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FIXED POINTS OF NONSELF TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN HYPERBOLIC SPACES

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Abstract. In this paper, we approximate fixed points of nonself $(\{\mu_n\},\{\nu_n\},\zeta)-$ total asymptotically nonexpansive mappings via new iterative scheme and establish strong and $\Delta-$convergence theorems in the setup of uniformly convex hyperbolic spaces. Our results presented in the paper extend and improve some recent results announced in the current literature.

Keywords: nonself $(\{\mu_n\},\{\nu_n\},\zeta)-$ total asymptotically nonexpansive mappings; new iteration process; uniformly convex hyperbolic spaces; strong and Δ-convergence theorems.

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1. Introduction

In this paper, $\mathbb{N}$ stands for the set of natural numbers. Let C be nonempty subset of a metric space (X,d). A mapping $T:C\to C$ is said to be

(i) *nonexpansive*, if $d(Tx,Ty)\le d(x,y)$ for all $x,y\in C$;

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(ii) *asymptotically nonexpansive* ([16]), if for each $n \in \mathbb{N}$ there exists a constant $k_n \geq 1$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n x, T^n y) \leq k_n d(x,y)$$

for all $x, y \in C$;

(iii) *asymptotically nonexpansive in the intermediate sense* ([6]), provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(d(T^n x, T^n y) - d(x,y)\right) \leq 0 \tag{1.1}$$

for all $x, y \in C$;

(iv) *a mapping of asymptotically nonexpansive type* ([33]), if

$$\limsup_{n\to\infty} \sup_{y\in C} (d(T^n x, T^n y) - d(x,y)) \leq 0$$

for all $x, y \in C$.

Being an important generalization of the class of nonexpansive self mappings, the class of asymptotically nonexpansive self mappings was introduced by Goebel et al. [16] whereas the class of asymptotically nonexpansive mappings in the intermediate sense which is essentially wider than that of asymptotically nonexpansive was introduced by Burk et al. [6].

On the other hand, if $c_n = \max\{\sup_{x\in C}(d(T^n x, T^n y) - d(x,y)), 0\}$, then (1.1) reduces to relation

$$d(T^n x, T^n y) \leq d(x,y) + c_n \tag{1.2}$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

In [4], Alber et al. introduced the class of total asymptotically nonexpansive mappings which is more general than the class of mapping satisfying (1.2).

Definition 1.1. Let C be nonempty subset of a metric space (X, d). A mapping $T : C \to C$ is said to be $(\{\mu_n\}, \{\nu_n\}, \zeta)-$ *total asymptotically nonexpansive*, if there exist nonnegative real sequences $\{\mu_n\}$ and $\{\nu_n\}$ with $\lim_{n\to\infty} \mu_n = \lim_{n\to\infty} \nu_n = 0$ and strictly increasing function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\zeta(0) = 0$ such that

$$d(T^n x, T^n y) \leq d(x,y) + \mu_n \zeta(d(x,y)) + \nu_n \tag{1.3}$$

for all $x,y \in C$ and $n \in \mathbb{N}$.

Remark 1.2. (1) If $\mu_n = 0$, then (1.3) reduces to (1.2).

(2) From the definitions, it is obvious that each nonexpansive mapping is an asymptotically nonexpansive with a sequence $\{k_n\} = 0$ and each asymptotically nonexpansive mapping is a $(\{\mu_n\},\{\nu_n\},\zeta)$-total asymptotically nonexpansive mapping with $\{\nu_n\} = 0$ and $\zeta(t) = t$ for $t > 0$.

In the above definition T remains self mapping, whereas the concept of asymptotically nonexpansive nonself mappings was introduced by Chidume et al. [7] in 2003 as the generalization of asymptotically nonexpansive self mappings in Banach space, we define a metric spaces version as follows:

Let (X,d) be a metric space and C be a nonempty subset of X. Recall that C is retract of X if there exists a continuous mappings $P : X \to C$ such that $Px = x$ for all $x \in C$. A mapping $P : X \to C$ is said to *retraction* if $P^2 = P$. It follows that if a mapping P is retraction, then $Py = y$ for all y in the range of P.

Definition 1.3. Let C be nonempty subset of a metric space (X,d) and $P : X \to C$ be the nonexpansive retraction of X onto C. A nonself mapping $T : C \to X$ is said to be

(1) *asymptotically nonexpansive nonself mapping* [7], if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \leq k_n d(x,y) \tag{1.4}$$

for all $x,y \in C$ and $n \in \mathbb{N}$;

(2) *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \leq Ld(x,y) \tag{1.5}$$

for all $x,y \in C$ and $n \in \mathbb{N}$;

(3) $(\{\mu_n\},\{\nu_n\},\zeta)$-*total asymptotically nonexpansive nonself mapping* [9, 10], if there exist nonnegative sequences $\{\mu_n\}$ and $\{\nu_n\}$ with $\mu_n \to 0$ and $\nu_n \to 0$ as $n \to \infty$ and a strictly increasing function $\zeta : [0,\infty) \to [0,\infty)$ with $\zeta(0) = 0$ such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \leq d(x,y) + \mu_n \zeta(d(x,y)) + \nu_n \tag{1.6}$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

Remark 1.4. 1. In view of Definition 1.3, every nonexpansive nonself mapping is a asymptotically nonexpansive nonself mapping (set sequence $\{k_n\} = 1$) and each asymptotically nonexpansive nonself mapping is a $(\{\mu_n\}, \{\nu_n\}, \zeta)$-total asymptotically nonexpansive nonself mapping (choose $\{\mu_n\} = 0$, $\{\nu_n\} = k_n - 1$ for all $n \geq 1$ and $\zeta(t) = t$, $t \geq 0$.)

