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RESULTS ON CYCLIC ϕ-WEAK CONTRACTIONS IN FUZZY METRIC SPACES

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Abstract. We develop the fixed point theorems for ϕ-weak contractions in fuzzy metric spaces. We also define ψ-weak contractive condition and establish the fixed point in G-complete fuzzy metric spaces.

Keywords: cyclic ϕ-weak contraction, fixed point, G-Cauchy, G-complete fuzzy metric space.

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1. Introduction

Grabiec [5] established the Banach contraction theorem and Edelstein fixed point theorem in fuzzy metric spaces. Vasuki [16] generalised Grabiec's fuzzy Banach contraction. In Vasuki [16] defined a generalization of Grabiecs fuzzy Banach contraction theorem and proved a common fixed point theorem for a sequence of mappings in a fuzzy metric space. Cho [4] defined the concept of compatible mappings and proved common fixed point theorems in fuzzy metric spaces. Pacurar and Rusin [12] introduced the concept of ϕ-contraction. They developed some fixed point theorems using cyclic ϕ- contraction in complete metric space. Based on these ideas Shen et.al [14] came up with notion of cyclic ϕ- contraction in fuzzy metric spaces. In addition,

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several problems in connection with the fixed point are investigated. In this paper, we generalize the fixed point theorems of Shen et. al in G-fuzzy metric spaces.

2. Preliminaries

Definition 2.1. [18] A binary operation $T : [0,1] \times [0,1] \to [0,1]$ is called a continuous triangular norm (in short, continuous t-norm) if it satisfies the following conditions:

(i) T is commutative and associative;

(ii) T is continuous;

(iii) $T(a,1) = a, \forall a \in [0,1]$;

(iv) $T(a,b) \leq T(c,d)$ whenever $a \leq b$ and $c \leq d$. $a,b,c,d \in [0,1]$.

Generally t-norm T can be expressed (by associativity) in a unique way to an n-ary operator taking for $(x_1,x_2,\cdots,x_n) \in [0,1]^n, n \in N$, the value $T(x_1,x_2,\cdots,x_n)$ is defined, in [11], by

$$T^0_{i=1} = T^n_{i=1} x_i = T(T^{n-1}_{i=1} x_i, x_n) = T(x_1,x_2,\cdots,x_n).$$

Definition 2.2.[12] Let X be a nonempty set, m a positive integer and $f : X \to X$ an operator. $X = \bigcup_{i=1}^m X_i$ is a cyclic representation of X with respect to f if

(1) $X_i, i = 1,2,\cdots,m$ are nonempty sets;

(2) $f(X_1) \subset X_2, \cdots, f(X_{m-1}) \subset X_m, f(X_m) \subset X_1$.

Definition 2.3. [3] A fuzzy metric space is an ordered triple (X,M,T) such that X is a nonempty set, T is a continuous t-norm and M is a fuzzy set on $X \times X \times (0,\infty)$ satisfying the following conditions, for all $x,y,z \in X, s,t > 0$:

(i) $M(x,y,t) > 0$;

(ii) $M(x,y,t) = 1$ if and only if $x = y$;

(iii) $M(x,y,t) = M(y,x,t)$;

(iv) $T\Big(M(x,y,t), M(y,z,s)\Big) \leq M(x,z,t+s)$;

(v) $M(x,y,\cdot) : (0,\infty) \to (0,1]$ is continuous.

Definition 2.4. [5] Let (X,M,T) be a fuzzy metric space. Then

(i) A sequence $\{x_n\}$ in X is said to converge to $x \in X$, denoted by $x_n \to x$, if and only if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$, i.e. for each $r \in (0,1)$ and $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x, t) > 1 - r$ for all $n \geq n_0$.

(ii) A sequence $\{x_n\}$ is a G-Cauchy sequence if and only if $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ for any $p > 0$ and $t > 0$.

