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FIXED POINTS OF GENERALIZED CONTRACTION MAPS IN METRIC SPACES

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Abstract. In this paper, we introduce a new class Ψ_1 of functions which are different from Ψ introduced by Hussain, Parvaneh, Samet and Vetro [9]. We define JS - Ψ_1- contraction for a single selfmap and prove the existence of fixed points. Also, we extend JS - Ψ_1- contraction to a pair of selfmaps and prove the existence of coincidence points and prove the existence of common fixed points by assuming the weakly compatible property. Further, we study the existence of common fixed points for a pair of weakly compatible selfmaps satisfying property (E. A). Examples are provided to illustrate our results.

Keywords: coincidence point; point of coincidence; common fixed point; property (E.A); weakly compatible maps.

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1. Introduction and preliminaries

It is well known that fixed point theory has wide applications in applied sciences. Banach contraction principle [6] which states that if (X,d) is complete metric space and $f : X \to X$ is a contraction map then f has a unique fixed point, is a fundamental result in this theory. Due to its importance and simplicity several authors have obtained many interesting extensions and

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generalizations of Banach contraction principle, some generalizations of contraction condition was obtained in ([7] - [10], [13]). Recently Hussain, Parvaneh, samet and Vetro [9] introduced a new contraction map, namely JS-contraction map and proved the existence and uniqueness of fixed points in complete metric spaces.

In 2002, Aamari and Moutawakil [1] introduced the notion of property (E. A). Different authors (G. V. R. Babu and G.N. Alemayehu [4], S.Mudgal [15], Talat Nazir and Mujahid Abbas [17]) applied this concept to prove the existence of common fixed points in metric spaces.

Throughout this paper, (X, d) denotes a metric space and we write it by X, f and g are selfmaps of X and $\mathbb{N}$ stands for the set of all natural numbers.

Definition 1.1. [13] Let f and g be selfmaps on a metric space (X,d). If $fx = gx = w$ for some $x \in X$, then x is called a coincidence point of f and g and the set of all coincidence points of f and g is denoted by $C(f,g)$, and w is called point of coincidence of f and g.

Definition 1.2. [11] A pair (f,g) of selfmaps on a metric space (X,d) is said to be compatible if $\lim_{n\to\infty} d(gfx_n, fgx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some z in X.

Definition 1.3. A pair (f,g) of selfmaps on a metric space (X,d) is said to be noncompatible if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some z in X but $\lim_{n\to\infty} d(gfx_n, fgx_n)$ is either non-zero or does not exist.

Definition 1.4. [1] A pair (f,g) of selfmaps on a metric space (X,d) is said to be satisfy property $(E.A)$ if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some z in X.

Remark 1.1. [3] Every pair of noncompatible selfmaps of a metric space (X,d) satisfies property (E. A), but its converse need not be true [See example 1.3 [3]].

Definition 1.5. [12] A pair (f,g) of selfmaps on a metric space (X,d) is said to be weakly compatible if $fgx = gfx$ whenever $fx = gx$ for any x in X.

Jleli and Samet [10] introduced the class of functions Φ, where Φ is the set of function $\phi : [0,\infty) \to [1,\infty)$ satisfying the conditions;

(i) ϕ is non-decreasing,

(ii) for each sequence $\{t_n\} \subseteq (0,\infty)$, $\lim_{n\to\infty}\phi(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$ and

(iii) there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that $\lim_{t\to 0^+} \frac{\phi(t)-1}{t^r} = \ell$, and proved the existence of fixed points in generalized metric spaces.

Theorem 1. 1. (Corollary 2.1 [10]) Let (X,d) be a complete metric space and $f : X \to X$ be a given map. Suppose that there exist $\phi \in \Phi$ and $k \in (0,1)$ such that $x,y \in X$,

$d(fx,fy) \neq 0 \implies \phi(d(fx,fy)) \leq [\phi(d(x,y))]^k$.

Then f has a unique fixed point.

The above theorem is a generalization of Banach contraction Principle.

In continuation to this study, Hussain, Parvaneh, Samet and Vetro[9] introduced a new class of functions Ψ and defined a new contraction condition, namely JS-contraction.

Ψ is the set of all functions $\psi : [0,\infty) \to [1,\infty)$ satisfying the following conditions:

(ψ_1) ψ is nondecreasing and $\psi(t) = 1$ if and only if $t = 0$;

(ψ_2) for each sequence $\{t_n\} \subseteq (0,\infty)$, $\lim_{n\to\infty}\psi(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;

(ψ_3) there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that $\lim_{t\to 0^+} \frac{\psi(t)-1}{t^r} = \ell$;

(ψ_4) $\psi(a+b) \leq \psi(a)\psi(b)$ for all $a,b > 0$.