As a matter of fact, if T is a self mapping, then P is an identity mapping. In addition, if $T : C \to X$ is $(\{\mu_n\}, \{\nu_n\}, \zeta)$-total asymptotically nonexpansive nonself mapping and $P : X \to C$ is a nonexpansive retraction, then $PT : C \to C$ is $(\{\mu_n\}, \{\nu_n\}, \zeta)$-*total asymptotically nonexpansive nonself mapping*. Indeed, for all $x, y \in C$ and $n \in \mathbb{N}$, it follows that

$$\begin{aligned} d(PT)^n x, (PT)^n y) &= d(PT(PT)^{n-1}x, PT(PT)^{n-1}y) \\ &\leq d(T(PT)^{n-1}x, T(PT)^{n-1}y) \\ &\leq d(x,y) + \mu_n \zeta(d(x,y)) + \nu_n. \end{aligned}$$

Thus, it is a more satisfactory definition of a nonself total asymptotically nonex- pansive mapping which is given as follows.

Definition 1.5. Let C be nonempty subset of a metric space (X, d) and $P : X \to C$ be the nonexpansive retraction of X onto C. A nonself mapping $T : C \to X$ is said to be $(\{\mu_n\}, \{\nu_n\}, \zeta)$-*total asymptotically nonexpansive nonself mapping* if there exist nonnegative sequences $\{\mu_n\}$ and $\{\nu_n\}$ with $\mu_n \to 0$ and $\nu_n \to 0$ as $n \to \infty$ and a strictly increasing function $\zeta : [0, \infty) \to [0, \infty)$ with $\zeta(0) = 0$ such that

$$d((PT)^n x, (PT)^n y) \leq d(x,y) + \mu_n \zeta(d(x,y)) + \nu_n \tag{1.7}$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

Approximating fixed points of nonlinear mappings using different iterative processes on different domains have remained at the heart of fixed point theory. Nonexpansive mappings constitute one of the most important classes of nonlinear mappings which have remained a crucial part of such studies. Then thereafter, there are numerous papers dealing with the approximation of fixed points of nonexpansive, asymptotically nonexpansive mappings, total asymptotically nonexpansive mapping and nearly asymptotically nonexpansive self maps in the course of modified Mann and Ishikawa, Noor and $S-$ iteration processes (see, e.g., [1, 4, 5, 12, 29, 37, 39, 40, 41] and references contained therein), for nonself mappings (see, e.g., [19, 28, 30, 44] and references contained therein) in uniformly convex Banach spaces and in CAT(0) spaces (see, e.g., [2, 3, 9, 11, 15, 24, 26, 43] and references contained therein).

In a linear domains with some geometric structure have been studied extensively, one of such structures is convexity. Since every Banach space is a vector space, it is easier to assign a convex structure to it. However, metric spaces do not enjoy this structure. Takahashi [42], introduced the notion of convex metric spaces and studied the fixed point theory for nonexpansive mappings in this setting. Later on, several attempts were made to introduce different convex structures on a metric space.

One such convex structure is available in hyperbolic spaces introduced by Kohlenbach [34]. Kohlenbach hyperbolic space [34] is more restrictive than the hyperbolic space introduced in [17] and more general than the concept of hyperbolic space in [38]. Spaces like CAT(0) and Banach are special cases of a hyperbolic space. The class of hyperbolic spaces also contains Hadamard manifolds, Hilbert ball equipped with the hyperbolic metric [18], $\mathbb{R}-$trees, and Cartesian products of Hilbert balls as special cases.

The modified convex structure neither contains Takahashi's convex structure as a special case nor is suitable to obtain an approximate fixed point sequence in a uniformly convex metric space; therefore accumulating this information authors have worked on hyperbolic spaces; (see for example, [10, 13, 14, 20, 21, 22, 23, 25, 27, 31, 46] and therefore the references cited in this) for self and nonself asymptotically nonexpansive mapping and wider classes of asymptotically nonexpansive mappings.

Recently, Abbas and Nazir [5] introduced a new three step iteration process and proved that its rate of convergence is comparatively faster than Picard and Aagrwal et al. [1] iteration processes for contraction mapping (see Theorem 3, page 226, [5]). They had also proved some weak and strong convergence theorems for nonexpansive mapppings. Moreover, they had applied their results to find solutions of constrained minimization problems and feasibility problems. In solving various numerical problems in pure and applied sciences the study of three steps iteration processes are very important.

Motivated by the above facts, we approximate fixed point of nonself $(\{\mu_n\},\{\nu_n\},\zeta)-$ total asymptotically nonexpansive mappings via new iterative scheme and establish strong and $\Delta-$convergence theorems in the setup of uniformly convex hyperbolic spaces. As mentioned above, the class of uniformly convex hyperbolic spaces include several types of spaces including those of uniformly convex Banach spaces as well as CAT(0) spaces. Thus, our results presented in the paper extend and improve some recent results announced in the current literature (related to in uniformly convex Banach spaces as well as CAT(0) spaces) in the setting of hyperbolic spaces, using one of the faster iterative process as compare to Picard and Aagrwal et al. [1] iteration processes.

2. Preliminaries

Throughout in this paper, we work in the setting of hyperbolic spaces introduced by Kohlenbach [34].

