z_{2n}))}\\
&= \frac{\alpha[d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})+d(z_{2n},z_{2n+1})]}{3}\\
&\quad+\frac{\beta[d(x_{2n},x_{2n+1})d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+2})d(x_{2n+1},x_{2n+1})]}{1+d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+1})}\\
&\quad= \frac{\alpha[d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})+d(z_{2n},z_{2n+1})]}{3}+\beta d(x_{2n},x_{2n+1})\\
&\quad= (\frac{\alpha}{3}+\beta)d(x_{2n},x_{2n+1})+(\frac{\alpha}{3})d(y_{2n},y_{2n+1})+(\frac{\alpha}{3})d(z_{2n},z_{2n+1}).
\end{aligned}$$

Which implies that

$$|d(x_{2n+1},x_{2n+2})| \leq (\frac{\alpha}{3}+\beta)|d(x_{2n},x_{2n+1})|+(\frac{\alpha}{3})|d(y_{2n},y_{2n+1})|+(\frac{\alpha}{3})|d(z_{2n},z_{2n+1})|. \tag{2}$$

Also

$$\begin{aligned}
d(y_{2n+1},y_{2n+2}) =& d(S(y_{2n},z_{2n},x_{2n}),T(y_{2n+1},z_{2n+1},x_{2n+1}))\\
&\precsim \frac{\alpha[d(y_{2n},y_{2n+1})+d(z_{2n},z_{2n+1})+d(x_{2n},x_{2n+1})]}{3}\\
&+\frac{\beta\left\{\begin{array}{l} d(y_{2n},S(y_{2n},z_{2n},x_{2n}))d(y_{2n},T(y_{2n+1},z_{2n+1},x_{2n+1}))+\\ \qquad d(y_{2n+1},T(y_{2n+1},z_{2n+1},x_{2n+1}))d(y_{2n+1},S(y_{2n},z_{2n},x_{2n}))\end{array}\right\}}{1+d(y_{2n},T(y_{2n+1},z_{2n+1},x_{2n+1}))+d(y_{2n+1},S(y_{2n},z_{2n},x_{2n}))}\\
&= \frac{\alpha[d(y_{2n},y_{2n+1})+d(z_{2n},z_{2n+1})+d(x_{2n},x_{2n+1})]}{3}\\
&+\frac{\beta[d(y_{2n},y_{2n+1})d(y_{2n},y_{2n+2})+d(y_{2n+1},y_{2n+2})d(y_{2n+1},y_{2n+1})]}{1+d(y_{2n},y_{2n+2})+d(y_{2n+1},y_{2n+1})}\\
&= \frac{\alpha[d(y_{2n},y_{2n+1})+d(z_{2n},z_{2n+1})+d(x_{2n},x_{2n+1})]}{3}+\beta d(y_{2n},y_{2n+1})\\
&= (\frac{\alpha}{3})d(x_{2n},x_{2n+1})+(\frac{\alpha}{3}+\beta)d(y_{2n},y_{2n+1})+(\frac{\alpha}{3})d(z_{2n},z_{2n+1})
\end{aligned}$$

which implies that

$$(3)\qquad |d(y_{2n+1},y_{2n+2})| \leq (\frac{\alpha}{3})|d(x_{2n},x_{2n+1})| + (\frac{\alpha}{3}+\beta)|d(y_{2n},y_{2n+1})| + (\frac{\alpha}{3})|d(z_{2n},z_{2n+1})|.$$

Similarly we can prove

$$(4)\qquad |d(z_{2n+1},z_{2n+2})| \leq (\frac{\alpha}{3})|d(x_{2n},x_{2n+1})| + (\frac{\alpha}{3})|d(y_{2n},y_{2n+1})| + (\frac{\alpha}{3}+\beta)|d(z_{2n},z_{2n+1})|.$$