Definition 1.6. [9] Let (X,d) be a metric space. A selfmap $f : X \to X$ is said to be JS-contraction if there exist a function $\psi \in \Psi$ and positive real numbers k_1,k_2,k_3,k_4 with $0 \leq k_1+k_2+k_3+2k_4 < 1$ such that

$$\begin{aligned} \psi(d(fx,fy)) \leq & \big[\psi(d(x,y))\big]^{k_1}\big[\psi(d(x,fx))\big]^{k_2}\big[\psi(d(y,fy))\big]^{k_3} \\ & \times \big[\psi(d(x,fy)+d(y,fx))\big]^{k_4} \text{ for all } x,y \in X. \end{aligned} \tag{1}$$

Theorem 1.2. [9] Let (X,d) be a complete metric space and $f : X \to X$ be a continuous JS-contraction. Then f has a unique fixed point.

We observe the following from the class of function Ψ.

Proposition 1.1. If $\psi \in \Psi$ then ψ is continuous from the right.

proof. Let $t_0 \in (0,\infty)$. Let $t > t_0$, write $t = t_0 + h$ for some $h > 0$. Now,

$$\psi(t) - \psi(t_0) = \psi(t_0+h) - \psi(t_0) \leq \psi(t_0)\psi(h) - \psi(t_0) = \psi(t_0)(\psi(h)-1). \tag{2}$$

Hence $\lim_{t\to t_0^+}(\psi(t)-\psi(t_0)) \le \lim_{h\to 0^+}\psi(t_0)(\psi(h)-1) = \psi(t_0)(0) = 0$
so that $\lim_{t\to t_0^+}\psi(t) = \psi(t_0)$.
Hence ψ is continuous at t_0 from the right of t_0.

If we allow left continuity to $\psi \in \Psi$ then ψ is continuous. In this case (ψ_2) of Ψ follows trivially from the continuity of ψ. Hence we define a new class of functions Ψ_1 as follows:

$$\Psi_1 = \Big\{\psi : [0,\infty) \to [1,\infty) \Big| \text{ (i) } \psi \text{ is nondecreasing, (ii) } \psi \text{ is continuous, (iii) } \psi(t) = 1 \text{ if and onily if } t = 0, \text{ and (iv) } \psi(a+b) \le \psi(a)\psi(b) \text{ for all } a,b > 0\Big\}.$$

The following example suggests that the class Ψ_1 is different from the class Ψ.

Example 1. *We define $\psi : [0,\infty) \to [1,\infty)$ by $\psi(t) = e^t$, $t \ge 0$. Then, clearly $\psi \in \Psi_1$, but $\psi \notin \Psi$, since (ψ_3) fails to hold; for, let $r \in (0,1)$, we consider*
$\lim_{t\to 0^+}\frac{\psi(t)-1}{t^r} = \lim_{t\to 0^+}\frac{e^t-1}{t^r} = \lim_{t\to 0^+}\frac{e^t}{rt^{r-1}} = \frac{1}{r}\lim_{t\to 0^+}e^t t^{1-r} = 0$
so that $\ell = \lim_{t\to 0^+}\frac{\psi(t)-1}{t^r} = 0 \notin (0,\infty]$. Hence $\Psi_1 \nsubseteq \Psi$.

In Section 2, we define JS - Ψ_1 - contraction for a single selfmap and give some fixed point results for JS - Ψ_1 - contraction maps in complete metric spaces. In Section 3, we define JS -Ψ_1 - contraction for a pair of selfmaps and prove the existence of coincidence points and extending this to the existence of common fixed points by using weakly compatible property. In the last section we study the existence of common fixed points for a pair of weakly compatible selfmaps satisfying property (E. A).

The following lemma is useful in our subsequent discussions.

Lemma 1.1. [5] Suppose (X,d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n,x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k \ge k$ such that $d(m_k,n_k) \ge \varepsilon$.
For each $k > 0$, corresponding to m_k, we can choose n_k to be the smallest positive integer such that $d(x_{m_k},x_{n_k}) \ \ge \ \varepsilon, d(x_{m_k},x_{n_k-1}) < \varepsilon$ and

(i) $\lim_{k\to\infty} d(x_{n_k-1},x_{m_k+1}) = \varepsilon$ (ii) $\lim_{k\to\infty} d(x_{n_k},x_{m_k}) = \varepsilon$
(iii) $\lim_{k\to\infty} d(x_{m_k-1},x_{n_k}) = \varepsilon$ and (iv) $\lim_{k\to\infty} d(x_{n_k},x_{m_k+1}) = \varepsilon$.