A *hyperbolic space* (X,d,W) is a metric space (X,d) together with a convexity mapping $W : X^2 \times [0,1] \to X$ satisfying

(W_1) $d(u,W(x,y,\alpha)) \leq \alpha d(u,x)+(1-\alpha)d(u,y)$;

(W_2) $d(W(x,y,\alpha),W(x,y,\beta)) = |\alpha-\beta|d(x,y)$;

(W_3) $W(x,y,\alpha) = W(y,x,1-\alpha)$;

(W_4) $d(W(x,z,\alpha),W(y,w,\alpha)) \leq (1-\alpha)d(x,y)+\alpha d(z,w)$,

for all $x,y,z,w \in X$ and $\alpha,\beta \in [0,1]$.

A metric space is said to be a *convex metric space* in the sense of Takahashi [42], where a triple (X,d,W) satisfy only (W_1).

All normed spaces and their subsets are hyperbolic spaces as well as convex metric spaces. The class of hyperbolic spaces is properly contained in the class of convex metric spaces ([32, 34]). For the definition of a CAT(0) space, basic properties and details to introduce a convex structure in it, we refer to [11, 24]. It is remarked that every CAT (0) space is a hyperbolic space [20] (see also [10, 22, 31]).

If $x,y \in X$ and $\lambda \in [0,1]$, then we use the notation $(1-\lambda)x \oplus \lambda y$ for $W(x,y,\lambda)$. The following holds even for the more general setting of convex metric space, for all $x,y \in X$ and $\lambda \in [0,1]$,

$$d(x,(1-\lambda)x \oplus \lambda y) = \lambda d(x,y) \text{ and } d(y,(1-\lambda)x \oplus \lambda y) = (1-\lambda)d(x,y).$$

A hyperbolic space (X,d,W) is *uniformly convex* [35], if for any $r > 0$ and $\varepsilon \in (0,2]$, there exists $\delta \in (0,1]$ such that for all $a,x,y \in X$,

$$d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \leq (1-\delta)r.$$

provided $d(x,a) \leq r, d(y,a) \leq r$ and $d(x,y) \geq \varepsilon r$.

A mapping $\eta : (0,\infty) \times (0,2] \to (0,1]$, which providing such a $\delta = \eta(r,\varepsilon)$ for given $r > 0$ and $\varepsilon \in (0,2]$, is called as a *modulus of uniform convexity.* We call η *monotone* if it decreases with r (for fixed ε).

In [35], Leuştean proved that CAT(0) spaces are uniformly convex hyperbolic spaces with modulus of uniform convexity $\eta(r,\varepsilon) = \frac{\varepsilon^2}{8}$ quadratic in ε. Thus, the class of uniformly convex hyperbolic spaces are a natural generalization of both uniformly convex Banach spaces and CAT(0) spaces.

Now, we recall the concept of Δ-convergence besides collecting some of its basic properties.

Let C be a nonempty subset of metric space X and $\{x_n\}$ be any bounded sequence in C. Consider a continuous functional $r_a(\cdot,\{x_n\}) : X \to \mathbb{R}^+$ defined by

$$r_a(x,\{x_n\}) = \limsup_{n\to\infty} d(x_n,x), \quad x \in X.$$

Then,

(a) the infimum of $r_a(.,\{x_n\})$ over C is said to be the *asymptotic radius* of $\{x_n\}$ with respect to C and is denoted by $r_a(C,\{x_n\})$.

(b) A point $z \in C$ is said to be an *asymptotic center* of the sequence $\{x_n\}$ with respect to C if

$$r_a(z,\{x_n\}) = \inf\{r_a(x,\{x_n\}) : x \in C\},$$

the set of all asymptotic centers of $\{x_n\}$ with respect to C is denoted by $Z_a(C,\{x_n\})$.

(c) If the asymptotic radius and the asymptotic center are taken with respect to X, then these are simply denoted by $r_a(X,\{x_n\}) = r_a(\{x_n\})$ and $Z_a(X,\{x_n\}) = Z_a(\{x_n\})$ respectively.

It is known that uniformly convex Banach spaces and even CAT(0) spaces enjoy the property that bounded sequences have unique asymptotic centers with respect to closed convex subsets. The following Lemma is due to Leuştean [36] and ensures that this property also holds in a complete uniformly convex hyperbolic space.

Lemma 2.1. [36] Let (X,d,W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset C of X.

Recall that a sequence $\{x_n\}$ in X is said to Δ-*converge* to $x \in X$, if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Lemma 2.2. [8] Let (X,d,W) be a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η, and let $\{x_n\}$ be a bounded sequence in X with $Z_a(C,\{x_n\}) = \{p\}$. Suppose $\{u_n\}$ is a subsequence of $\{x_n\}$ with $Z_a(C,\{u_n\}) = \{u\}$ and $\{d(x_n,u)\}$ converges, then $p = u$.

Lemma 2.3. ([23]) Let (X,d,W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{t_n\}$ be a sequence in $[a,b]$ for some $a,b \in (0,1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\limsup_{n\to\infty} d(x_n,x) \le c, \quad \limsup_{n\to\infty} d(y_n,x) \le c.$$
$$\lim_{n\to\infty} d(W(x_n,y_n,t_n),x) = c$$

for some $c \ge 0$, then $\lim_{n\to\infty} d(x_n,y_n) = 0$.