Adding (2),(3) and (4), we get

$$\begin{aligned}
&|d(x_{2n+1},x_{2n+2})| + |d(y_{2n+1},y_{2n+2})| + |d(z_{2n+1},z_{2n+2})| \\
&\qquad \leq (\alpha+\beta)|d(x_{2n},x_{2n+1})| + (\alpha+\beta)|d(y_{2n},y_{2n+1})| + (\alpha+\beta)|d(z_{2n},z_{2n+1})| \\
&\qquad = (\alpha+\beta)[|d(x_{2n},x_{2n+1})| + |d(y_{2n},y_{2n+1})| + |d(z_{2n},z_{2n+1})|] \\
&\qquad = h[|d(x_{2n},x_{2n+1})| + |d(y_{2n},y_{2n+1})| + |d(z_{2n},z_{2n+1})|].
\end{aligned}$$

Since, $h = \alpha+\beta < 1$ then from above, we have

$$\begin{aligned}
&|d(x_n,x_{n+1})| + |d(y_n,y_{n+1})| + |d(z_n,z_{n+1})| \\
&\qquad \leq h[|d(x_{n-1},x_n)| + |d(y_{n-1},y_n)| + |d(z_{n-1},z_n)|] \\
&\qquad \leq ... \leq h^n[|d(x_0,x_1)| + |d(y_0,y_1)| + |d(z_0,z_1)|].
\end{aligned}$$

Now if $|d(x_n,x_{n+1})| + |d(y_n,y_{n+1})| + |d(z_n,z_{n+1})| = \delta_n$ then
$\delta_n \leq h\delta_{n-1} \leq ... \leq h^n\delta_0$.
Without the loss of generality, we can take $m > n$ since $0 \leq h < 1$, so we get

$$\begin{aligned}
&|d(x_n,x_m)| + |d(y_n,y_m)| + |d(z_n,z_m)| \\
&\qquad \leq [|d(x_n,x_{n+1})| + |d(y_n,y_{n+1})| + |d(z_n,z_{n+1})|] \\
&\qquad + \leq ... \leq [|d(x_{m-1},x_m)| + |d(y_{m-1},y_m)| + |d(z_{m-1},z_m)|] \\
&\qquad \leq [h^n\delta_0 + h^{n+1}\delta_0 + ... + h^{m-1}\delta_0] \\
&\qquad = \sum_{i=n}^{i=m-1} h^i\delta_0 \to 0 \; as \; m,n \to \infty.
\end{aligned}$$

This implies that $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ are Cauchy sequence in X. Since X is complete, there exist $x,y,z \in X$ such that $x_n \to x, y_n \to y$ *and* $z_n \to z$ as $n \to \infty$. Now we show that $x = S(x,y,z), y =$

$S(y,z,x)$ *and* $z = S(z,x,y)$. We suppose on contrary that $x \neq S(x,y,z), y \neq S(y,z,x)$ *and* $z \neq S(z,x,y)$, then

$$\begin{aligned}
d(x,S(x,y,z)) &\precsim d(x,x_{2n+2}) + d(x_{2n+2},S(x,y,z)) \\
&= d(x,x_{2n+2}) + d(T(x_{2n+1},y_{2n+1},z_{2n+1}),S(x,y,z)) \\
&= d(x,x_{2n+2}) + \frac{\alpha[d(x,x_{2n+1}) + d(y,y_{2n+1}) + d(z,z_{2n+1})]}{3} \\
&+ \frac{\beta\left\{\begin{array}{c} d(x,S(x,y,z))d(x,T(x_{2n+1},y_{2n+1},z_{2n+1})) \\ +d(x_{2n+1},T(x_{2n+1},y_{2n+1},z_{2n+1}))d(x_{2n+1},S(x,y,z)) \end{array}\right\}}{1 + d(x,T(x_{2n+1},y_{2n+1},z_{2n+1}) + d(x_{2n+1},S(x,y,z))} \\
&= d(x,x_{2n+2}) + \frac{\alpha[d(x,x_{2n+1}) + d(y,y_{2n+1}) + d(z,z_{2n+1})]}{3} \\
&+ \frac{\beta[d(x,S(x,y,z))d(x,x_{2n+2}) + d(x_{2n+1},x_{2n+2})d(x_{2n+1},S(x,y,z))]}{1 + d(x,x_{2n+2} + d(x_{2n+1},S(x,y,z))}.
\end{aligned}$$

$\Rightarrow |d(x,S(x,y,z))| \to 0$ as $n \to +\infty$ which implies that $S(x,y,z) = x$.