2. Fixed points of JS - Ψ_1 - contractions

In the following, we introduce a JS -Ψ_1 - contraction by using a function $\psi \in \Psi_1$ and prove the existence of fixed points in complete metric spaces.

Definition 2.1. Let (X,d) be a metric space and $f: X \to X$ be selfmap. If there exist a function $\psi \in \Psi_1$ and nonnegative real numbers k_1, k_2, k_3 and k_4 with $0 \leq k_1 + k_2 + k_3 + 2k_4 < 1$ such that

$$\begin{aligned}\psi(d(fx,fy)) \leq & \big[\psi(d(x,y))\big]^{k_1}\big[\psi(d(x,fx))\big]^{k_2}\big[\psi(d(y,fy))\big]^{k_3} \\ & \times \big[\psi(d(x,fy)+d(y,fx))\big]^{k_4} \text{ for all } x,y \in X,\end{aligned} \tag{3}$$

then we say that f is a JS - Ψ_1 - contraction.

Here we observe that every contraction map with contraction constant $k \in [0,1)$ is a JS - Ψ_1 - contraction with $\psi(t) = e^t$, $t \geq 0$. But its converse is not true (Example 2).

Theorem 2.1. Let (X,d) be a complete metric space and $f: X \to X$ be a JS - Ψ_1 - contraction. Then f has a unique fixed point.

Proof. Let x_0 be an arbitrary point in X. We define an iterative sequence $\{x_n\}$ by $x_1 = fx_0$, $x_{n+1} = fx_n$ for $n = 0,1,2,\ldots$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then we have $fx_n = x_n$ so that x_n is a fixed point of f and we are through.

Without loss of generality, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.
First we show that $d(x_n,x_{n+1}) \to 0$ as $n \to \infty$. Since f is JS - Ψ_1 - contraction, we have

$$\begin{aligned}\psi(d(x_n,x_{n+1})) &= \psi(d(fx_{n-1},fx_n)) \\ &\leq \big[\psi(d(x_{n-1},x_n))\big]^{k_1}\big[\psi(d(x_{n-1},fx_{n-1}))\big]^{k_2} \\ &\quad \times \big[\psi(d(x_n,fx_n))\big]^{k_3}\big[\psi(d(x_{n-1},fx_n)+d(x_n,fx_{n-1}))\big]^{k_4} \\ &= \big[\psi(d(x_{n-1},x_n))\big]^{k_1}\big[\psi(d(x_{n-1},x_n))\big]^{k_2}\big[\psi(d(x_n,x_{n+1}))\big]^{k_3} \\ &\quad \times \big[\psi(d(x_{n-1},x_{n+1})+d(x_n,x_n))\big]^{k_4} \\ &\leq \big[\psi(d(x_{n-1},x_n))\big]^{k_1}\big[\psi(d(x_{n-1},x_n))\big]^{k_2}\big[\psi(d(x_n,x_{n+1}))\big]^{k_3} \\ &\quad \times \big[\psi(d(x_{n-1},x_n)+d(x_n,x_{n+1}))\big]^{k_4}\end{aligned}$$

$$\leq \big[\psi(d(x_{n-1},x_n))\big]^{k_1+k_2}\big[\psi(d(x_n,x_{n+1}))\big]^{k_3}\big[\psi(d(x_{n-1},x_n))\big]^{k_4}$$
$$\times \big[\psi(d(x_n,x_{n+1}))\big]^{k_4}$$
$$= \big[\psi(d(x_{n-1},x_n))\big]^{k_1+k_2+k_4}\big[\psi(d(x_n,x_{n+1}))\big]^{k_3+k_4}.$$

Hence it follows that

$$\psi(d(x_n,x_{n+1})) \leq \big[\psi(d(x_{n-1},x_n))\big]^{\frac{k_1+k_2+k_4}{1-k_3-k_4}} \leq \big[\psi(d(x_{n-2},x_{n-1}))\big]^{(\frac{k_1+k_2+k_4}{1-k_3-k_4})^2}$$
$$\leq \cdots \leq \big[\psi(d(x_0,x_1))\big]^{(\frac{k_1+k_2+k_4}{1-k_3-k_4})^n} \to 1 \text{ as } n\to\infty,$$

so that $\psi(d(x_n,x_{n+1})) \to 1$ as $n\to\infty$.