Lemma 2.4. ([37]) Let $\{\delta_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences of nonnegative numbers such that

$$\delta_{n+1} \le \beta_n \delta_n + \gamma_n.$$

for all $n \in \mathbb{N}$. If $\beta_n \geq 1$ for all $n \in \mathbb{N}$, $\sum_{n=1}^{\infty}(\beta_n - 1) < \infty$ and $\sum_{n=1}^{\infty}\gamma_n < \infty$, then $\lim_{n\to\infty}\delta_n$ exists.

Recently, Wan [45] proved the demiclosed principle for a $(\{\mu_n\},\{\nu_n\},\zeta)$-total asymptotically nonexpansive nonself mappings in hyperbolic spaces as follows:

Theorem 2.5. ([45]) Let (X,d,W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let C be a nonempty closed and convex subset of X. Let $T : C \to X$ be a uniformly L-Lipschitzian and a $(\{\mu_n\},\{\nu_n\},\zeta)$-total asymptotically nonexpansive nonself mapping. P is a nonexpansive retraction of X onto C. Let $\{x_n\} \subset C$ be a bounded approximate fixed point sequence for T (i.e, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$) and $x_n \rightharpoonup p$. Then we have $T(p) = p$.

3. Strong and Δ-convergence Theorems in Hyperbolic space

Now we introduce the iteration process in a hyperbolic space which involve $(\{\mu_n\},\{\nu_n\},\zeta)$-*total asymptotically nonexpansive nonself mapping* in the sense of the Definition 1.5.

Let C be a nonempty closed convex subset of a hyperbolic space X and T be a mapping defined in the Definition 1.5. For arbitrarily chosen $x_1 \in C$, we construct the sequence $\{x_n\}$ as:

$$\begin{aligned} x_{n+1} &= PW((PT)^n y_n, (PT)^n z_n, \alpha_n), \\ y_n &= PW((PT)^n x_n, (PT)^n z_n, \beta_n), \\ z_n &= PW(x_n, (PT)^n x_n, \gamma_n), \quad n \in \mathbb{N}, \end{aligned} \tag{3.1}$$

where $\{\alpha_n\},\{\beta_n\}$ and$\{\gamma_n\}$ are sequences in $(0,1)$.

Now we establish strong and $\Delta-$convergence theorems for iterative scheme defined by (3.1). The following Proposition is trivially holds.

Proposition 3.1. Let $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing function and $M, M_0 \geq 0$ be constants such that $\zeta(r) \leq rM_0$ for $r \geq M$, then

$$\zeta(r) \leq \zeta(M) + rM_0.$$

Lemma 3.2. Let C be a nonempty closed convex subset of a uniformly convex hyperbolic space X and P be a nonexpansive retraction of X onto C. Let $T : C \to X$ be $(\{\mu_n\},\{\nu_n\},\zeta)-$ total asymptotically nonexpansive mapping with sequences $\{\mu_n\}$ and $\{\nu_n\}$ for all $n \in \mathbb{N}$ such that $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \nu_n < \infty$ and a strictly increasing function $\zeta : [0,\infty) \to [0,\infty)$ with $\zeta(0) = 0$. Let $\{x_n\}$ be sequence defined by (3.1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ such that $0 < a \leq \alpha_n \leq \beta_n \leq b < 1$, then $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F(T)$, where $F(T)$ is the set of fixed points of T.

Proof. In fact, for each $p \in F(T)$, using (3.1), (W_1) and Proposition 3.1, we have

$$\begin{aligned}
d(x_{n+1}, p) &= d(PW((PT)^n y_n, (PT)^n z_n, \alpha_n), p) \\
&\leq (1-\alpha_n) d((PT)^n y_n, p) + \alpha_n d((PT)^n z_n, p) \\
&\leq (1-\alpha_n)[d(y_n, p) + \mu_n \zeta(d(y_n, p)) + \nu_n] \\
&+ \alpha_n [d(z_n, p) + \mu_n \zeta(d(z_n, p)) + \nu_n] \\
&\leq (1-\alpha_n)[(1+\mu_n M_0) d(y_n, p) + \mu_n \zeta(M)] \\
&+ \alpha_n [(1+\mu_n M_0) d(z_n, p) + \mu_n \zeta(M)] + \nu_n \\
&\leq (1+\mu_n M_0)\Big[(1-\alpha_n) d(y_n, p) + \alpha_n d(z_n, p)\Big] \\
&+ \mu_n \zeta(M) + \nu_n.
\end{aligned} \tag{3.2}$$

Using (3.1), (W_1) and Proposition 3.1, we have

$$\begin{aligned}
d(z_n, p) &= d(PW(x_n, (PT)^n x_n, \gamma_n), p) \\
&\leq (1-\gamma_n) d(x_n, p) + \gamma_n d((PT)^n x_n, p) \\
&\leq (1-\gamma_n) d(x_n, p) + \gamma_n [d(x_n, p) + \mu_n \zeta(d(x_n, p)) + \nu_n] \\
&\leq (1+\mu_n M_0) d(x_n, p) + \mu_n \zeta(M) + \nu_n.
\end{aligned} \tag{3.3}$$

using (3.1), (W_1) and Proposition 3.1, we have

$$\begin{aligned} d(y_n,p) &= d(PW((PT)^n x_n,(PT)^n z_n,\beta_n),p) \\ &\leq (1-\beta_n)d((PT)^n x_n,p)+\beta_n d((PT)^n z_n,p) \\ &\leq (1-\beta_n)[d(x_n,p)+\mu_n\zeta(d(x_n,p))+\nu_n)] \\ &+\beta_n[d(z_n,p)+\mu_n\zeta(d(z_n,p))+\nu_n)] \\ &\leq (1+\mu_n M_0)\Big[(1-\beta_n)d(x_n,p)+\beta_n d(z_n,p)\Big]+\mu_n\zeta(M)+\nu_n, \end{aligned} \tag{3.4}$$

using (3.3) and (3.4), we have

$$d(y_n,p) \leq \quad (1+\mu_n M_0)^2 d(x_n,p)+(2+\mu_n M_0)(\nu_n+\mu_n\zeta(M)). \tag{3.5}$$

