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KAKUTANI'S FIXED POINT THEOREM FOR MULTI-FUNCTIONS WITH SEQUENTIALLY AT MOST ONE FIXED POINT AND THE MINIMAX THEOREM FOR TWO-PERSON ZERO-SUM GAMES: A CONSTRUCTIVE ANALYSIS

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Abstract. In this paper we constructively prove Kakutani's fixed point theorem for multi-functions with sequentially at most one fixed point and uniformly closed graph, and apply this result to prove the minimax theorem for two-person zero-sum games with finite strategies. We follow the Bishop style constructive mathematics.

Keywords: Kakutani's fixed point theorem; constructive mathematics; sequentially at most one fixed point; minimax theorem.

2000 AMS Subject Classification: 26E40; 91A10

1. Introduction

It is well known that Brouwer's fixed point theorem can not be constructively proved[1].

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[1] [6] provided a *constructive* proof of Brouwer's fixed point theorem. But it is not constructive from the view point of constructive mathematics à la Bishop. It is sufficient to say that one dimensional case of Brouwer's fixed point theorem, that is, the intermediate value theorem is non-constructive (See [4] or [8]).

Thus, Kakutani's fixed point theorem for multi-functions (multi-valued functions or correspondences) also can not be constructively proved. On the other hand, Sperner's lemma which is used to prove Brouwer's theorem, however, can be constructively proved. Some authors have presented a constructive (or an approximate) version of Brouwer's theorem using Sperner's lemma (See [8] and [9]). Also Dalen in [8] states a conjecture that a uniformly continuous function f from a simplex to itself, with property that each open set contains a point x such that x is not equal to $f(x)$ ($x \neq f(x)$) and on the boundaries of the simplex $x \neq f(x)$, has an exact fixed point. Recently Berger and Ishihara[2] showed that the following theorem is equivalent to Brouwer's fan theorem.

> Each uniformly continuous function from a compact metric space into itself with at most one fixed point has a fixed point.

By reference to the notion of *sequentially at most one maximum* in Berger, Bridges and Schuster[1] we require a stronger condition that a function has *sequentially at most one fixed point*, and in [7] we have shown the following result.

> Each uniformly continuous function from a compact metric space into itself with *sequentially at most one fixed point* has a fixed point,

without the fan theorem. It is a partial answer to Dalen's conjecture. The property of sequentially at most one fixed point is stronger than the condition that a function has *at most one fixed point* in [2].

In this paper we extend the property of sequentially at most one fixed point to multi-functions, and will prove Kakutani's fixed point theorem for compact and convex valued multi-functions with sequentially at most one fixed point and uniformly closed graph in an n-dimensional simplex. The uniformly closed graph property of multi-functions is a stronger version of the closed graph property. And we apply this result to prove the minimax theorem for two-person zero-sum games with finite strategies.

In the next section we prove Kakutani's fixed point theorem for compact and convex valued multi-functions with sequentially at most one fixed point and uniformly closed graph. In Section 4 we prove the minimax theorem for zero-sum games with finite strategies. We follow the Bishop style constructive mathematics according to [3], [4] and [5].

2. Kakutani's fixed point theorem for multi-functions with sequentially at most one fixed point and uniformly closed graph

In constructive mathematics a nonempty set is called an *inhabited* set. A set S is inhabited if there exists an element of S.

> Note that in order to show that S is inhabited, we cannot just prove that it is impossible for S to be empty: we must actually construct an element of S (see page 12 of [5]).

Also in constructive mathematics compactness of a set means *total boundedness with completeness*. First define finite enumerability of a set and an ε-approximation to a set. A set S is *finitely enumerable* if there exist a natural number N and a mapping of the set $\{1, 2, \dots, N\}$ onto S. An ε-approximation to S is a subset of S such that for each $x \in S$ there exists y in that ε-approximation with $|x - y| < \varepsilon$($|x - y|$ is the distance between x and y). S is totally bounded if for each $\varepsilon > 0$ there exists a finitely enumerable ε-approximation to S. Completeness of a set, of course, means that every Cauchy sequence in the set converges.