Hence by the property (ii) and (iii) of ψ we have $d(x_n,x_{n+1}) \to 0$ as $n\to\infty$.

Now, we show that $\{x_n\}$ is a Cauchy sequence. Suppose $\{x_n\}$ is not Cauchy. Then by Lemma 1.1 there exist $\varepsilon > 0$ and sequence of positive integers $\{n_k\}$ and $\{m_k\}$ such that $n_k > m_k \geq k$ satisfying

$$d(x_{m_k},x_{n_k}) \geq \varepsilon. \tag{4}$$

Let us choose the smallest n_k satisfying (4), then we have $n_k > m_k \geq k$ with $d(x_{m_k},x_{n_k}) \geq \varepsilon$ and $d(x_{m_k},x_{n_k-1}) < \varepsilon$ satisfying (i)- (iv) of Lemma 1.1 . Now, by using the inequality (3) we have

$$\psi(d(x_{n_k},x_{m_k})) = \psi(d(fx_{n_k-1},fx_{m_k-1}))$$
$$\leq \big[\psi(d(x_{n_k-1},x_{m_k-1}))\big]^{k_1}\big[\psi(d(fx_{n_k-1},x_{n_k-1}))\big]^{k_2}\big[\psi(d(fx_{m_k-1},x_{m_k-1}))\big]^{k_3}$$
$$\times \big[\psi(d(fx_{n_k-1},x_{m_k-1})+d(x_{n_k-1},fx_{m_k-1}))\big]^{k_4}$$
$$= \big[\psi(d(x_{n_k-1},x_{m_k-1}))\big]^{k_1}\big[\psi(d(x_{n_k},x_{n_k-1}))\big]^{k_2}\big[\psi(d(x_{m_k},x_{m_k-1}))\big]^{k_3}$$
$$\times \big[\psi(d(x_{n_k},x_{m_k-1})+d(x_{n_k-1},x_{m_k}))\big]^{k_4}$$
$$\leq \big[\psi(d(x_{n_k-1},x_{m_k-1}))\big]^{k_1}\big[\psi(d(x_{n_k},x_{n_k-1}))\big]^{k_2}\big[\psi(d(x_{m_k},x_{m_k-1}))\big]^{k_3}$$
$$\times \big[\psi(d(x_{n_k},x_{m_k-1})\big]^{k_4}\big[\psi(d(x_{n_k-1},x_{m_k}))\big]^{k_4}.$$

On letting $k\to\infty$ we have

$$\psi(\varepsilon) \leq \big[\psi(\varepsilon)\big]^{k_1}\big[\psi(0)\big]^{k_2}\big[\psi(0)\big]^{k_3}\big[\psi(\varepsilon)\big]^{k_4}\big[\psi(\varepsilon)\big]^{k_4}.$$

By using the property $\psi(0) = 1$, we have

$$\psi(\varepsilon) \leq \big[\psi(\varepsilon)\big]^{k_1+2k_4} < \psi(\varepsilon), \text{ a contradiction.}$$

Hence $\{x_n\}$ a Cauchy sequence.

Since (X,d) is complete, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

We now show that u is a fixed point of f.

Suppose $u \neq fu$. From the inequality (3) we have

$$\begin{aligned}
\psi(d(x_n,fu)) = \psi(d(fx_{n-1},fu)) &\leq \big[\psi(d(x_{n-1},u))\big]^{k_1}\big[\psi(d(x_{n-1},fx_{n-1}))\big]^{k_2} \\
&\quad \times \big[\psi(d(u,fu))\big]^{k_3}\big[\psi(d(x_{n-1},fu)+d(u,fx_{n-1}))\big]^{k_4} \\
&= \big[\psi(d(x_{n-1},u))\big]^{k_1}\big[\psi(d(x_{n-1},x_{n+1}))\big]^{k_2} \\
&\quad \times \big[\psi(d(u,fu))\big]^{k_3}\big[\psi(d(x_{n-1},fu)+d(u,x_n))\big]^{k_4}.
\end{aligned}$$

On letting $n \to \infty$ we have

$$\begin{aligned}
\psi(d(u,fu)) &\leq \big[\psi(d(u,u))\big]^{k_1}\big[\psi(d(u,u))\big]^{k_2}\big[\psi(d(u,fu))\big]^{k_3}\big[\psi(d(u,fu)+d(u,u))\big]^{k_4} \\
&= \big[\psi(0)\big]^{k_1+k_2+k_4}\big[\psi(d(u,fu))\big]^{k_3+k_4} \\
&= \big[\psi(d(u,fu))\big]^{k_3+k_4} < \psi(d(u,fu)),
\end{aligned}$$

a contradiction.