Substituting values form (3.3) and (3.5) into (3.2), we have

$$d(x_{n+1},p) \quad \leq \quad (1+\mu_n A)d(x_n,p)+B\mu_n+C\nu_n, \quad n\in\mathbb{N},$$

for some A,B and $C\geq 0$, where

$$\begin{aligned} A &= \mu_n^2 M_0^3+3M_0+3\mu_n M_0^2, \\ B &= 3\zeta(M)+4\mu_n\zeta(M)+M_0 \text{ and } C=3+4\mu_n M_0. \end{aligned}$$

Since $\sum_{n=1}^{\infty}\mu_n<\infty$ and $\sum_{n=1}^{\infty}\nu_n<\infty$, so that $\sum_{n=1}^{\infty}h_n<\infty$, where $h_n=(\mu_n B+\nu_n C)$. Then above inequality can be writen as

$$d(x_{n+1},p) \leq \quad (1+\mu_n A)d(x_n,p)+h_n \quad n\in\mathbb{N}, \tag{3.6}$$

using, Lemma 2.4, we observe that $\lim_{n\to\infty} d(x_n,p)$ exists for each $p\in F(T)$. This completes the proof.

Lemma 3.3. Let C be a nonempty closed convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and P be a nonexpansive retraction of X onto C. Let $T: C\to X$ be a uniformly L-Lipschitzian and $(\{\mu_n\},\{\nu_n\},\zeta)-$total asymptotically nonexpansive mapping with sequences $\{\mu_n\}$ and $\{\nu_n\}$, $n\in\mathbb{N}$ such that $\sum_{n=1}^{\infty}\mu_n<\infty$ and $\sum_{n=1}^{\infty}\nu_n<\infty$ and a strictly increasing function $\zeta:[0,\infty)\to[0,\infty)$ with $\zeta(0)=0$. Let $\{x_n\}$ be sequence defined by (3.1). Then $\lim_{n\to\infty} d(x_n,Tx_n)=0$.

Proof. It follows from Lemma 3.2 that $\lim_{n\to\infty} d(x_n, p)$ exists, for $p \in F(T)$. Without loss of generality we may assume that $\lim_{n\to\infty} d(x_n, p) = c > 0$. From (3.3), we have

$$d(z_n, p) \leq (1+\mu_n M_0)d(x_n, p) + \mu_n \zeta(M) + \nu_n \quad n \in \mathbb{N},$$

taking $\limsup_{n\to\infty}$, both the sides, we get

$$\limsup_{n\to\infty} d(z_n, p) \leq c. \tag{3.7}$$

Since T is $(\{\mu_n\}, \{v_n\}, \zeta)$-*total asymptotically nonexpansive nonself mapping*, it follows that

$$\begin{aligned} d((PT)^n z_n, p) &= d(PT)^n z_n, (PT)^n p) \\ &\leq d(z_n, p) + \mu_n \zeta(d(z_n, p)) + \nu_n \\ &\leq (1+M_0\mu_n)d(z_n, p) + \mu_n \zeta(M) + \nu_n, \quad n \in \mathbb{N}, \end{aligned}$$

taking $\limsup_{n\to\infty}$, both the sides, we get

$$\limsup_{n\to\infty} d((PT)^n z_n, p) \leq c. \tag{3.8}$$

From (3.5), we have

$$d(y_n, p) \leq (1+\mu_n M_0)^2 d(x_n, p) + (2+\mu_n M_0)(\nu_n + \mu_n \zeta(M)) \quad n \in \mathbb{N},$$

taking $\limsup_{n\to\infty}$, both the sides, we get

$$\limsup_{n\to\infty} d(y_n, p) \leq c \tag{3.9}$$

and

$$\begin{aligned} d((PT)^n y_n, p) &= d(PT)^n y_n, (PT)^n p) \\ &\leq d(y_n, p) + \mu_n \zeta(d(y_n, p)) + \nu_n \\ &\leq (1+M_0\mu_n)d(y_n, p) + \mu_n \zeta(M) + \nu_n, \quad n \in \mathbb{N}, \end{aligned}$$

taking $\limsup_{n\to\infty}$, both the sides and using (3.9), we get

$$\limsup_{n\to\infty} d((PT)^n y_n, p) \leq \quad c. \tag{3.10}$$

Futher,

$$\begin{aligned} c &= \lim_{n\to\infty} d(x_{n+1}, p) \\ &= \lim_{n\to\infty} \left\{ d(PW((PT)^n z_n, (PT)^n y_n, \alpha_n), p) \right\} \\ &\leq \lim_{n\to\infty} \left\{ (1-\alpha_n) \limsup_{n\to\infty} d((PT)^n z_n, p) + \alpha_n \limsup_{n\to\infty} d((PT)^n y_n, p) \right\} \end{aligned}$$