Let x be a point in a compact metric space X, and f be a uniformly continuous function f from X into itself. According to [8] and [9] f has an approximate fixed point. It means

$$\text{For each } \varepsilon > 0 \text{ there exists } x \in X \text{ such that } |x - f(x)| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\inf_{x \in X} |x - f(x)| = 0.$$

The notion that f has at most one fixed point in [2] is defined as follows:

Definition 2.1. (at most one fixed point) *For all $x, y \in X$, if $x \neq y$, then $f(x) \neq x$ or $f(y) \neq y$.*

Next by reference to the notion of *sequentially at most one maximum* in [1], we define the notion that f has *sequentially at most one fixed point* as follows;

Definition 2.2. (sequentially at most one fixed point) *All sequences* $(x_n)_{n\geq 1}$, $(y_n)_{n\geq 1}$ *in* X *such that* $|f(x_n) - x_n| \longrightarrow 0$ *and* $|f(y_n) - y_n| \longrightarrow 0$ *are eventually close in the sense that* $|x_n - y_n| \longrightarrow 0$.

We extend this definition to a case of multi-functions. Consider an n-dimensional simplex Δ as a compact metric space. Let F be a compact and convex valued multi-function from Δ to the set of its inhabited subsets. Denote the distance between $F(x)$ and $x \in \Delta$ by $|F(x) - x|$, that is,

$$|F(x) - x| = \inf_{y \in F(x)} |y - x|.$$

It can be defined since $F(x)$ is a compact subset of a compact metric space, and so it is located (see [5]). An inhabited subset S of a metric space X is called located if for each $x \in X$ the distance

$$|x - S| = \inf_{t \in S} |x - t|$$

exists.

The definition of the property that a multi-function has sequentially at most one fixed point is as follows;

Definition 2.3. (sequentially at most one fixed point for multi-function) *All sequences* $(x_n)_{n\geq 1}$, $(y_n)_{n\geq 1}$ *in* Δ *such that* $|F(x_n) - x_n| \longrightarrow 0$ *and* $|F(y_n) - y_n| \longrightarrow 0$ *are eventually close in the sense that* $|x_n - y_n| \longrightarrow 0$.

A graph of a multi-function F from Δ to the set of its inhabited subsets is

$$G(F) = \cup_{x \in \Delta} \{x\} \times F(x).$$

If $G(F)$ is a closed set, we say that F has a closed graph. It implies the following fact.

> If sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ are such that for each n $y_n \in F(x_n)$, and if $x_n \longrightarrow x$, then $y_n \longrightarrow y$ for some $y \in F(x)$.

According to [5] this means

For each $\varepsilon > 0$ if there exists n_0 such that $|x_n - x| < \varepsilon$ when $n \geq n_0$, then there exists n'_0 such that $|y_n - F(x)| < \varepsilon$, that is, $|y_n - y| < \varepsilon$ for some $y \in F(x)$ when $n \geq n'_0$.

n_0 and n'_0 depend on x and ε. Further we require a uniform version of this property for multi-functions, and call such a multi-function *a multi-function with uniformly closed graph*, or say that a multi-function has a uniformly closed graph. It means that n_0 and n'_0 depend on only ε not on x. Now we show the following lemma, which is based on Lemma 2 of [1].

Lemma 2.1. *Let F be a compact and convex valued multi-function with sequentially at most one fixed point and uniformly closed graph from Δ to the set of its inhabited subsets. Assume* $\inf_{x\in\Delta} |F(x) - x| = 0$. *If the following property holds,*

For each $\delta > 0$ there exists $\varepsilon > 0$ such that if $x, y \in \Delta$, $|F(x) - x| < \varepsilon$ and $|F(y) - y| < \varepsilon$, then $|x - y| \leq \delta$,

then, there exists a point $z \in \Delta$ such that $z \in F(z)$, that is , F has a fixed point.

Proof.

Choose a sequence $(x_n)_{n\geq 1}$ in Δ such that $|F(x_n) - x_n| \longrightarrow 0$. Compute N such that $|F(x_n) - x_n| < \varepsilon$ for all $n \geq N$. Then, for $m, n \geq N$ we have $|x_m - x_n| \leq \delta$. Since $\delta > 0$ is arbitrary, $(x_n)_{n\geq 1}$ is a Cauchy sequence in Δ, and converges to a limit $z \in \Delta$. The uniformly closed graph property of F yields $z \in F(z)$.

This completes the proof.