Similarly we can prove $S(y,z,x) = y,\ S(z,,x,y) = z$. It follows similarly that $T(x,y,z) = x,\ T(y,z,x) = y,\ T(z,x,y) = z$. So we have proved (x,y,z) is common tripled fixed point of S *and* T. Now we show that S *and* T have a unique common tripled fixed point. For this assume $(x^*,y^*,z^*) \in X$ is another common tripled fixed point of S *and* T. Then

$$\begin{aligned}
d(x,x^*) =& d(S(x,y,z),T(x^*,y^*,z^*)) \\
&\precsim \frac{\alpha[d(x,x^*) + d(y,y^*) + d(z,z^*)]}{3} + \\
&\frac{\beta[d(x,S(x,y,z))d(x,T(x^*,y^*,z^*)) + d(x^*,T(x^*,y^*,z^*))d(x^*,S(x,y,z))]}{1 + d(x,T(x^*,y^*,z^*)) + d(x^*,S(x,y,z))} \\
&\precsim \frac{\alpha[d(x,x^*) + d(y,y^*) + d(z,z^*)]}{3} + \frac{\beta[d(x,x)d(x,x^*) + d(x^*,x^*)d(x^*,x)]}{1 + d(x,x^*) + d(x^*,x)} \\
&\precsim \frac{\alpha[d(x,x^*) + d(y,y^*) + d(z,z^*)]}{3}.
\end{aligned} \tag{5}$$

Implies that

$$|d(x,x^*)| \leq (\frac{\alpha}{3})|[d(x,x^*) + d(y,y^*) + d(z,z^*)]|.$$

Similarly we have

$$|d(y,y^*)| \leq (\frac{\alpha}{3})|[d(x,x^*) + d(y,y^*) + d(z,z^*)]|$$

and

$$|d(z,z^*)| \leq (\frac{\alpha}{3})|[d(x,x^*)+d(y,y^*)+d(z,z^*)]|.$$

Which implies that

$$|d(x,x^*)|+|d(y,y^*)|+|d(z,z^*)| \leq \alpha|[d(x,x^*)+d(y,y^*)+d(z,z^*)]|$$

$$\leq \alpha[|d(x,x^*)|+|d(y,y^*)|+|d(z,z^*)|].$$

Which is a contradiction, since $\alpha+\beta<1 \Rightarrow \alpha<1$.

Hence $|d(x,x^*)|+|d(y,y^*)|+|d(z,z^*)|=0 \Rightarrow x=x^*,\ y=y^*\ and\ z=z^*$, which prove the uniqueness of common tripled fixed point of $S\ and\ T$.

Theorem 2.2. Let (X,d) be a complete complex valued metric space. Suppose that the mapping $F : X\times X\times X \to X$, satisfies

(6) $$d(F(x,y,z),F(u,v,w)) \precsim \alpha d(x,u)+\beta d(y,v)+\gamma d(z,w)+\delta\frac{d(F(x,y,z),u)d(F(u,v,w),x)}{1+d(x,u)+d(y,v)+d(z,w)}$$

for all $x,y,z,u,v,w \in X$ where $\alpha,\beta,\gamma,\delta$ are non-negative reals with $\alpha+\beta+\gamma+\delta<1$. Then F has a unique tripled fixed point.