Using (3.8) and (3.10), we get

$$c \leq \lim_{n\to\infty} ((1-\alpha_n)c + \alpha_n c) = c.$$

Thus,

$$c = \lim_{n\to\infty} d(W((PT)^n z_n, (PT)^n y_n, \alpha_n), p),$$

for $c > 0$. Hence, from Lemma 2.3, we have

$$\lim_{n\to\infty} d((PT)^n z_n, (PT)^n y_n) = 0. \tag{3.11}$$

Since

$$\begin{aligned} d((PT)^n x_n, p) &= d((PT)^n x_n, (PT)^n p) \\ &\leq (1 + M_0 \mu_n) d(x_n, p) + \mu_n \zeta(M) + \nu_n \quad n \in \mathbb{N}, \end{aligned}$$

therefore, taking $\limsup_{n\to\infty}$, both the sides, we get

$$\limsup_{n\to\infty} d((PT)^n x_n, p) \leq c. \tag{3.12}$$

Next,

$$\begin{aligned} d(x_{n+1}, p) &= d(PW((PT)^n y_n, (PT)^n z_n, \alpha_n), p) \\ &\leq (1-\alpha_n) d((PT)^n y_n, p) + \alpha_n d((PT)^n z_n, p) \\ &\leq (1-\alpha_n) d((PT)^n y_n, p) + \alpha_n d((PT)^n z_n, (PT)^n y_n) + \alpha_n d((PT)^n y_n, p) \\ &\leq d((PT)^n y_n, p) + \alpha_n d((PT)^n z_n, (PT)^n y_n) \\ &\leq (1 + \mu_n M_0) d(y_n, p) + \mu_n \zeta(M) + \nu_n + \alpha_n d((PT)^n z_n, (PT)^n y_n), \end{aligned}$$

taking $\liminf_{n\to\infty}$, both the sides, we get

$$c \leq \liminf_{n\to\infty} d(y_n, p). \tag{3.13}$$

From (3.9) and(3.13), we get $\lim_{n\to\infty} d(y_n, p) = c$, means that

$$\begin{aligned} c &= \lim_{n\to\infty} \Big\{ d(PW((PT)^n x_n, (PT)^n z_n, \beta_n), p) \Big\} \\ &\leq \lim_{n\to\infty} \Big\{ (1-\beta_n) \limsup_{n\to\infty} d((PT)^n x_n, p) + \beta_n \limsup_{n\to\infty} d((PT)^n z_n, p) \Big\}. \end{aligned}$$

Using (3.8) and (3.12) we have

$$c \leq \lim_{n\to\infty} ((1-\beta_n)c + \beta_n c) = c.$$

Thus,

$$c = \lim_{n\to\infty} d(PW((PT)^n x_n, (PT)^n z_n, \beta_n), p),$$

for $c > 0$. Hence, from Lemma 2.3, we have

$$\lim_{n\to\infty} d((PT)^n x_n, (PT)^n z_n) = 0. \tag{3.14}$$

Using (3.11) and (3.14), we have

$$\begin{aligned} \lim_{n\to\infty} d((PT)^n x_n, (PT)^n y_n) \leq \lim_{n\to\infty} d((PT)^n z_n, (PT)^n y_n) + \lim_{n\to\infty} d((PT)^n y_n, (PT)^n x_n) \\ = 0. \end{aligned} \tag{3.15}$$

Again from (3.1), we have

$$\begin{aligned} d(y_n, p) &= d(PW((PT)^n x_n, (PT)^n z_n, \beta_n), p) \\ &\leq (1-\beta_n) d((PT)^n x_n, p) + \beta_n d((PT)^n z_n, p) \\ &\leq d((PT)^n z_n, p) + (1-\beta_n) d((PT)^n x_n, (PT)^n z_n) \\ &\leq (1+\mu_n M_0) d(z_n, p) + \mu_n \zeta(M) + \nu_n + (1-\beta_n) d((PT)^n x_n, (PT)^n z_n), \end{aligned}$$

using (3.14) and taking $\liminf_{n\to\infty}$, both the sides, we get

$$c \leq \liminf_{n\to\infty} d(z_n, p). \tag{3.16}$$

Hence, from(3.6) and (3.16), we have $\lim_{n\to\infty} d(z_n, p) = c$, means that

$$\begin{aligned} c &= \lim_{n\to\infty} \left\{ d(PW(x_n,(PT)^n x_n,\gamma_n),p) \right\} \\ &\le \lim_{n\to\infty} \left\{ (1-\gamma_n)\limsup_{n\to\infty} d(x_n,p) + \gamma_n \limsup_{n\to\infty} d((PT)^n x_n,p) \right\}. \end{aligned}$$

Using (3.12), we have

$$c \le \lim_{n\to\infty}((1-\gamma_n)c+\gamma_n c) = c.$$

Thus,

$$c = \lim_{n\to\infty} d(PW(x_n,(PT)^n x_n,\gamma_n),p),$$

for $c > 0$. Hence, from Lemma 2.3, we have

$$\lim_{n\to\infty} d(x_n,(PT)^n x_n) = 0. \tag{3.17}$$

Next, we show that $\lim_{n\to\infty} d((PT)^n z_n, x_n) = 0, \lim_{n\to\infty} d((PT)^n y_n, x_n) = 0$. Apply triangle inequality, using (3.14) and (3.17), we get

$$\begin{aligned} \lim_{n\to\infty} d((PT)^n z_n,x_n) &\le \lim_{n\to\infty} \left[d((PT)^n z_n,(PT)^n x_n) + d((PT)^n x_n,x_n) \right] \\ &\le \lim_{n\to\infty} d((PT)^n z_n,(PT)^n x_n) + \lim_{n\to\infty} d((PT)^n x_n,x_n) \\ &= 0. \end{aligned} \tag{3.18}$$