A fixed point of a multi-function is defined as follows;

Definition 2.1. *x is a fixed point of a multi-function F if $x \in F(x)$.*

We define an approximate fixed point of a multi-function F as follows;

Definition 2.1. *For each $\varepsilon > 0$ x is an approximate fixed point of a multi-function F if $|x - F(x)| < \varepsilon$.*

We constructively show that if the value of a multi-function F from Δ to the set of inhabited subsets of Δ with sequentially at most one fixed point and uniformly closed graph is compact and convex, it has a fixed point. If a set X is homeomorphic to Δ (so

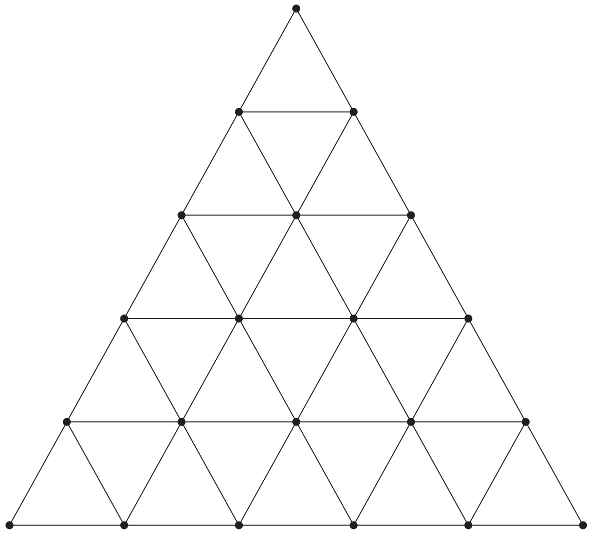

FIGURE 1. Subdivision of 2-dimensional simplex

X is also compact), we can show the same result for a multi-function from X to the set of inhabited subsets of X.

Our Kakutani's fixed point theorem is as follows;

Theorem 2.1. *If F is a compact and convex valued multi-function with sequentially at most one fixed point and uniformly closed graph from an n-dimensional simplex Δ to the set of its inhabited subsets, then it has a fixed point.*

Proof.

(1) Let Δ be an n-dimensional simplex, and consider m-th subdivision of Δ. Subdivision in a case of 2-dimensional simplex is illustrated in Figure 1. In a 2-dimensional case we divide each side of Δ in m equal segments, and draw the lines parallel to the sides of Δ. Then, the 2-dimensional simplex is partitioned into m^2 triangles. We consider subdivision of Δ inductively for cases of higher dimension. In a 3 dimensional case each face of Δ is a 2-dimensional simplex, and so it is partitioned into m^2 triangles in the way above mentioned, and draw the planes parallel to the faces of Δ. Then, the 3-dimensional simplex is partitioned into m^3 trigonal pyramids. And similarly for cases of higher dimension.

Consider sufficiently fine partition of Δ, and define a uniformly continuous function $f^m: \Delta \longrightarrow \Delta$ as follows. If x is a vertex of a simplex constructed by m-th subdivision of Δ, $f^m(x) = y$ for some $y \in F(x)$. For other $x \in \Delta$ we define f^m by a convex combination of the values of F at vertices of a simplex $x_0^m, x_1^m, \dots, x_n^m$. Let $\sum_{i=0}^n \lambda_i = 1$, $\lambda_i \geqq 0$,

$$f^m(x) = \sum_{i=0}^n \lambda_i f^m(x_i^m) \text{ with } x = \sum_{i=0}^n \lambda_i x_i^m.$$

Since f^m is clearly uniformly continuous, it has an approximate fixed point according to [8] and [9]. Let x^* be an approximate fixed point of f^m, then for each $\frac{\varepsilon}{2} > 0$ there exists $x^* \in \Delta$ which satisfies

$$|x^* - f^m(x^*)| < \frac{\varepsilon}{2}.$$

If the partition of Δ is sufficiently fine, the distance between vertices of a simplex, $|x_i^m - x_j^m|$, $i \neq j$, is sufficiently small. Since F has a uniformly closed graph, for each $y_i \in F(x_i^m)$ and some $y_j \in F(x_j^m)$ we have $|y_i - y_j| < \frac{\varepsilon}{2}$, and for each $y_j \in F(x_j^m)$ and some $y_i \in F(x_i^m)$ we have $|y_i - y_j| < \frac{\varepsilon}{2}$. Since x^* is expressed as $x^* = \sum_{i=0}^n \lambda_i x_i^m$, if $|x_i^m - x_j^m|$ is sufficiently small for each i and j, $|x^* - x_i^m|$ is also sufficiently small for each i. Therefore, for each $y_i \in F(x_i^m)$ and some $y_i^* \in F(x^*)$ we have $|y_i - y_i^*| < \frac{\varepsilon}{2}$. y_i^*'s for different x_i^m's may be different. But, since $F(x^*)$ is convex, we have