Proof. Choose $x_0,y_0,z_0 \in X$ and set

$$x_1=F(x_0,y_0,z_0), y_1=F(y_0,z_0,x_0), z_1=F(z_0,x_0,x_0)$$

$$\cdots$$

$$\cdots$$

$$x_{n+1}=F(x_n,y_n,z_n), y_{n+1}=F(y_n,z_n,x_n), z_{n+1}=F(z_n,x_n,y_n).$$

From (6), we have

$$\begin{aligned} d(x_n,x_{n+1}) &= d(F(x_{n-1},y_{n-1},z_{n-1}),F(x_n,y_n,z_n)). \\ &\precsim \alpha d(x_{n-1},x_n)+\beta d(y_{n-1},y_n)+\gamma d(z_{n-1},z_n) \\ &+\delta\frac{d(F(x_{n-1},y_{n-1},z_{n-1}),x_n)d(F(x_n,y_n,z_n),x_{n-1})}{1+d(x_{n-1},x_n)+d(y_{n-1},y_n)+d(z_{n-1},z_n)} \\ &= \alpha d(x_{n-1},x_n)+\beta d(y_{n-1},y_n)+\gamma d(z_{n-1},z_n) \\ &+\delta\frac{d(x_n,x_n)d(x_{n+1},x_{n-1})}{1+d(x_{n-1},x_n)+d(y_{n-1},y_n)+d(z_{n-1},z_n)} \\ &= \alpha d(x_{n-1},x_n)+\beta d(y_{n-1},y_n)+\gamma d(z_{n-1},z_n). \end{aligned}$$

Similarly

$$\begin{aligned}
d(y_n, y_{n+1}) &= d(F(y_{n-1}, z_{n-1}, x_{n-1}), F(y_n, z_n, x_n)) \\
&\precsim \alpha d(y_{n-1}, y_n) + \beta d(z_{n-1}, z_n) + \gamma d(x_{n-1}, x_n) \\
&+ \delta \frac{d(F(y_{n-1}, z_{n-1}, x_{n-1}), y_n) d(F(y_n, z_n, x_n), y_{n-1})}{1 + d(y_{n-1}, y_n) + d(z_{n-1}, z_n) + d(x_{n-1}, x_n)} \\
&= \alpha d(y_{n-1}, y_n) + \beta d(z_{n-1}, z_n) + \gamma d(x_{n-1}, x_n) \\
&+ \delta \frac{d(y_n, y_n) d(y_{n+1}, y_{n-1})}{1 + d(y_{n-1}, y_n) + d(z_{n-1}, z_n) + d(x_{n-1}, x_n)} \\
&= \alpha d(y_{n-1}, y_n) + \beta d(z_{n-1}, z_n) + \gamma d(x_{n-1}, x_n)
\end{aligned}$$

and

$$\begin{aligned}
d(z_n, z_{n+1}) &= d(F(z_{n-1}, x_{n-1}, y_{n-1}), F(z_n, x_n, y_n)) \\
&\precsim \alpha d(z_{n-1}, z_n) + \beta d(x_{n-1}, x_n) + \gamma d(y_{n-1}, y_n) \\
&+ \delta \frac{d(F(z_{n-1}, x_{n-1}, y_{n-1}), z_n) d(F(z_n, x_n, y_n), z_{n-1})}{1 + d(z_{n-1}, z_n) + d(x_{n-1}, x_n) + d(y_{n-1}, y_n)} \\
&= \alpha d(z_{n-1}, z_n) + \beta d(x_{n-1}, x_n) + \gamma d(y_{n-1}, y_n) \\
&+ \delta \frac{d(z_n, z_n) d(z_{n+1}, z_{n-1})}{1 + d(z_{n-1}, z_n) + d(x_{n-1}, x_n) + d(y_{n-1}, y_n)} \\
&= \alpha d(z_{n-1}, z_n) + \beta d(x_{n-1}, x_n) + \gamma d(y_{n-1}, y_n).
\end{aligned}$$