(iii) The fuzzy metric space (X, M, T) is called G-complete if every G-Cauchy sequence is convergent.

Definition 2.5. [14] A function $\phi : [0,1] \to [0,1]$ is called a comparison function if it satisfies

(1) ϕ is nondecreasing and left continuous;

(2) $\phi(t) > t$ for all $t \in (0,1)$.

Lemma 2.1. [14] *If ϕ be a comparison function, then*

(i) $\phi(1) = 1$

(ii) $\lim_{n\to+\infty} \phi^n(t) = 1$ *for all $t \in (0,1)$, where $\phi^n(t)$ denotes the composition of $\phi(t)$ with itself n times.*

With the inspiration from cyclic ϕ-contraction in [14] we present a contraction in fuzzy metric space, with P_{cl}, the collection of closed subsets of X.

Definition 2.6. Let (X, M, T) be a fuzzy metric space, m a positive integer, $A_1, A_2, \cdots, A_m \in P_{cl}(X), Y = \bigcup_{i=1}^m A_i$ and $f : Y \to Y$ an operator. If

(i) $\bigcup_{i=1}^m A_i$ is cyclic representation of Y with respect to f;

(ii) there exists a comparison function $\phi : [0,1] \to [0,1]$ such that

$$M(fx, fy, t) \geq \phi(\min\{M(x,y,t), M(x, fx, t), M(y, fy, t)\})$$

for any $x \in A_i, y \in A_{i+1}$ and $t > 0$, where $A_{m+1} = A_1$, then f is called cyclic ϕ- weak contraction in the fuzzy metric space (X, M, T).

Definition 2.7. [14] Let (X, M, T) be a fuzzy metric space and let $\{f_n\}$ be a sequence of self-mappings on X. $f_0 : X \to X$ is a given mapping. The sequence $\{f_n\}$ is said to converge uniformly to f_0 if for each $\varepsilon \in (0,1)$ and $t > 0$, there exists $n_0 \in N$ such that

$$M(f_n(x), f_0(x), t) > 1 - \varepsilon$$

for all $n \geq n_0$ and $x \in X$.

3. Main results

Theorem 3.1. *Let* (X,M,T) *be a G-complete fuzzy metric space, m a positive integer,* $A_1,A_2,\cdots,A_m \in P_{cl}(X), Y=\bigcup_{i=1}^m A_i, \phi:[0,1]\to[0,1]$ *a comparison function and* $f:Y\to Y$ *an operator. Assume that*

(i) $\bigcup_{i=1}^m A_i$ *is cyclic representation of* Y *with respect to* f*;*

(ii) f *is a cyclic* ϕ*-weak contraction.*

Then f *has a unique fixed point* $x' \in \bigcap_{i=1}^m A_i$ *and the iterative sequence* $\{x_n\}_{n\geq 0}, (x_n = f(x_{n-1})n \in N)$ *converges to* x' *for any starting point* $x_0 \in Y$.

Proof. Let $x_0 \in Y = \bigcap_{i=1}^m A_i$ be starting point, since $x_n = f(x_{n1})(n \geq 1)$, we have

$$M(x_n,x_{n+1},t) = M(f(x_{n-1}),f(x_n),t) \ for\ any\ t > 0.$$

For any $n \geq 0$, there exists $i_n \in 1,2,\cdots,m$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_{n+1}}$. Therefore, we can get

$$\begin{aligned} M(x_n,x_{n+1},t) &= M(f(x_{n-1}),f(x_n),t) \\ &\geq \phi(\min\{(M(x_{n-1},x_n,t),M(x_{n-1},fx_{n-1},t),M(x_n,fx_n,t))\}) \\ &= \phi(\min\{M(x_{n-1},x_n,t),M(x_{n-1},x_n,t),M(x_n,x_{n+1},t)\}) \\ &= \phi(\min\{M(x_{n-1},x_n,t),M(x_n,x_{n+1},t)\}). \end{aligned}$$