$$y^* = \sum_{i=0}^n \lambda_i y_i^* \in F(x^*).$$

Since, for each i $|y_i - y_i^*| < \frac{\varepsilon}{2}$ and $f^m(x^*) = \sum_{i=0}^n \lambda_i f^m(x_i^m) = \sum_{i=0}^n \lambda_i y_i$,we have

$$|f^m(x^*) - y^*| < \frac{\varepsilon}{2}.$$

Since $|x^* - f^m(x^*)| < \frac{\varepsilon}{2}$, we obtain

$$|x^* - y^*| < \varepsilon.$$

This means

$$|x^* - F(x^*)| < \varepsilon.$$

Since ε is arbitrary,

$$\inf_{x^* \in \Delta} |x^* - F(x^*)| = 0.$$

(2) Choose a sequence $(z_n)_{n\geq 1}$ in Δ such that $|z_n - F(z_n)| \longrightarrow 0$. In view of Lemma 2.1 it is enough to prove that the following condition holds.

For each $\delta > 0$ there exists $\varepsilon > 0$ such that if $x, y \in \Delta$, $|F(x) - x| < \varepsilon$ and $|F(y) - y| < \varepsilon$, then $|x - y| \leq \delta$.

Assume that the set

$$K = \{(x, y) \in \Delta \times \Delta : \ |x - y| \geq \delta\}$$

is inhabited and compact[2]. Since the mapping $(x, y) \longrightarrow \max(|F(x) - x|, |F(y) - y|)$ is uniformly continuous, we can construct an increasing binary sequence $(\lambda_n)_{n\geq 1}$ such that

$$\lambda_n = 0 \Rightarrow \inf_{(x,y)\in K} \max(|F(x) - x|, |F(y) - y|) < 2^{-n},$$

$$\lambda_n = 1 \Rightarrow \inf_{(x,y)\in K} \max(|F(x) - x|, |F(y) - y|) > 2^{-n-1}.$$

It suffices to find n such that $\lambda_n = 1$. In that case, if $|F(x) - x| < 2^{-n-1}$, $|F(y) - y| < 2^{-n-1}$, we have $(x, y) \notin K$ and $|x - y| \leq \delta$. Assume $\lambda_1 = 0$. If $\lambda_n = 0$, choose $(x_n, y_n) \in K$ such that $\max(|F(x_n) - x_n|, |F(y_n) - y_n|) < 2^{-n}$, and if $\lambda_n = 1$, set $x_n = y_n = z_n$. Then, $|F(x_n) - x_n| \longrightarrow 0$ and $|F(y_n) - y_n| \longrightarrow 0$, so $|x_n - y_n| \longrightarrow 0$. Computing N such that $|x_N - y_N| < \delta$, we must have $\lambda_N = 1$.

This completes the proof.

3. Minimax Theorem

In this section we derive the minimax theorem of zero-sum games by our Kakutani's fixed point theorem in the previous section. The minimax theorem can also be proved by Brouwer's fixed point theorem[3]. But the proof by Kakutani's fixed point theorem is more smart. consider a two person zero-sum game. There are two players A and B.

[2]See Theorem 2.2.13 of [5].

[3]See [7].

Player A has m alternative pure strategies, and the set of his pure strategies is denoted by $S_A = \{a_1, a_2, \dots, a_m\}$. Player B has n alternative pure strategies, and the set of his pure strategies is denoted by $S_B = \{b_1, b_2, \dots, b_n\}$. m and n are finite natural numbers. The payoff of player A when a combination of players' strategies is (a_i, b_j) is denoted by $M(a_i, b_j)$. Since we consider a zero-sum game, the payoff of player B is equal to $-M(a_i, b_j)$. Let p_i be a probability that A chooses his strategy a_i, and q_j be a probability that B chooses his strategy b_j. A mixed strategy of A is represented by a probability distribution over S_A, and is denoted by $x = (p_1, p_2, \dots, p_m)$ with $\sum_{i=1}^m p_i = 1$. Similarly, a mixed strategy of B is denoted by $y = (q_1, q_2, \dots, q_n)$ with $\sum_{j=1}^n q_j = 1$. A combination of mixed strategies (x, y) is called a *profile*. The expected payoff of player A at a profile (x, y) is written as follows,

$$M(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i M(a_i, b_j) q_j.$$

We assume that $M(a_i, b_j)$ is finite. Then, since $M(x, y)$ is linear with respect to probability distributions over the sets of pure strategies of players, it is a uniformly continuous function. The expected payoff of A when he chooses a pure strategy a_i and B chooses a mixed strategy y is $M(a_i, y) = \sum_{j=1}^n M(a_i, b_j) q_j$, and his expected payoff when he chooses a mixed strategy x and B chooses a pure strategy b_j is $M(x, b_j) = \sum_{i=1}^m p_i M(a_i, b_j)$. The set of all mixed strategies of A is denoted by P, and that of B is denoted by Q. P is an $m-1$-dimensional simplex, and Q is an $n-1$-dimensional simplex.

