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FIXED POINT THEOREM OF A PAIR OF MULTIVALUED MAPPINGS SATISFYING SPECIAL TYPE OF CONTRACTIVE CONDITION

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Abstract: We establish a common fixed point theorem of mutivalued mapping in partial metric space.

Keywords: partial Hausdorff metric; common fixed point; set-valued mappings; partial metric space.

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1. Introduction and preliminaries

In the last thirty years, the theory of multivalued functions has advanced in a variety of ways. In 1969, the systematic study of Banach-type fixed theorems of multivalued mappings started with the work of Nadler [1], who proved that a multivalued contractive mapping of a complete metric space X into the family of closed bounded subsets of X has a fixed point. His findings were followed by Agarwal *et al.* [2], Azam *et al.* [3] and many others (see, *e.g.*, [4-9]).

In 1994, Matthews [10], introduced the concept of a partial metric space and obtained a Banach-type fixed point theorem on complete partial metric spaces. Later on, several authors (see, e.g.. [11-17]) proved fixed point theorems of single-valued mappings in partial metric spaces. Recently Aydi *et al.* [18] proved a fixed point result for multivalued mappings in partial metric spaces. Haghi *et al.* [19] established that some metric fixed point generalizations to partial metric spaces can be obtained from the corresponding results in metric spaces. In this paper we

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obtain common fixed points of contractive-type multivalued mappings on partial metric spaces which cannot be deduced from the corresponding results in metric spaces. An example is also established to show that our result is a real generalization of analogous results for metric spaces [1, 9, 10, 18, 20].

We start with recalling some basic definitions and lemmas on a partial metric space.

Definition 1 A partial metric on a nonempty set X is a function $p : X \times X \rightarrow [0, \infty)$ such that for all $x, y, z \in X$:

(P_1) $p(x,x) = p(y,y) = p(x.y)$ if and only if $x = y$.

(P_2) $p(x,x) \leq p(x,y)$,

(P_3) $p(x,y) = p(y,x)$.

(P_4) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

The pair (X,p) is then called a partial metric space. Also, each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls $\{B_p(x, r): x \in X, r > 0)$, where $B_p(x, r) = \{y \in X : p(x,y) < p(x, x) + r\}$. If (X,p) is a partial metric space, then the function $p^s : X \times X \rightarrow \mathrm{R}^+$ given by $p^s(x,y) = 2p(x,y) - p(x, x) - p(y,y)$, $x,y \in X$, is a metric on X. A basic example of a partial metric space is the pair (R^+,p), where $p(x,y) = max\{x,y\}$ for all $x,y \in \mathrm{R}^+$.

Lemma 2 [10] *Let (X,p) be a partial metric space, then we have the following.*

1. *A sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in \mathrm{X}$ if and only if $\lim_{n\rightarrow\infty}(x,x_n) = p(x,x)$.*

2. *A sequence $\{x_n\}$ in a partial metric space (X,p) is called a Cauchy sequence if the $\lim_{n,m\rightarrow\infty}(x_n,x_m)$ exists and is finite.*

3. *A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$, that is, $p(x, x) = \lim_{n,m\rightarrow\infty} p(x_n,x_m)$.*

4. *A partial metric space (X,p) is complete if and only if the metric space (X,p^s) is complete. Furthermore, $\lim_{n\rightarrow\infty} p^s(x_n,z) = 0$ if and only if*

$$p(z,z) = \lim_{n\rightarrow\infty} p(x_n\text{-}z) = \lim_{n,m\rightarrow\infty} p(x_n,x_m).$$

A subset A of X is called closed in (X,p) if it is closed with respect to τ_p. A is called bounded in (X,p) if there is $x_0 \in X$ and $M > 0$ such that $a \in B_p(x_0,M)$ for all $a \in A$, *i.e.*, $p(x_0,a) < p(x_0,x_0) + M$ for all $a \in A$.

Let $CB^P(X)$ be the collection of all nonempty, closed and bounded subsets of X with respect to the partial metric p. For $A \in CB^P(X)$, we define

$$p(x,A) = \inf_{y\in A} p(x,y).$$

For $A,B \in CB^p(X)$,

$$\delta_p(A,B) = \sup_{a\in A}(a,B),$$

$$\delta_p(B,A) = \sup_{b\in B}(b,A),$$

$$H_p(A,B) = \max\{\delta_p(A,B),\delta_p(B,A)\}.$$

Note that [18] $p(x,A) = 0 \Rightarrow p^s(x, A) = 0$, where $p^s(x, A) = \inf_{y\in A} p^s(x,y)$.