If $\min\{M(x_{n-1},x_n,t),M(x_n,x_{n+1},t)\} = M(x_n,x_{n+1},t)$ It leads a contradiction that $M(x_n,x_{n+1},t) > M(x_n,x_{n+1},t)$. Hence $\min\{M(x_{n-1},x_n,t),M(x_n,x_{n+1},t)\} = M(x_{n-1},x_n,t)$. Thus we get

$$M(x_n,x_{n+1},t) \geq \phi(M(x_{n-1},x_n,t)).$$

Using the definition of ϕ, we get by induction that

$$M(x_n,x_{n+1},t) \geq \phi^n(M(x_0,x_1,t)).$$

Therefore, for any $p > 0$, we have

$$\begin{aligned} M(x_n,x_{n+p},t) &\geq T(M(x_n,x_{n+1},t/p),M(x_{n+1},x_{n+2},t/p),\cdots,M(x_{n+p-1},x_{n+p},t/p)) \\ &\geq T(\phi^n(M(x_0,x_1,t/p)),\phi^{n+1}(M(x_0,x_1,t/p)),\cdots,\phi^{n+p-1}(M(x_0,x_1,t/p)) \\ &= T_{i=0}^{p-1}\phi^{n+i}(M(x_0,x_1,t/p)). \end{aligned}$$

According to Lemma [2.1], for every $i \in 0,1,\cdots,p-1$, we obtain that $\lim_{n\to\infty}\phi^{n+i}(M(x_0,x_1,t/p)) = 1$. As T is continuous t-norm, $M(x_n,x_{n+p},t) \to 1$ as $n \to \infty$. It shows that $\{x_n\}_{n\geq 0}$ is a G-Cauchy sequence in the G-complete subspace Y. Hence there is $x' \in Y$ such that $\lim_{n\to\infty} x_n = x'$.

Using the condition (i) in this theorem, it follows that the iterative sequence $\{x_n\}_{n\geq 0}$ has an infinite number of terms in each $A_i, i = 1,2,\cdots,m$. Since Y is G-complete, from each $A_i, i = 1,2,\cdots,m$, we can extract a subsequence of $\{x_n\}_{n\geq 0}$ which converges to x' as well. Because each $A_i, i = 1,2,\cdots,m$ is closed, we conclude that $x' \in \bigcap_{i=1}^m A_i$ and thus $\bigcap_{i=1}^m A_i$ is non empty. Set $Z = \bigcap_{i=1}^m A_i$ Obviously, Z is also closed and G-complete. Consider the restriction of f to Z, that is, $f|_Z : Z \to Z$. Next, we will prove that $f|_Z$ has a unique fixed point in $Z \subset Y$. Now $x' \in Z$, since $f|_Z(x') \in Z$ and $x_n \in A_{i_n}$, we can choose $A_{i_{n+1}}$ such that $f|_Z(x') \in A_{i_{n+1}}$. Hence, for any $t > 0$, we have

$$\begin{aligned} M(f|_Z(x'),x',t) &= M(f(x'),x',t) \\ &\geq T(M(f(x'),f(x_n),t/2),M(x_{n+1},x',t/2)) \\ &\geq T(\phi(x',x_n,t/2),M(x_{n+1},x',t/2)) \to T(1,1) = 1(n\to\infty). \end{aligned}$$

Clearly, we get $f|_Z(x') = x'$, namely, x' a fixed point, which is obtained by iteration from starting point x_0. To show uniqueness, we assume that $z \in \bigcap_{i=1}^m A_i$ is another fixed point of $f|_Z$. Since $x', z \in A_i$ for all $i \in N$, we can obtain

$$\begin{aligned} M(x',z,t) &= M(f|_Z(x'),f|_Z(z),t) \\ &= M(f(x'),f(z),t) \\ &\geq \phi(\min M(x',z,t),M(x',f(x'),t),M(z,f(z),t)) \\ &> M(x',z,t). \end{aligned}$$