Therefore by letting $d_n = d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1})$, we have

$$\begin{aligned}
d_n =& d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) \\
&\precsim (\alpha + \beta + \gamma) d(x_{n-1}, x_n) + (\alpha + \beta + \gamma) d(y_{n-1}, y_n) + (\alpha + \beta + \gamma) d(z_{n-1}, z_n) \\
&= (\alpha + \beta + \gamma) d(x_{n-1}, x_n) + d(y_{n-1}, y_n) + d(z_{n-1}, z_n) \\
d_n &\precsim (\alpha + \beta + \gamma) d_{n-1}
\end{aligned}$$

i.e, $d_n \precsim p d_{n-1}$ where $p = \alpha + \beta + \gamma < 1$.

In general, we have $n = 0, 1, 2, 3...$

$$d_n \precsim p d_{n-1} \precsim p^2 d_{n-2} \precsim ... \precsim p^n d_0.$$

Now, for all $m > n$

$$d(x_m,x_n) \precsim d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + ... + d(x_{m-1},x_m)$$

and

$$d(y_m,y_n) \precsim d(y_n,y_{n+1}) + d(y_{n+1},y_{n+2}) + ... + d(y_{m-1},y_m)$$

also

$$d(z_m,z_n) \precsim d(z_n,z_{n+1}) + d(z_{n+1},z_{n+2}) + ... + d(z_{m-1},z_m).$$

Therefore, we have $d(x_m,x_n) + d(y_m,y_n) + d(z_m,z_n) \precsim p^n d_0 + p^{n+1} d_0 + ... + p^{m-1} d_0$,

thus

$$|d(x_m,x_n) + d(y_m,y_n) + d(z_m,z_n)| \leq \frac{p^n}{1-p}|d_0| \to 0 \quad as \;\; n \to \infty$$

which implies that $\{x_n\},\{y_n\}$ *and* $\{z_n\}$ are Cauchy sequence in X. Since X is complete. Therefore there exist $x^*, y^*, z^* \in X$ such that $\lim_{n\to\infty} x_n = x^*, \lim_{n\to\infty} y_n = y^*$ and $\lim_{n\to\infty} z_n = z^*$.

Thus we have

$$\begin{aligned} d(F(x^*,y^*,z^*),x^*) &\precsim d\left(F(x^*,y^*,z^*),x_{N+1}\right) + d(x_{N+1},x^*) \\ &= d(F(x^*,y^*,z^*),F(x_N,y_N,z_N)) + d(x_{N+1},x^*) \\ &\precsim \alpha d(x_N,x^*) + \beta d(y_N,y^*) + \gamma d(z_N,z^*) \\ &+ \frac{\delta d\left(F(x^*,y^*,z^*),x_N\right) d\left(F(x_N,y_N,z_N),x^*\right)}{1 + d(x_N,x^*) + d(y_N,y*) + d(z_N,z^*)} + d(x_{N+1},x^*) \\ &= \alpha(x_N,x^*) + \beta d(y_N,y^*) + \gamma d(z_N,z^*) \\ &+ \frac{\delta d\left(F(x^*,y^*,z^*),x_N\right) d(x_{N+1},x^*)}{1 + d(x_N,x^*) + d(y_N,y*) + d(z_N,z^*)}. \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus we have

$d(F(x^*,y^*,z^*),x^*) = 0$ and hence $F(x^*,y^*,z^*), = x^*$.

Similarly we have $F(y^*,z^*,x^*), = y^*$ *and* $F(z^*,x^*,y^*), = z^*$.

Hence (x^*,y^*,z^*) is tripled fixed point of F.