Proposition 3 [18] *Let* (X,p) *be a partial metric space. For any* $A,B,C\in CB^p(X)$, *we have*

(i): $\delta_p(A,A) = \sup\{p(a,a): a\in A\}$;

(ii): $\delta_p(A,A) \leq \delta_p(A, B)$;

(iii): $\delta_p(A, B) = 0$ *implies that* $A \subseteq B$;

(iv): $\delta_p(A, B) \leq \delta_p(A,C) + \delta_p(C,B) - \inf_{c\in C} p(c, c)$.

Proposition 4 [18] *Let* (X,p) *be a partial metric space. For any* $A,B,C\in CB^p(X)$, *we have*

(h_1): $H_p(A,A) \leq H_p(A,B)$;

(h_2): $H_p(A,B) = H_p(B,A)$;

(h_3): $H_p(A,B) \leq H_p(A,C) + H_p(C,B) - \inf_{c\in C} p(c, c)$..

It is immediate [18] to check that $H_p(A,B) = 0 \Rightarrow A = B$. But the converse does not hold always.

Remark 5 [18] Let (X,p) be a partial metric space and A be a nonempty set in (X,p), then $a \in \overline{A}$ if and only if

$$p(a,A) = p(a,a),$$

where $\overline{A}$ denotes the closure of A with respect to the partial metric p. Note that A is closed in (X,p) if and only if $\overline{A} = A$.

Lemma 6 [21] *Let A and B be nonempty, closed and bounded subsets of a partial metric space* (X,p) *and* $0 < h \in \mathbb{R}$. *Then, for every* $a \in A$, *there exists* $b \in B$ *such that* $p(a, b) \leq H_p(A,B) + h$.

Definition 7 [22] A function $\varphi : [0, +\infty) \to [0,1)$ is said to be an *MT*-function if it satisfies Mizoguchi and Takahashi's condition (*i.e.*, $\limsup_{n \to t^+} \varphi(r) < 1$ for all $t \in [0, +\infty)$). Clearly, if $\varphi : [0,+\infty) \to [0,1)$ is a nondecreasing function or a nonincreasing function, then it is an *MT*-function. So, the set of *MT*-functions is a rich class.

Proposition 8 [22] *Let* $\varphi : [0,+\infty) \to [0,1)$ *be a function. Then the following statements are equivalent.*

1. φ *is an MT-function.*
2. *For each* $t \in [0,\infty)$, *there exist* $r_t^{(1)} \in [0,1)$ *and* $\varepsilon_t^{(1)} > 0$ *such that* $\varphi(s) \leq r_t^{(1)}$ *for all* $s \in (t, t + \varepsilon_t^{(1)})$.
3. *For each* $t \in [0,\infty)$, *there exist* $r_t^{(2)} \in [0,1)$ *and* $\varepsilon_t^{(2)} > 0$ *such that* $\varphi(s) \leq r_t^{(2)}$ *for all* $s \in (t, t + \varepsilon_t^{(2)})$.
4. *For each* $t \in [0,\infty)$, *there exist* $r_t^{(3)} \in [0,1)$ *and* $\varepsilon_t^{(3)} > 0$ *such that* $\varphi(s) \leq r_t^{(3)}$ *for all* $s \in (t, t + \varepsilon_t^{(3)})$.
5. *For each* $t \in [0,\infty)$, *there exist* $r_t^{(4)} \in [0,1)$ *and* $\varepsilon_t^{(4)} > 0$ *such that* $\varphi(s) \leq r_t^{(4)}$ *for all* $s \in (t, t + \varepsilon_t^{(4)})$.
6. *For any nonincreasing sequence* $\{x_n\}_{n \in \mathbb{N}}$ *in* $[0,\infty)$, *we have* $0 \leq \sup_{n \in \mathbb{N}} \varphi(x_n) < 1$.

7. φ *is a function of contractive factor* [23], *that is, for any strictly decreasing sequence* $\{x_n\}_{n\in N}$ *in* $[0,\infty)$, *we have* $0 \leq \sup_{n\in N} \varphi(x_n) < 1$.

2. Main results

Mizoguchi and Takahashi proved the following theorem in [20].