This leads to a contradiction. Thus, x' is the unique fixed point of $f|_Z$ for any starting point $x_0 \in Z \subset Y$. Now, we still have to prove that the iterative sequence $x_n, n \geq 0$ converges to x' for any initial point $x_0 \in Y$. Let $x \in Y = \bigcup_{i=1}^{m} A_i$, there exists $i_0 \in 1,2,\cdots,m$ such that $x \in A_{i_0}$. As $x' \in \bigcap_{i=1}^{m} A_i$, it follows that $x' \in A_{i_0+1}$ as well. Then, for any $t > 0$, we have

$$M(f(x), f(x'), t) \geq \phi(M(x, x', t)).$$

By induction and Definition [2.6], we can obtain

$$\begin{aligned} M(x_n, x', t) &= M(f_n(x_0), x', t) \\ &= M(f_n(x_0), f(x'), t) \\ &= M(f(f_{n-1}(x_0)), f(x'), t) \\ &\geq \phi(\min\{(M(f_{n-1}(x_0), x', t), M(f_{n-1}(x_0), f(f_{n-1}(x_0)), t), M(x', f(x'), t))\}) \\ &\geq \phi(M(f_{n-1}(x_0), x', t)) \\ &\geq \phi^n(M(x_0, x', t)). \end{aligned}$$

Supposing $x_0 \neq x'$, it follows immediately that $x_n \to x'$ as $n \to \infty$. So the iterative sequence $\{x_n\}, n \geq 0$ converges to the unique fixed point x' of f for any starting point $x_0 \in Y$.

Definition 3.1. Let (X, M, T) be a fuzzy metric space, m a positive integer, $A_1, A_2, \cdots, A_m \in P_{cl}(X), Y = \bigcup_{i=1}^{m} A_i$ and $f : Y \to Y$ an operator. If

(i) $\bigcup_{i=1}^{m} A_i$ is cyclic representation of Y with respect to f;

(ii) there exists a comparison function $\psi : [0,1] \to [0,1]$ such that

$$M(fx, fy, t) \geq \psi(\min\{M(x, y, t), M(x, fx, t), M(y, fx, t)\}),$$

for any $x \in A_i, y \in A_{i+1}$ and $t > 0$, where $A_{m+1} = A_1$, then f is called cyclic ψ-contraction in the fuzzy metric space (X, M, T).

Theorem 3.2. *Let (X, M, T) be a G-complete fuzzy metric space, m a positive integer, $A_1, A_2, \cdots, A_m \in P_{cl}(X), Y = \bigcup_{i=1}^{m} A_i, \phi : [0,1] \to [0,1]$ a comparison function and $f : Y \to Y$ an operator. Assume that*

(i) $\bigcup_{i=1}^{m} A_i$ is cyclic representation of Y with respect to f;

(ii) f is a cyclic ψ-contraction.

Then f has a unique fixed point $x' \in \bigcap_{i=1}^m A_i$ and the iterative sequence $\{x_n\}_{n\geq 0}, (x_n = f(x_{n-1})n \in N)$ converges to x' for any starting point $x_0 \in Y$.

Proof. Let the point $x_0 \in Y = \bigcap_{i=1}^m A_i$ be a starting point. Since $x_n = f(x_{n-1})(n \geq 1)$, we have $M(x_n, x_{n+1}, t) = M(f(x_{n-1}), f(x_n), t)$ for any $t > 0$. Besides, for any $n \geq 0$, there exists $i_n \in 1, 2, \cdots, m$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_{n+1}}$. Therefore, we can get

$$\begin{aligned} M(x_n, x_{n+1}, t) &= M(f(x_{n-1}), f(x_n), t) \\ &\geq \psi(\min\{(M(x_{n-1}, x_n, t), M(x_{n-1}, fx_{n-1}, t), M(x_n, fx_{n-1}, t))\}) \\ &= \psi(\min\{M(x_{n-1}, x_n, t), M(x_{n-1}, x_n, t), M(x_n, x_n, t)\}) \\ &= \psi(\min\{M(x_{n-1}, x_n, t), 1\}) \\ &= \psi(M(x_(n-1), x_n, t)). \end{aligned}$$