We call $v_A(x) = \inf_{y \in Q} M(x, y)$ the *guaranteed payoff* of A at x. And we define v_A^* as follows,

$$v_A^* = \sup_{x \in P} \inf_{y \in Q} M(x.y)$$

This is a constructive version of the maximin payoff. Similarly, we call $v_B(y) = \sup_{x \in P} M(x, y)$ the guaranteed payoff of player B at y, and define v_B^* as follows,

$$v_B^* = \inf_{y \in Q} \sup_{x \in P} M(x, y).$$

This is a constructive version of the minimax payoff. For a fixed x we have $\inf_{y\in Q} M(x,y) \leq M(x,y)$ for all y, and so

$$\sup_{x\in P}\inf_{y\in Q} M(x,y) \leq \sup_{x\in P} M(x,y) \text{ for all } y$$

holds. Then, we obtain $\sup_{x\in P}\inf_{y\in Q} M(x,y) \leq \inf_{y\in Q}\sup_{x\in P} M(x,y)$. This is rewritten as

$$v_A^* \leq v_B^*. \tag{1}$$

Now, consider the following set for player A given y;

$$\{a_i \in S_A | M(a_i, y) \geq M(a_i', y) \text{ for all } a_i' \in S_A\}.$$

Since S_A is finite, we can find a_i which realizes $\max_{a_i\in S_A} M(a_i, y)$. Linearity of the expected payoff function implies that if there are multiple pure strategies which satisfy this condition, convex combinations of those pure strategies also satisfy it. Denote the set of such mixed strategies by

$$\Gamma_A(y) = \{x \in P | M(x,y) \geq M(x',y) \text{ for all } x' \in P\},$$

Similarly for player B consider the following set given x;

$$\{b_j \in S_B | M(x, b_j) \leq M(x, b_j') \text{ for all } b_j' \in S_B\}.$$

If there are multiple pure strategies which satisfy this condition, convex combinations of those pure strategies also satisfy it. Denote the set of such mixed strategies by

$$\Gamma_B(x) = \{y \in Q | M(x,y) \leq M(x,y') \text{ for all } y' \in Q\}.$$

Define a multi-function from $P \times Q$ to the set of inhabited subsets of $P \times Q$ by

$$\Theta(x,y) = (\Gamma_A(y), \Gamma_B(x)).$$

Since $P \times Q$ is the product of two simplices, it is convex. And since there are $m+n-2$ independent vectors in $P \times Q$, $P \times Q$ is homeomorphic to an $m+n-2$-dimensional simplex.

We assume the following conditions about payoff functions.

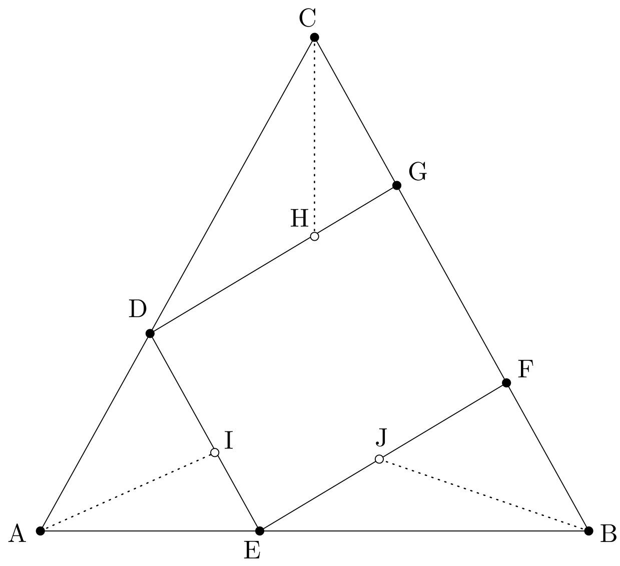

Figure 2. Homeomorphism between simplex and combination of strategies

Assumption 3.1. *All sequences* $((x_n, y_n))_{n\geq 1}$, $((x'_n, y'_n))_{n\geq 1}$ *in* $P\times Q$ *such that* $\max(M(a_i, y_n)-M(x_n, y_n), 0) \longrightarrow 0$, $\max(M(x_n, y_n)-M(x_n, b_j), 0) \longrightarrow 0$, $\max(M(a_i, y'_n)-M(x'_n, y'_n), 0) \longrightarrow 0$ *and* $\max(M(x'_n, y'_n)-M(x'_n, b_j), 0) \longrightarrow 0$ *for all* i *and* j *are eventually close in the sense that* $|(x_n, y_n) - (x'_n, y'_n)| \longrightarrow 0$.