Similarly, using (3.11) and (3.18), we get

$$\begin{aligned} \lim_{n\to\infty} d((PT)^n y_n,x_n) &\le \lim_{n\to\infty} \left[d((PT)^n y_n,(PT)^n z_n) + d((PT)^n z_n,x_n) \right] \\ &\le \lim_{n\to\infty} d((PT)^n y_n,(PT)^n z_n) + \lim_{n\to\infty} d((PT)^n z_n,x_n) \\ &= 0. \end{aligned} \tag{3.19}$$

using (3.1) and (3.17), we get

$$\begin{aligned}\lim_{n\to\infty} d(z_n,x_n) &= \lim_{n\to\infty} d(PW(x_n,(PT)^n x_n,\gamma_n),x_n)\\ &\le \lim_{n\to\infty}\Big[(1-\gamma_n)d(x_n,x_n)+\gamma_n d((PT)^n x_n,x_n)\Big]\\ &\le \gamma_n \lim_{n\to\infty} d((PT)^n x_n,x_n)\\ &= 0.\end{aligned} \tag{3.20}$$

Using triangle inequality, (3.14), (3.17) and (3.20), we have

$$\begin{aligned}\lim_{n\to\infty} d(y_n,z_n) &\le \lim_{n\to\infty}\Big[d(y_n,x_n)+d(x_n,z_n)\Big]\\ &\le \lim_{n\to\infty}\Big[d(PW((PT)^n x_n,(PT)^n z_n,\beta_n),x_n)+d(x_n,z_n)\Big]\\ &\le \lim_{n\to\infty}\Big[(1-\beta_n)d((PT)^n x_n,x_n)+\beta_n d((PT)^n z_n,(PT)^n x_n)\\ &+\beta_n d((PT)^n x_n,x_n)+d(x_n,z_n)\Big]\\ &\le \lim_{n\to\infty} d((PT)^n x_n,x_n)+\lim_{n\to\infty} d(x_n,z_n)+\beta_n \lim_{n\to\infty} d((PT)^n z_n,(PT)^n x_n)\\ &= 0.\end{aligned} \tag{3.21}$$

Using (3.14) and (3.15), we get

$$\begin{aligned}\lim_{n\to\infty} d((PT)^n y_n,y_n) &= \lim_{n\to\infty} d(PW((PT)^n x_n,(PT)^n z_n,\beta_n),(PT)^n y_n)\\ &\le (1-\beta_n)\lim_{n\to\infty} d((PT)^n x_n,(PT)^n y_n)+\beta_n \lim_{n\to\infty} d((PT)^n z_n,(PT)^n y_n)\\ &= 0,\end{aligned} \tag{3.22}$$

using (3.14), we have

$$\begin{aligned}\lim_{n\to\infty} d((PT)^n z_n,y_n) &= \lim_{n\to\infty} d(PW((PT)^n x_n,(PT)^n z_n,\beta_n),(PT)^n z_n)\\ &\le (1-\beta_n)\lim_{n\to\infty} d((PT)^n x_n,(PT)^n z_n)\\ &= 0.\end{aligned} \tag{3.23}$$

Hence, from (3.1), (3.22) and (3.23), we get

$$\begin{aligned}\lim_{n\to\infty} d(x_{n+1},y_n) &= \lim_{n\to\infty} d(PW((PT)^n y_n,(PT)^n z_n,\alpha_n),y_n)\\ &\leq (1-\alpha_n)\lim_{n\to\infty} d((PT)^n y_n,y_n)+\alpha_n \lim_{n\to\infty} d((PT)^n z_n,y_n) \qquad (3.24)\\ &= 0.\end{aligned}$$

Apply triangle inequality, (3.20), (3.21) and (3.24), we have

$$\begin{aligned}\lim_{n\to\infty} d(x_{n+1},x_n) &\leq \lim_{n\to\infty}\Big[d(x_{n+1},y_n)+d(y_n,z_n)+d(z_n,x_n)\Big]\\ &\leq \lim_{n\to\infty} d(x_{n+1},y_n)+\lim_{n\to\infty} d(y_n,z_n)+\lim_{n\to\infty} d(z_n,x_n) \qquad (3.25)\\ &= 0.\end{aligned}$$

Since T be a uniformly L-Lipschitzian, so finally we compute

$$\begin{aligned}d(x_n,Tx_n) &\leq d(x_n,x_{n+1})+d(x_{n+1},(PT)^{n+1}x_{n+1})\\ &\quad +d((PT)^{n+1}x_{n+1},(PT)^{n+1}x_n)+d((PT)^{n+1}x_n,Tx_n)\\ &\leq (1+L)d(x_n,x_{n+1})+d(x_{n+1},(PT)^{n+1})x_n)+Ld(x_n,(PT)^n x_n),\end{aligned}$$

apply (3.17), (3.25) and taking limit as $n\to\infty$ in the above inequality, we have

$$\lim_{n\to\infty} d(x_n,Tx_n) = 0.$$

This completes the proof.

Theorem 3.4. Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and P be a nonexpansive retraction of X onto C. Let $T:C\to X$ be a uniformly L-Lipschitzian and $(\{\mu_n\},\{\nu_n\},\zeta)-$total asymptotically nonexpansive mapping with sequences $\{\mu_n\}$ and $\{\nu_n\}$, $n\in\mathbb{N}$ such that $\sum_{n=1}^{\infty}\mu_n<\infty$ and $\sum_{n=1}^{\infty}\nu_n<\infty$ and a strictly increasing function $\zeta:[0,\infty)\to[0,\infty)$ with $\zeta(0)=0$. Let $\{x_n\}$ be sequence defined by (3.1). where $\{\alpha_n\},\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$. Then $\{x_n\}$ is Δ-convergent to an element of $F(T)$.