Consider the definition of ψ, we get by induction that

$$M(x_n, x_{n+1}, t) \geq \psi^n(M(x_0, x_1, t)).$$

Thus, for any $p > 0$, we have

$$\begin{aligned} M(x_n, x_{n+p}, t) &\geq T(M(x_n, x_{n+1}, t/p), M(x_{n+1}, x_{n+2}, t/p), \cdots, M(x_{n+p-1}, x_{n+p}, t/p)) \\ &\geq T(\psi^n(M(x_0, x_1, t/p)), \psi^{n+1}(M(x_0, x_1, t/p)), \cdots, \psi^{n+p-1}(M(x_0, x_1, t/p)) \\ &= T_{i=0}^{p-1} \psi^{n+i}(M(x_0, x_1, t/p)). \end{aligned}$$

Using Lemma [2.1], for every $i \in 0, 1, \cdots, p-1$, we obtain that $\lim_{n\to\infty} \psi^{n+i}(M(x_0, x_1, t/p)) = 1$. As T is continuous t-norm, $M(x_n, x_{n+p}, t) \to 1$ as $n \to \infty$. It shows that $\{x_n\}_{n\geq 0}$ is a G-Cauchy sequence in the G-complete subspace Y. So there exists $x' \in Y$ such that $\lim_{n\to\infty} x_n = x'$.

Now using the condition (i) in this theorem, it follows that the iterative sequence $\{x_n\}_{n\geq 0}$ has an infinite number of terms in each $A_i, i = 1, 2, \cdots, m$. Since Y is G-complete, from each $A_i, i = 1, 2, \cdots, m$, one can extract a subsequence of $\{x_n\}_{n\geq 0}$ which converges to x' as well. Because each $A_i, i = 1, 2, \cdots, m$ is closed, we conclude that $x' \in \bigcap_{i=1}^m A_i$ and thus $\bigcap_{i=1}^m A_i$ is non empty. Set $Z = \bigcap_{i=1}^m A_i$ Obviously, Z is also closed and G-complete. Consider the restriction of

f to Z, that is, $f|_Z : Z \to Z$. Next, we will prove that $f|_Z$ has a unique fixed point in $Z \subset Y$. For the foregoing $x' \in Z$, since $f|_Z(x') \in Z$ and $x_n \in A_{i_n}$, we can choose $A_{i_{n+1}}$ such that $f|_Z(x') \in A_{i_{n+1}}$. Hence, for any $t > 0$, we have

$$\begin{aligned} M(f|_Z(x'),x',t) &= M(f(x'),x',t) \\ &\geq T(M(f(x'),f(x_n),t/2),M(x_{n+1},x',t/2)) \\ &\geq T(\psi(x',x_n,t/2),M(x_{n+1},x',t/2)) \to T(1,1) = 1(n \to \infty). \end{aligned}$$

Clearly, we get $f|_Z(x') = x'$ namely, x' a fixed point, which is obtained by iteration from starting point x_0. To show uniqueness, we assume that $z \in \bigcap_{i=1}^m A_i$ is another fixed point of $f|_Z$. Since $x', z \in A_i$ for all $i \in N$, we can obtain

$$\begin{aligned} M(x',z,t) &= M(f|_Z(x'),f|_Z(z),t) \\ &= M(f(x'),f(z),t) \\ &\geq \psi(\min M(x',z,t),M(x',f(x'),t),M(z,f(x'),t)) \\ &> M(x',z,t). \end{aligned}$$