We call this condition sequential non-constancy of payoff functions.

Let us consider a homeomorphism between an $m + n - 2$-dimensional simplex and the space of players' mixed strategies, which is denoted by $\mathbf{P}$. Figure 2 depicts an example of a case of two players with two pure strategies for each player. $\mathbf{P}$ is represented by a rectangle DEFG. Vertices D, E, F and G represent states where two players choose pure strategies, and points on edges DE, EF, FG and GD represent states where one player chooses a pure strategy. In this homeomorphism, vertices of the simplex do not correspond to any vertex of $\mathbf{P}$. Vertices of the simplex and points on faces (simplices whose dimension is lower than $m+n-2$) of the simplex correspond to the points on faces of $\mathbf{P}$. For example, in Figure 2 A, B and C correspond, respectively, to I, J and H. On the other hand, each vertex of $\mathbf{P}$, D, E, F and G corresponds, respectively, to itself on a face of the simplex which contains them.

Let us check that $\Theta(x, y)$ satisfies the conditions for our Kakutani's fixed point theorem.

(1) $P \times Q$ is clearly a compact and convex set.

(2) $\Theta(x, y)$ is a multi-function from $P \times Q$ to the set of inhabited subsets of $P \times Q$.

(3) We show convexity of $\Theta(x, y)$. It is sufficient to show convexity of $\Gamma_A(y)$. Suppose that $x^1 \in \Gamma_A(y)$ and $x^2 \in \Gamma_A(y)$. Then,

$$M(x^1, y) \geq M(a_i, y) \text{ for all } a_i \in S_A$$

and

$$M(x^2, y) \geq M(a_i, y) \text{ for all } a_i \in S_A$$

hold. Since $M(x, y)$ is linear with respect to probability distributions over the sets of pure strategies of players, for $0 \leq \lambda \leq 1$ we have

$$\lambda M(x^1, y) + (1 - \lambda)M(x^2, y) = M(\lambda x^1 + (1 - \lambda)x^2, y) \geq M(a_i, y) \text{ for all } a_i \in S_A.$$

Thus, we obtain $\lambda x^1 + (1-\lambda)x^2 \in \Gamma_A(y)$, and $\Gamma_A(y)$ is convex. Convexity of $\Gamma_B(x)$ is similarly proved.

(4) We show that $\Theta(x, y)$ has a uniformly closed graph. Let x'' be a mixed strategy of player A, y'' be a mixed strategy of player B and $x \in \Gamma_A(y)$. Uniform continuity of $M(x, y)$ implies that, for a positive number $\frac{\varepsilon}{2}$, we can select $\delta > 0$ and $\delta' > 0$ so that when $|(x'', y'') - (x, y)| < \delta$ and $|(x', y'') - (x', y)| < \delta'$, we have $|M(x'', y'') - M(x, y)| < \frac{\varepsilon}{2}$ and $|M(x', y'') - M(x', y)| < \frac{\varepsilon}{2}$. Since $M(x, y) \geq M(x', y)$ for all $x' \in P$, we have

$$M(x'', y'') > M(x, y) - \frac{\varepsilon}{2} \geq M(x', y) - \frac{\varepsilon}{2} > M(x', y'') - \varepsilon \text{ for all } x'.$$

Thus, $x'' \in V(\Gamma_A(y''), \varepsilon)$. About $\Gamma_B(x)$ we can show a similar result, that is, $y'' \in V(\Gamma_B(x''), \varepsilon)$. $V(\Gamma_A(y''), \varepsilon)$ and $V(\Gamma_A(y''), \varepsilon)$ are ε neighborhoods of $\Gamma_A(y'')$ and $\Gamma_B(x'')$. This completes the proof that $\Theta(x, y)$ has a uniformly closed graph.