Now if $(x^{'},y^{'},z^{'})$ is another tripled fixed point of F, then

$$\begin{aligned} d(x^{'},x^*) &= d(F(x^{'},y^{'},z^{'}),F(x^*,y^*,z^*)) \\ &\precsim \alpha d(x^{'},x^*)+\beta d(y^{'},y^*)+\gamma d(z^{'},z^*) \\ &+\frac{\delta d(F(x^{'},y^{'},z^{'}),x^*)d(F(x^*,y^*,z^*),x^{'})}{1+d(x^{'},x^*)+d(y^{'},y*)+d(z^{'},z^*)} \end{aligned}$$

$$d(x^{'},x^*)\precsim \alpha d(x^{'},x^*)+\beta d(y^{'},y^*)+\gamma d(z^{'},z^*)+\frac{\delta d(x^{'},x^*)d(x^*,x^{'})}{1+d(x^{'},x^*)+d(y^{'},y^*)+d(z^{'},z^*)}$$

or

$$d(x^{'},x^*)\precsim \alpha d(x^{'},x^*)+\beta d(y^{'},y^*)+\gamma d(z^{'},z^*)+\delta d(x^{'},x^*)$$

or

$$d(x^{'},x^*)\precsim (\alpha+\delta)d(x^{'},x^*)+\beta d(y^{'},y^*)+\gamma d(z^{'},z^*)$$

similarly

$$d(y^{'},y^*)\precsim (\alpha+\delta)d(y^{'},y^*)+\beta d(z^{'},z^*)+\gamma d(x^{'},x^*)$$

and

$$d(z^{'},z^*)\precsim (\alpha+\delta)d(z^{'},z^*)+\beta d(x^{'},x^*)+\gamma d(y^{'},y^*)$$

which implies that $|[1-(\alpha+\beta+\gamma+\delta)][d(x^{'},x^*)+d(y^{'},y^*)+d(z^{'},z^*)]|\leq 0$ i. e. $d(x^{'},x^*)+d(y^{'},y^*)+d(z^{'},z^*)=0$. Thus we have $(x^{'},y^{'},z^{'})=(x^*,y^*,z^*)$. Therefore F has a unique tripled fixed point.

Theorem 2.3. Let (X,d) be a complete complex valued metric space. Suppose that the mapping $F:X\times X\times X\to X$, satisfies

$$\begin{aligned} d(F(x,y,z),F(u,v,w)) &\precsim \alpha d(F(x,y,z),x)+\beta d(F(u,v,w),u) \\ &+\gamma\frac{d(F(x,y,z),u)d(F(u,v,w),x)}{1+d(F(x,y,z),u)+d(F(u,v,w),x)+d(x,u)+d(y,v)+d(z,w)} \end{aligned} \tag{7}$$

for all $x,y,z,u,v,w\in X$ where α,β,γ are non-negative reals with $\alpha+\beta+\gamma<1$.

Then F has a unique tripled fixed point.

Proof. Choose $x_0, y_0, z_0 \in X$ and set $x_1 = F(x_0,y_0,z_0), y_1 = F(y_0,z_0,x_0)$,
$z_1 = F(z_0,x_0,x_0) \cdots x_{n+1} = F(x_n,y_n,z_n), y_{n+1} = F(y_n,z_n,x_n), z_{n+1} = F(z_n,x_n,y_n)$.
From (7), we have

$$
\begin{aligned}
d(x_n,x_{n+1}) &= d(F(x_{n-1},y_{n-1},z_{n-1}),F(x_n,y_n,z_n)).\\
&\precsim \alpha d(F(x_{n-1},y_{n-1},z_{n-1}),x_{n-1}) + \beta d(F(x_n,y_n,z_n),x_n).\\
&\frac{\gamma d(F(x_{n-1},y_{n-1},z_{n-1}),x_n) d(F(x_n,y_n,z_n),x_{n-1})}{\left\{\begin{array}{c} 1+d(F(x_{n-1},y_{n-1},z_{n-1}),x_n)+d(F(x_n,y_n,z_n),x_{n-1})+ \\ d(x_{n-1},x_n)+d(y_{n-1},y_n)+d(z_{n-1},z_n)\end{array}\right\}}\\
&\precsim \alpha d(x_n,x_{n-1}) + \beta d(x_{n+1},x_n)\\
&+\frac{\gamma d(x_n,x_n) d(x_{n+1},x_{n-1})}{1+d(x_n,x_n)+d(x_{n+1},x_{n-1})+d(x_{n-1},x_n)+d(y_{n-1},y_n)+d(z_{n-1},z_n)}\\
\Rightarrow d(x_n,x_{n+1}) &\precsim \alpha d(x_n,x_{n-1}) + \beta d(x_{n+1},x_n)\\
\Rightarrow d(x_n,x_{n+1}) &\precsim \frac{\alpha}{1-\beta} d(x_n,x_{n-1})\\
\Rightarrow d(x_n,x_{n+1}) &\precsim p d(x_n,x_{n-1}) \textit{ where } p = \frac{\alpha}{1-\beta} < 1.
\end{aligned}
$$