Proof. For the sequence $\{x_n\}$, Δ-convergent to an element of $F(T)$, it suffices to show that

$$W_\Delta(\{x_n\}) = \bigcup_{\{u_n\}\subset\{x_n\}} Z_a(C,\{u_n\})\subset F(T)$$

and $W_\Delta(\{x_n\})$ consists of exactly one point.

Let $u \in W_\Delta(\{x_n\})$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $Z_a(C,\{u_n\}) = \{u\}$. By Lemma 2.1, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v$ for some $v \in C$. By Theorem 2.5, $v \in F(T)$. Since $\{d(u_n,v)\}$ converges, by Lemma 2.2, we have $u = v \in F(T)$. This shows that $W_\Delta(x_n) \subset F(T)$.

Next, we claim that $W_\Delta(\{x_n\})$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemma 2.1, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v$ for some $v \in C$. Let $Z_a(C,\{u_n\}) = \{u\}$ and $Z_a(C,\{x_n\}) = \{x\}$. We have already seen that $u = v$ and $v \in F(T)$. Finally, we claim that $x = v$. Suppose not, then by the existence of $\lim_{n\to\infty} d(x_n,v)$ and uniqueness of asymptotic centers, we have

$$\begin{aligned}\limsup_{n\to\infty} d(v_n,v) &< \limsup_{n\to\infty} d(v_n,x) \le \limsup_{n\to\infty} d(x_n,x)\\ &< \limsup_{n\to\infty} d(x_n,v) = \limsup_{n\to\infty} d(v_n,v),\end{aligned}$$

a contradiction and hence $x = v \in F(T)$. This shows that $W_\Delta(\{x_n\}) = \{x\}$. This completes the proof.

Theorem 3.5. Under the assumption of Theorem 3.4, the sequence $\{x_n\}$ defined by (3.1). converges strongly to a fixed point of T if and only if $\lim_{n\to\infty} d(x_n,F(T)) = 0$.

Proof. Necessity is obvious. We only prove the sufficiency. Suppose that $\liminf_{n\to\infty} d(x_n,F(T)) = 0$. From (3.6), we have

$$d(x_{n+1},F(T)) \le (1+A\mu_n)d(x_n,F(T)) + h_n, \quad n \in \mathbb{N},$$

for some $A \ge 0$. By Lemma 2.4, we have $\lim_{n\to\infty} d(x_n,F(T))$ exists. It follows that $\lim_{n\to\infty} d(x_n,F(T)) = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. The Following arguments similar to those given in ([2, Theorem 4.3] and [12, Lemma 5]), we obtained the following inequality

$$d(x_{n+m},p) \le L\left[d(x_n,p) + \sum_{j=n}^{\infty} h_j\right]$$

for every $p \in F(T)$ and for all $m,n \in \mathbb{N}$, where $L = e^{A(\sum_{j=n}^{n+m-1} \mu_j)} > 0$ and $b_j = h_j$ As, $\sum_{n=1}^{\infty} \mu_n < \infty$ so $L^* = e^{A(\sum_{n=1}^{\infty} \mu_n)} \ge L = e^{A(\sum_{j=n}^{n+m-1} \mu_j)} > 0$. Let $\varepsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n,F) =$

0 and $\sum_{n=1}^{\infty} h_n < \infty$, there exists a positive integer n_0 such that

$$d(x_n, F) \quad < \quad \frac{\varepsilon}{4L^*} \quad \text{and} \quad \sum_{j=n_0}^{\infty} h_j < \frac{\varepsilon}{6L^*}, \quad \forall n \geq n_0.$$

In particular, $\inf\{d(x_{n_0}, p) : p \in F\} < \frac{\varepsilon}{4L^*}$. Thus, there must exist $p^* \in F$ such that

$$d(x_{n_0}, p^*) < \frac{\varepsilon}{3L^*}.$$

Hence, for $n \geq n_0$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p^*) + d(p^*, x_n) \\ &\leq 2L^* \left[d(x_{n_0}, p^*) + \sum_{j=n_0}^{\infty} h_j \right] \\ &< 2L^* \left(\frac{\varepsilon}{3L^*} + \frac{\varepsilon}{6L^*} \right) = \varepsilon. \end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence in closed subset C of a complete uniformly convex hyperbolic space and so it must converge strongly to a point q in C. Now, $\lim_{n\to\infty} d(x_n, F(T)) = 0$ gives that $d(q, F(T)) = 0$. Since $F(T)$ is closed, we have $q \in F(T)$.

Theorem 3.6. Under the assumption of Theorem 3.4, if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for $t \in (0, \infty)$ such that

$$d(x, Tx) \geq f(d(x, F(T))),$$

for all $x \in C$, then the sequence defined by (3.1) converges strongly to a fixed point of T.

Proof. As proved in Theorem 3.5, $\lim_{n\to\infty} d(x_n, F(T))$ exists. Also by Lemma 3.3 we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then, we have

$$\lim_{n\to\infty} f(d(x_n, F(T))) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Since f is a non-decreasing function with $f(0) = 0$, and $f(t) > 0$ for $t \in (0, \infty)$, we have

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Hence, the rest of the result follows from Theorem 3.5.

Conflict of Interests

The authors declare that there is no conflict of interests.

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