Theorem 9 *Let* (X,d) *be a complete metric space,* $S : X \to CB(X)$ *be a multivalued map and* $\varphi : [0, +\infty) \to [0,1)$ *be an MT-function. Assume that*

$$H(Sx,Sy) \leq \varphi(d(x,y))\, d(x,y) \tag{2.1}$$

for all $x,y \in X$, *then* S *has a fixed point in* X.

In the following we show that in partial metric spaces Mizoguchi and Takahashi's contractive condition (2.1) is useful to achieve common fixed points of two distinct mappings. Whereas this condition is not feasible to obtain a common fixed point of two distinct mappings on a metric space.

Theorem 10 *Let* (X,p) *be a complete partial metric space,* $S, T: X \to CB^P(X)$ *be multivalued mappings and* $\varphi : [0, +\infty) \to [0,1)$ *be an MT-function. Assume that*

$$H_p(Sx, Ty) \leq \varphi(p(x,y))\, p(x,y) \tag{2.2}$$

for all $x,y \in X$, *then there exists* $z \in X$ *such that* $z \in Sz$ *and* $z \in Tz$.

Proof Let $x_0 \in X$ and $x_1 \in Sx_0$. *If* $p(x_0,x_1) = 0$, then $x_0 = x_1$ and

$$H_p(Sx_0, Tx_1) \leq \varphi(p(x_0,x_1))\, p(x_0,x_1) = 0.$$

Thus $Sx_0=Tx_1$, which implies that

$$x_1 = x_0 \in Sx_0 = Tx_1 = Tx_0$$

and we finished. Assume that $p(x_0, x_1) > 0$. By Lemma 6, we can take $x_2 \in Tx_1$ such that

$$p(x_1,x_2) \leq \frac{H_p(Sx_0,Tx_1) + p(x_0,x_1)}{2}. \tag{2.3}$$

If $p(x_1, x_2) = 0$, then $x_1 = x_2$ and

$$H_p(Tx_1, Sx_2) \leq \varphi\,(p(x_1,x_2))\; p(x_1,x_2) = 0.$$

Then, $Tx_1 = Sx_2$. That is,

$$x_2 = x_1 \in Tx_1 = Sx_2 = Sx_2$$

and we finished. Assume that $p(x_1,x_2) > 0$. Now we choose $x_3 \in Sx_2$ such that

$$p(x_2,x_3) \leq \frac{H_p(Sx_1, Tx_2) + p(x_1, x_2)}{2}. \tag{2.4}$$

By repeating this process, we can construct a sequence x_n of points in X and a sequence A_n of elements in $CB^P(X)$ such that

$$x_{j+1} \in A_j = \begin{cases} Sx_j, & j = 2k, k \geq 0, \\ Tx_j, & j = 2k+1, k \geq 0, \end{cases} \tag{2.5}$$

and

$$p(x_j,x_{j+1}) \leq \frac{H_p(A_{j-1}, A_j) + p(x_{j-1}, x_j)}{2} \text{ with } j \geq 0, \tag{2.6}$$

along with the assumption that $p(x_j,x_{j+1}) > 0$ for *each* $j \geq 0$. Now, for $j = 2k + 1$, we have

$$\begin{aligned} p(x_j,x_{j+1}) &\leq \frac{H_p(A_{j-1}, A_j) + p(x_{j-1}, x_j)}{2} \\ &\leq \frac{H_p(Sx_{2k}, Tx_{2k+1}) + p(x_{2k}, x_{2k+1})}{2} \\ &\leq \frac{\psi(p(x_{2k}, x_{2k+1}))p(x_{2k}, x_{2k+1}) + p(x_{2k}, x_{2k+1})}{2} \\ &\leq \left(\frac{\varphi(p(x_{j-1}, x_j)) + 1}{2}\right) p(x_{j-1}, x_j) \\ &\leq p(x_{j-1}, x_j). \end{aligned}$$

Similarly, for $j = 2k + 2$, we obtain

$$p(x_j,x_{j+1}) \leq \frac{H_p(A_{2k+1}, Sx_{2k+2}) + p(x_{j-1}, x_j)}{2}$$

$$\leq \left(\frac{\varphi(p(x_{j-1}, x_j))+1}{2}\right)p(x_{j-1}, x_j)$$

$$\leq p(x_{j-1}, x_j).$$

It follows that the sequence $\{p(x_n, x_{n+1})\}$ is decreasing and converges to a nonnegative real number $t \geq 0$. Define a function $\psi : [0, \infty) \to [0, 1)$ as follows:

$$\psi(\xi) = \frac{\psi(\xi)+1}{2},$$

Then

$$\limsup_{\xi \to t^+} \psi(\xi) < 1.$$

Using Proposition 8, for $t \geq 0$, we can find $\delta(t) > 0$, $\lambda_t < 1$, such that $t \leq r \leq \delta(t) + t$ implies $\psi(r) < \lambda_t$ and there exists a natural number N such that $t \leq p(x_n, x_{n+1}) \leq \delta(t) + t$, whenever $n > N$. Hence

$$\psi(p(x_n, x_{n+1})) < \lambda_t, \quad \text{whenever } n > N.$$

Then, for $n = 1, 2, 3, \ldots\ldots,$

$$p(x_n,x_{n+1}) \leq \left(\frac{\varphi(p(x_{n-1}, x_n))+1}{2}\right)p(x_{n-1}, x_n) \leq \psi(p(x_{n-1}, x_n))\, p(x_{n-1}, x_n)$$

$$\leq \max\left\{\max_{n=1}^{N} \psi(p(x_{n-1}, x_n)), \lambda_t\right\} p(x_{n-1}, x_n)$$

$$\leq \left[\max\left\{\max_{n=1}^{N} \psi(p(x_{n-1}, x_n)), \lambda_t\right\}\right]^2 p(x_{n-2}, x_{n-1})$$

$$\leq \left[\max\left\{\max_{n=1}^{N} \psi(p(x_{n-1}, x_n)), \lambda_t\right\}\right]^n p(x_0, x_1).$$

Put $\max\left\{\max_{n=1}^{N} \psi(p(x_{n-1}, x_n)), \lambda_t\right\} = \Omega$, then $\Omega < 1$,

$$p(x_n,x_{n+1}) \leq \Omega^n\, p(x_0,x_1)$$

and $$p(x_n,x_{n+m}) \leq \sum_{i=1}^{m} p(x_{n+i-1}, x_{n+i}) - \sum_{i=1}^{m} p(x_{n+i}, x_{n+i})$$

$$\leq p(x_n,x_{n+1}) + p(x_{n+1},x_{n+2}) + \ldots\ldots + p(x_{n+m-1}, x_{n+m})$$

$$\leq (\Omega^n + \Omega^{n+1} + \ldots\ldots + \Omega^{n+m-1})\, p(x_0,x_1)$$

$$\leq \left(\frac{\Omega^n}{1-\Omega}\right) p(x_0,x_1) \to 0 \text{ as } n \to \infty \text{ (since } 0 < \Omega < 1).$$

By the definition of p^s, we get, for any $m \in N$,

$$p^s(x_n,x_{n+m}) \leq 2p(x_n,x_{n+m}) \to 0 \quad \text{as } n \to +\infty.$$

Which implies that (x_n) is a Cauchy sequence in (X, p^s). Since (X, p) is complete, so the corresponding metric space (X, p^s) is also complete. Therefore, the sequence (x_n) converges to some $z \in X$ with respect to the metric p^s, that is, $\lim_{n\to\infty} p^s(X_n, z) = 0$. Since,

$$p(x_n,x_n) \leq p(x_n,x_{n+1}) \leq \Omega^n\, p(x_0,x_1) \to 0 \quad \text{as } n \to +\infty.$$

Therefore

$$p(Sz, z) \leq p(Sz, x_{2n+2}) + p(x_{2n+2}, z) - p(x_{2n+2}, x_{2n+2})$$

$$\leq p(x_{2n+2}, Sz) + p(x_{2n+2}, z)$$

$$\leq \sup_{u \in Tx_{2n+1}} p(u, Sz) + p(x_{2n+2}, z)$$

$$\leq \delta_p\, (Tx_{2n+1}, Sz) + p(x_{2n+2}, z)$$

$$\leq H_p\, (Tx_{2n+1}, Sz) + p(x_{2n+2}, z)$$

$$\leq \varphi\, (p(x_{2n+1}, z))\, p(x_{2n+1}, z) + p(x_{2n+2}, z)$$

$$\leq p(x_{2n+1}, z) + p(x_{2n+2}, z) \tag{2.8}$$

Taking limit as $n \to \infty$, we get

$$p(Sz, z) = 0. \tag{2.9}$$

Thus from (2.8) and (2.9), we get

$$p(z, z) = p(Sz, z).$$

Thus by Remark 5, we get that $z \in Sz$. It follows similarly that $z \in Tz$. This completes the proof of the theorem.

Conflict of Interests

The authors declare that there is no conflict of interests.

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