This leads to a contradiction. Thus, x' is the unique fixed point of $f|_Z$ for any starting point $x_0 \in Z \subset Y$. Now, we still have to prove that the iterative sequence $x_n n \geq 0$ converges to x' for any initial point $x_0 \in Y$. Let $x \in Y = \bigcup_{i=1}^m A_i$, there exists $i_0 \in 1,2,\cdots,m$ such that $x \in A_{i_0}$. As $x' \in \bigcap_{i=1}^m A_i$, it follows that $x' \in A_{i_0+1}$ as well. Then, for any $t > 0$, we have

$$M(f(x),f(x'),t) \geq \psi(M(x,x',t)).$$

By induction and Definition [2.6], we can obtain

$$\begin{aligned} M(x_n,x',t) &= M(f_n(x_0),x',t) \\ &= M(f_n(x_0),f(x'),t) \\ &= M(f(f_{n1}(x_0)),f(x'),t) \end{aligned}$$

$$\geq \psi(\min\{(M(f_{n-1}(x_0),x',t),M(f_{n-1}(x_0),f(f_{n-1}(x_0)),t),M(x',f(f_{n-1}(x_0)),t))\})$$
$$\geq \psi(M(f_{n-1}(x_0),x',t))$$
$$\geq \psi^n(M(x_0,x',t)).$$

Supposing $x_0 \neq x'$, it follows immediately that $x_n \to x'$ as $n \to \infty$. So the iterative sequence $\{x_n\}, n \geq 0$ converges to the unique fixed point x' of f for any starting point $x_0 \in Y$.

Theorem 3.3. *Let $f : Y \to Y$ be a self-mapping as in Theorem [3.1]. If there exists an iterative sequence $\{y_n\} n \in N$ in Y such that $M(y_n, f(y_n),t) \to 1$ as $n \to \infty$ for any $t > 0$, then $y_n \to x'$ as $n \to \infty$.*

Proof. In view the proof of Theorem [3.1], we can find x' as unique fixed point of f for any starting point $x_0 \in Y$. Therefore, for any $t > 0$, we have

$$1 \geq M(y_n,x',t) \geq T(M(y_n, f(y_n),t/2),M(f(y_n), f(x'),t/2))$$
$$\geq T(M(y_n, f(y_n),t/2),\phi(\min\{M(y_n,x',t/2),M(y_n, f(y_n),t/2),M(x', f(x'),t/2)\}))$$
$$T(M(y_n, f(y_n),t/2),\phi^n(M(x_0,x',t/2))).$$

Since $M(y_n, f(y_n),t/2) \to 1$ and $\phi^n(M(x_0,x',t/2)) \to 1$ as $n \to \infty$, it shows that $M(y_n,x',t) \to 1$ which is equivalent to $y_n \to x'$ as $n \to \infty$.

Theorem 3.4. *Let $f : Y \to Y$ be a self-mapping as in Theorem [3.1]. If there exists a convergent sequence $\{y_n\}_{n\in N}$ in Y such that $M(y_{n+1}, f(y_n),t) \to 1$ as $n \to \infty$ for any $t > 0$, then there exists $x_0 \in Y$ such that $M(y_n, f^n(x_0),t) \to 1$ as $n \to \infty$.*

Proof. For any $t > 0$, let $y_n \in Y, n \in N$ such that $M(y_{n+1}, f(y_n),t) \to 1, n \to \infty$. Set y as a limit of $\{y_n\}_{n\in N}$. By the proof of previous Theorem we note that $x' \in \bigcap_{i=1}^m A_i$ is the unique fixed point of f for any starting point $x_0 \in Y$ and $t > 0$. Therefore, for any $t = t_1 +t_2$ with $t_1,t_2 > 0$ and $n \geq 0$, we have

$$M(y_{n+1},x',t) \geq T(M(y_{n+1}, f(y_n),t_1),M(f(y_n), f(x'),t_2)).$$

Now, Suppose that $M(y_{n+1},x',t) \neq 1, n \to \infty$, there exists $0 < \varepsilon < 1$ and $t > 0$ such that