(5) Consider sequences (x_n, y_n) and (x'_n, y'_n). If $|\Theta(x_n, y_n) - (x_n, y_n)| \longrightarrow 0$ and $|\Theta(x'_n, y'_n) - (x'_n, y'_n)| \longrightarrow 0$, then $\max(M(a_i, y_n) - M(x_n, y_n)) \longrightarrow 0$, $\max(M(x_n, b_j) - M(x_n, y_n)) \longrightarrow 0$, $\max(M(a_i, y_n) - M(x_n, y_n)) \longrightarrow 0$ and $\max(M(x'_n, b_j) - M(x'_n, y'_n)) \longrightarrow 0$ for all i and j. Assumption 3.1 implies $|(x_n, y_n) - (x'_n, y'_n)| \longrightarrow 0$. Thus, Θ has sequentially at most one fixed point.

Therefore, the conditions of our Kakutani's fixed point theorem are satisfied by $\Theta(x, y)$, and it has a fixed point. Let denote the fixed point by (x^*, y^*). Then,

$$M(x', y^*) \leq M(x^*, y^*) \leq M(x^*, y') \text{ for all } (x', y')$$

holds. This means

$$\sup_x M(x, y^*) \leq M(x^*, y^*) \leq \inf_y M(x^*, y). \tag{2}$$

Since

$$\sup_x M(x, y^*) \geq \inf_y \sup_x M(x, y) = v_B^*, \ \inf_y M(x^*, y) \leq \sup_x \inf_y M(x, y) = v_A^*,$$

(2) implies

$$v_B^* \leq M(x^*, y^*) \leq v_A^*. \tag{3}$$

With (1) and (3) we obtain

$$v_A^* = v_B^*.$$

This v_A^* or v_B^* is the *value of the game*. Summarizing the results,

Theorem 3.1. *The value of a two person zero-sum game with sequentially non-constant payoff functions is determined by $M(x^*, y^*)$. Since we can constructively find a fixed point of a multi-function which satisfies the conditions of our Kakutani's fixed point theorem, we can constructively get the value of the game.*

		Player 2	
		X	Y
Player	X	2, -2	-1, 1
1	Y	-1, 1	1, -1

Table 1. Example of game

Consider an example. See a game in Table 1. It is a modified version of the so-called Matching-Pennies Game. Pure strategies of Player 1 and 2 are X and Y. The left side number in each cell represents the payoff of Player 1 and the right side number represents the payoff of Player 2. Let p_X and $1 - p_X$ denote the probabilities that Player 1 chooses, respectively, X and Y, and q_X and $1-q_X$ denote the probabilities for Player 2. Denote the expected payoff of Player 1 by $M(p_X, q_X)$. Since we consider a zero-sum game, the expected payoff of Player 2 is $-M(p_X, q_X)$. We have

$$\begin{aligned} M(p_X, q_X) &= 2p_Xq_X - (1-p_X)q_X - p_X(1-q_X) + (1-p_X)(1-q_X) \\ &= p_X(5q_X-2) + 1 - 2q_X \end{aligned}$$

Denote the payoff of Player 1 when he chooses X by $M(X, q_X)$, and that when he chooses Y by $M(Y, q_X)$. Similarly for Player B. Then,

$$M(X, q_X) = 3q_X - 1,\ M(Y, q_X) = 1-2q_X,\ -M(p_X, X) = 1-3p_X,\ -M(p_X, Y) = 2p_X - 1,$$

$$M(X, q_X) - M(p_X, q_X) = (1-p_X)(5q_X-2),\ M(Y, q_X) - M(p_X, q_X) = -p_X(5q_X-2),$$

$$-M(p_X, X) + M(p_X, q_X) = (q_X-1)(5p_X-2),\ -M(p_X, Y) + M(p_X, q_X) = q_X(5p_X-2).$$

And we have

$$\text{When } q_X > \frac{2}{5},\ M(X, q_X) > M(Y, q_X) \text{ and } M(X, q_X) > M(p_X, q_X) \text{ for } p_X < 1,$$

$$\text{When } q_X < \frac{2}{5},\ M(Y, q_X) > M(X, q_X) \text{ and } M(Y, q_X) > M(p_X, q_X) \text{ for } p_X > 0,$$

$$\text{When } p_X > \frac{2}{5},\ -M(p_X, Y) > -M(p_X, X) \text{ and } -M(p_X, Y) > -M(p_X, q_X) \text{ for } q_X > 0,$$

$$\text{When } p_X < \frac{2}{5},\ -M(p_X, X) > -M(p_X, Y) \text{ and } -M(p_X, X) > -M(p_X, q_X) \text{ for } q_X < 1.$$

Consider sequences $(p_X(n))_{n\geq 1}$ and $(q_X(n))_{n\geq 1}$, and let $0 < \varepsilon < \frac{2}{5}$, $0 < \delta < \varepsilon$. There are the following cases.