Similarly we have $d(y_n,y_{n+1}) \precsim pd(y_n,y_{n-1})$ and $d(z_n,z_{n+1}) \precsim pd(z_n,z_{n-1})$. Which implies that $\{x_n\},\{y_n\}$ and $\{z_n\}$ are Cauchy sequence in X, therefore by completeness of X there exists $x^*, y^*, z^* \in X$, such that
$\lim_{n\to\infty} x_n = x^*,\ \lim_{n\to\infty} y_n = y^* \textit{ and } \lim_{n\to\infty} z_n = z^*$.
Let $c \in C$ with $0 \prec c$. For every $m \in \mathbb{N}$, exist $N \in \mathbb{N}$ such that $d(x_n,x^*) < \frac{1-\beta}{3m} c$, for all $n \geq N$.
Thus we have

$$
\begin{aligned}
d(F(x^*,y^*,z^*),x^*) &\leq d(x_{N+1},F(x^*,y^*,z^*)) + d(x_{N+1},x^*)\\
&= d(F(x_N,y_N,z_N),F(x^*,y^*,z^*)) + d(x_{N+1},x^*)\\
&\leq \alpha d(F(x_N,y_N,z_N),x_N) + \beta d(F(x^*,y^*,x^*),z^*) + d(x_{N+1},x^*)
\end{aligned}
$$

$$+\frac{\gamma d(F(x_N,y_N,z_N),x^*)d(F(x^*,y^*,z^*),x_N)}{\left\{\begin{array}{c}1+d(F(x_N,y_N,z_N),x^*)+d(F(x^*,y^*,z^*),x^*)+d(x_N,x^*)+\\ d(y_N,y^*)+d(z_N,z^*)\end{array}\right\}}$$

so we have

$$d(F(x^*,y^*,z^*),x^*)\leq\frac{\alpha}{1-\beta}(d(x_{N+1},x_N)+d(x_{N+1},x^*)+Q).$$

$$Where\ Q=\gamma\frac{\gamma d(F(x_N,y_N,z_N),x^*)d(F(x^*,y^*,z^*),x_N)}{\left\{\begin{array}{c}1+d(F(x_N,y_N,z_N),x^*)+d(F(x^*,y^*,z^*),x_N)+d(x_N,x^*)+\\ d(y_N,y^*)+d(z_N,z^*)\end{array}\right\}}$$

$$<\frac{c}{m}$$

$\Rightarrow |d(F(x^*,y^*,z^*),x^*)|<\frac{c}{m}$ for all $m\geq 1$ since m is arbitrary, therefore $d(F(x^*,y^*,z^*),x^*)=0$ and hence $F(x^*,y^*,z^*)=x^*$. Similarly we have $F(y^*,z^*,x^*)=y^*$ and $F(z^*,x^*,y^*)=z^*$. Hence (x^*,y^*,z^*) is the tripled fixed point of F. Now if (x',y',z') is another triple fixed point of F then by applying (7), we have

$$\begin{aligned}d(x',x^*)&=d(F(x',y',z'),F(x^*,y^*,z^*))\\&\precsim \alpha d(F(x',y',z'),x')+\beta d(F(x^*,y^*,z^*),x^*)\\&+\frac{\gamma d(F(x',y',z'),x^*)F(x^*,y^*,z^*),x')}{1+d(F(x',y',z'),x^*)+d(F(x^*,y^*,z^*),x')+d(x',x^*)+d(y',y^*)+d(z',z^*)}.\end{aligned}$$

This implies that $d(x',x^*)=0$. Thus we have $x'=x^*$. Similarly $y'=y^*$ and $z'=z^*$. Therefore F has unique tripled fixed point.