$$\lim_{n\to\infty} M(y_{n+1},x',t) = M(y,x',t) = 1-\varepsilon.$$

Then there exists $0 < t_0 < t$ such that

$$M(y,x',t_0) \leq 1-\varepsilon$$

and

$$\limsup_{n\to\infty} M(y_n,x',t_0) = 1-\varepsilon.$$

Since $y_n \in Y = \bigcup_{i=1}^{m} A_i$ for each $n \geq 0$, there is $i_n \in 1,2,\cdots,m$ such that $y_n \in A_{i_n}$. But $x' \in \bigcap_{i=1}^{m} A_i$, so we can select one A_{i_n+1} such that $x' \in A_{i_{n+1}}$. Therefore, we can obtain

$$M(y_{n+1},x',t) \geq T(M(y_{n+1},f(y_n),t-t_0),\phi(M(y_n,x',t_0))),n \geq 0.$$

As T is continuous t-norm , we have

$$\begin{aligned}
1-\varepsilon = \lim_{n\to\infty} M(y_{n+1},x',t) &= M(y,x',t) \\
&\geq \limsup_{n\to\infty} T(M(y_{n+1},f(y_n),t-t_0),\phi(M(y_n,x',t_0))) \\
&= T(\limsup_{n\to\infty} M(y_{n+1},f(y_n),t-t_0),\limsup_{n\to\infty} \phi(M(y_n,x',t_0))) \\
&= T(1,\limsup_{n\to\infty} \phi(M(y_n,x',t_0))) \\
&= T(1,\limsup_{n\to\infty} \phi(M(y_n,x',t_0))) \\
&= \phi(1-\varepsilon) > 1-\varepsilon,
\end{aligned}$$

which is a contradiction. Hence, $M(y,x',t) = 1$, namely, $y = x'$. Thus, for any $t > 0$, we have $M(y_n,f^n(x_0),t) \to M(y,x',t)$ as $n \to \infty$.

Theorem 3.5. *Let $f : Y \to Y$ be a self-mapping as in Theorem [3.1] and $f_n : Y \to Y, n \in N$. Moreover if the following three conditions hold:*

(i) there exists a fixed point x'_n for each f_n;

(ii) $\{f_n\}n \in N$ converges uniformly to f;

(iii) the sequence $x'_n, n \in N$ is convergent.

Then, $x'_n \to x'$ as $n \to \infty$.

Proof. Suppose that $x'_n n \in N$ converges to x''. Since $\{f_n\}, n \in N$ converges uniformly to f, for any $\varepsilon \in (0,1)$ and $t > 0$, there exists an $n_0 \in N$ such that $M(f_n(x), f(x),t) > 1-\varepsilon$ for all

$n \geq n_0$ and $x \in Y$. That is, for every $x \in Y, M(f_n(x), f(x), t) \to 1$ as $n \to \infty$. By induction, for any $t = t_1 + t_2$ with $t_1, t_2 > 0$, we can easily get

$$\begin{aligned}
M(x'_n, x', t) &= M(f_n(x'_n), f(x'), t_1 + t_2) \\
&\geq T(M(f_n(x'_n), f(x'_n), t_1), M(f(x'_n), f(x'), t_2)) \\
&\geq T(M(f_n(x'_n), f(x'_n), t_1), \phi(\min\{(M(x'_n, x', t_2), M(x'_n, f(x'_n), t_2), M(x', f(x'), t_2))\})) \\
&= T(M(f_n(x'_n), f(x'_n), t_1), \phi(M(x'_n, x', t_2))).
\end{aligned}$$