(1) (a) If $p_X(n) > \frac{2}{5} + \delta$ and $q_X(n) > \frac{2}{5} + \delta$, or
(b) $p_X(n) > \frac{2}{5} + \delta$ and $q_X(n) < \frac{2}{5} - \delta$, or
(c) $p_X(n) < \frac{2}{5} - \delta$ and $q_X(n) < \frac{2}{5} - \delta$, or
(d) $p_X(n) < \frac{2}{5} - \delta$ and $q_X(n) > \frac{2}{5} + \delta$, or
(e) $p_X(n) > \frac{2}{5} + \delta$ and $\frac{2}{5} - \varepsilon < q_X(n) < \frac{2}{5} + \varepsilon$, or
(f) $p_X(n) < \frac{2}{5} - \delta$ and $\frac{2}{5} - \varepsilon < q_X(n) < \frac{2}{5} + \varepsilon$, or
(g) $\frac{2}{5} - \varepsilon < p_X(n) < \frac{2}{5} + \varepsilon$, and $q_X(n) > \frac{2}{5} + \delta$ or

(h) $\frac{2}{5}-\varepsilon < p_X(n) < \frac{2}{5}+\varepsilon$, and $q_X(n) < \frac{2}{5}-\delta$,

then there exists no pair of $(p_X(n), q_X(n))$ such that

$M(X, q_X(n)) - M(p_X(n), q_X(n)) \longrightarrow 0$, $M(Y, q_X(n)) - M(p_X(n), q_X(n)) \longrightarrow 0$,

$-[M(p_X(n), X) - M(p_X(n), q_X(n))] \longrightarrow 0$ and

$-[M(p_X(n), Y) - M(p_X(n), q_X(n))] \longrightarrow 0$.

(2) If $\frac{2}{5}-\varepsilon < p_X(n) < \frac{2}{5}+\varepsilon$ and $\frac{2}{5}-\varepsilon < q_X(n) < \frac{2}{5}+\varepsilon$ with $0 < \varepsilon < \frac{2}{5}$,

$M(X, q_X(n)) - M(p_X(n), q_X(n)) \longrightarrow 0$, $M(Y, q_X(n)) - M(p_X(n), q_X(n)) \longrightarrow 0$,

$-[M(p_X(n), X) - M(p_X(n), q_X(n))] \longrightarrow 0$ and

$-[M(p_X(n), Y) - M(p_X(n), q_X(n))] \longrightarrow 0$, then

$(p_X(n), q_X(n)) \longrightarrow (\frac{2}{5}, \frac{2}{5})$ for any sequence $(p_X(n), q_X(n))$.

Therefore, the payoff functions satisfy Assumption 3.1.

References

[1] J. Berger, D. Bridges, and P. Schuster, The fan theorem and unique existence of maxima, Journal of Symbolic Logic, 71 (2006), 713-720.

[2] J. Berger and H. Ishihara, Brouwer's fan theorem and unique existence in constructive analysis, Mathematical Logic Quarterly, 51 (2005) 360-364.

[3] E. Bishop and D. Bridges, Constructive Analysis, Springer, 1985.

[4] D. Bridges and F. Richman, Varieties of Constructive Mathematics, Cambridge University Press, 1987.

[5] D. Bridges and L. Vîţă, Techniques of Constructive Mathematics, Springer, 2006.

[6] R. B. Kellogg, T. Y. Li, and J. Yorke, A constructive proof of Brouwer fixed-point theorem and computational results, SIAM Journal on Numerical Analysis, 13 (1976) 473-483.

[7] Y. Tanaka. Brouwer's fixed point theorem with sequentially at most one fixed point: a constructive analysis, Advances in Fixed Point Theory, 2 (2012) 9-17.

[8] D. van Dalen, Brouwer's ε-fixed point from Sperner's lemma, Theoretical Computer Science, 412 (2011) 3140-3144.

[9] W. Veldman, Brouwer's approximate fixed point theorem is equivalent to Brouwer's fan theorem, In S. Lindström, E. Palmgren, K. Segerberg, and V. Stoltenberg-Hansen, editors, Logicism, Intuitionism and Formalism. Springer, 2009.