If we put $\beta=0$ in Theorem (2.1) we get the following Corollary (2.1)

Corollary 2.1. Let (X,d) be a complete complex valued metric space and let the mappings $S,T:X\times X\times X\to X$, satisfies

$d(S(x,y,z),T(u,v,w))\precsim\frac{\alpha[d(x,u)+d(y,v)+d(z,w)]}{3}$

for all $x,y,z,u,v,w\in X$ and α is a non negative real with $\alpha<1$.

Then S *and* T have a unique common tripled fixed point.

In Theorem (2.2) if $\alpha=\beta=\gamma=\delta$. We have the following corollary.

Corollary 2.2. Let (X,d) be a complete complex valued metric space. Suppose that the mapping $F:X\times X\times X\to X$, satisfies

$d(F(x,y,z),F(u,v,w)) \precsim \alpha\left\{ d(x,u)+d(y,v)+d(z,w)+\frac{d(F(x,y,z),u)d(F(u,v,w),x)}{1+d(x,u)+d(y,v)+d(z,w)} \right\}$

for all $x,y,z,u,v,w \in X$ where α is a non-negative real with $\alpha < \frac{1}{4}$. Then F has a unique tripled fixed point.

In Theorem (2.3) if we put $\alpha = \beta = \gamma$ then we get the following corollary.

Corollary 2.3. Let (X,d) be a complete complex valued metric space. Suppose that the mapping $F : X \times X \times X \to X$, satisfies

$$d(F(x,y,z),F(u,v,w)) \precsim \alpha[d(F(x,y,z),x)+d(F(u,v,w),u)+ \frac{d(F(x,y,z),u)d(F(u,v,w),x)}{1+d(F(x,y,z),u)+d(F(u,v,w),x)+d(x,u)+d(y,v)+d(z,w)}]$$

for all $x,y,z,u,v,w \in X$ where α is a non-negative real with $\alpha < \frac{1}{3}$.

Then F has a unique tripled fixed point.

Example 2.1. Let $X = [0,1]$. Define $d : X \times X \to C$ by

$$d(x,y) = |x-y|e^{\frac{i\pi}{6}}.$$

Consider the mapping $S,T : X^3 \to X$ defined by

$$S(x,y,z) = \frac{x}{8}+\frac{y}{10}+\frac{z}{20}; \quad \forall x,y,z \in X$$

$$T(u,v,w) = \frac{u}{10}+\frac{v}{50}+\frac{w}{40}; \quad \forall u,v,w \in X.$$

By taking $\alpha = 0.9$ and $\beta = 0.08$, it can be easily verified that inequality (1) holds for all $x,y,z,u,v,w \in X = [0,1]$.

Hence $(0,0,0)$ is common tripled fixed point of S and T.

Example 2.2. Let $X = [0,1]$. Define $d : X \times X \to C$ by

$$d(x,y) = |x-y|e^{\frac{i\pi}{6}}.$$

Consider the mapping $F : X^3 \to X$ defined by

$$F(x,y,z) = \frac{xyz}{6} \quad \forall x,y,z \in X.$$

Fix $\alpha = 0.3, \beta = 0.1, \gamma = 0.2$ and $\delta = 0.05$, one can easily see that inequality (6) holds for all $x,y,z,u,v,w \in X = [0,1]$. Thus $(0,0,0)$ is the unique tripled fixed point of F.

Conflict of Interests

The authors declare that there is no conflict of interests.

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