Now, let us assume that $x'_n \neq x'$ as $n \to \infty$, i.e., there exist $\eta \in (0,1)$ and $t > 0$ such that

$$\lim_{n\to\infty} M(x'_n, x', t) = M(x'', x', t) = 1 - \eta.$$

Then there exists $0 < t_0 < t$ such that

$$M(x'', x', t_0) \leq 1 - \eta$$

and

$$\limsup_{n\to\infty} M(x'_n, x', t_0) = 1 - \eta.$$

Thus, we can have

$$\begin{aligned}
1 - \eta &= \lim_{n\to\infty} M(x'_n, x', t) = M(x'', x', t) \\
&\geq \limsup_{n\to\infty} T(M(f_n(x'_n), f(x,_n), t - t_0), \phi(\min\{(M(x'_n, x', t_0), M(x'_n, f(x'_n), t_0), M(x', f(x'), t_0))\}) \\
&= \limsup_{n\to\infty} T(M(f_n(x'_n), f(x,_n), t - t_0), \phi(M(x'_n, x', t_0)) \\
&= T(1, \limsup_{n\to\infty} \phi(M(x'_n, x', t_0)) \\
&= \limsup_{n\to\infty} \phi(M(x'_n, x', t_0)) \\
&= \phi(1 - \eta) > 1 - \eta,
\end{aligned}$$

which is not true. Hence, $M(x'_n, x', t) \to 1$ as $n \to \infty, i.e., x'_n \to x'$ as $n \to \infty$.

Theorem 3.6. *Let (X, M, T) be a G-complete fuzzy metric space, m a positive integer, $A_1, A_2, \cdots, A_m \in P_{cl}(X), Y = \bigcup_{i=1}^m A_i, \phi : [0,1] \to [0,1]$ a comparison function and $f : Y \to Y$ an operator. Assume that* *(i)* $\bigcup_{i=1}^{m} A_i$ *is cyclic representation of* Y *with respect to* f*;*

(ii) f *is a cyclic* ψ*-contraction.*

If there exists an iterative sequence $\{y_n\}_{n\in N}$ *in* Y *such that* $M(y_n, f(y_n), t) \to 1$ *as* $n \to \infty$ *for any* $t > 0$*, then* $y_n \to x'$ *as* $n \to \infty$*.*

Theorem 3.7. *Let* (X, M, T) *be a G-complete fuzzy metric space, m a positive integer,* $A_1, A_2, \cdots, A_m \in P_{cl}(X), Y = \bigcup_{i=1}^{m} A_i, \phi : [0,1] \to [0,1]$ *a comparison function and* $f : Y \to Y$ *an operator. Assume that*

(i) $\bigcup_{i=1}^{m} A_i$ *is cyclic representation of* Y *with respect to* f*;*

(ii) f *is a cyclic* ψ*-contraction.*

and $f_n : Y to Y, n \in N$*. Moreover if the following three conditions hold:*

(iii) there exists a fixed point x'_n *for each* f_n*;*

(iv) $\{f_n\} n \in N$ *converges uniformly to* f*;*

(v) the sequence $x'_n n \in N$ *is convergent.*

Then, $x'_n \to x'$ *as* $n \to \infty$*.*

Theorem 3.8. *Let* (X, M, T) *be a G-complete fuzzy metric space, m a positive integer,* $A_1, A_2, \cdots, A_m \in P_{cl}(X), Y = \bigcup_{i=1}^{m} A_i, \phi : [0,1] \to [0,1]$ *a comparison function and* $f : Y \to Y$ *an operator. Assume that*

(i) $\bigcup_{i=1}^{m} A_i$ *is cyclic representation of* Y *with respect to* f*;*

(ii) f *is a cyclic* ψ*-contraction.*

If there exists a convergent sequence $\{y_n\} n \in N$ *in* Y *such that* $M(y_{n+1}, f(y_n), t) \to 1$ *as* $n \to \infty$ *for any* $t > 0$*, then there exists* $x_0 \in Y$ *such that* $M(y_n, f^n(x_0), t) \to 1$ *as* $n \to \infty$*.*

Conflict of Interests

The authors declare that there is no conflict of interests.

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