Therefore $fu = u$, so that u is a fixed point of f.

Now, we show that u is a unique fixed point of f.

Let u and v be two fixed points of f with $u \neq v$. Hence $fu = u$ and $fv = v$.

$$\begin{aligned}
\psi(d(v,u)) &= \psi(d(fv,fu)) \\
&\leq \big[\psi(d(v,u))\big]^{k_1}\big[\psi(d(v,fv))\big]^{k_2}\big[\psi(d(u,fu))\big]^{k_3} \\
&\quad \times \big[\psi(d(v,fu)+d(u,fv))\big]^{k_4} \\
&= \big[\psi(d(v,u))\big]^{k_1}\big[\psi(d(v,v))\big]^{k_2}\big[\psi(d(u,u))\big]^{k_3}\big[\psi(d(v,u)+d(u,v))\big]^{k_4} \\
&\leq \big[\psi(d(v,u))\big]^{k_1}\big[\psi(d(v,v))\big]^{k_2}\big[\psi(d(u,u))\big]^{k_3}\big[\psi(d(v,u)\big]^{k_4}\big[\psi(d(u,v))\big]^{k_4}.
\end{aligned}$$

Hence $\psi(d(v,u)) \leq \big[\psi(d(v,u))\big]^{k_1+2k_4} < \psi(d(v,u))$,

a contradiction. Therefore $u = v$.

Hence f has a unique fixed point.

Corollary 2.2. Let (X,d) be a complete metric space and $f : X \to X$ be a selfmap. If there exist nonnegative real numbers k_1, k_2, k_3 and k_4 with $0 \le k_1 + k_2 + k_3 + 2k_4 < 1$ such that

$$
\begin{aligned}
d(fx, fy) \le\ & k_1 d(x,y) + k_2 d(x, fx) + k_3 d(y, fy) \\
& + k_4[d(x, fy) + d(y, fx)] \text{ for all } x, y \in X,
\end{aligned} \tag{5}
$$

then f has a unique fixed point.

Proof. From the inequality (5), we have

$$
e^{d(fx,fy)} \le e^{k_1 d(x,y)}.e^{k_2 d(x,fx)}.e^{k_3 d(y,fy)}.e^{k_4[d(x,fy)+d(y,fx)]} \text{ for all } x, y \in X. \tag{6}
$$

By choosing $\psi : [0,\infty) \to [1,\infty)$ defined by $\psi(t) = e^t,\ t \ge 0$, then $\psi \in \Psi_1$, and the inequality (6) is of the form (3). Thus by Theorem 2.1, f has a unique fixed point in X.

Example 2. *Let $X = [4,5]$ with the usual metric. We define $f : X \to X$ by*

$$
fx = \begin{cases} 5 - \frac{1}{x} & \text{if } x \in [4,5] \setminus \{4.25\}; \\ 5 & \text{if } x = 4.25. \end{cases}
$$

We define $\psi : [0,\infty) \to [1,\infty)$ by $\psi(t) = 2^t$. Clearly, $\psi \in \Psi_1$.
We now verify the inequality (3)*.*
Since $\psi(t) = 2^t$, f satisfies the inequality (3) *if and only if f satisfies the following inequality:*

$$
\begin{aligned}
|fx - fy| \ \le\ & k_1|x-y| + k_2|x - fx| + k_3|y - fy| \\
& + k_4\big[|x - fy| + |y - fx|\big].
\end{aligned} \tag{7}
$$

Hence, for the function f defined in this example, we verify the inequality (7) *with $k_1 = \frac{1}{16}$, $k_2 = \frac{1}{3}$, $k_3 = \frac{1}{3}$, $k_4 = \frac{1}{48}$*
<u>*Case*</u> *(i): $x, y \in [4,5] \setminus \{4.25\}$.*
In this case, $f(x) = 5 - \frac{1}{x}$ and $f(y) = 5 - \frac{1}{y}$.

Now, we have

$$\begin{aligned}|fx-fy| = |\frac{1}{x}-\frac{1}{y}| = |\frac{1}{xy}|x-y| &\leq \frac{1}{16}|x-y| \\ &\leq \frac{1}{16}|x-y| + \frac{1}{3}\,|5-\frac{x^2+1}{x}| + \frac{1}{3}|5-\frac{y^2+1}{y}| \\ &\quad + \frac{1}{48}\left[|x+\frac{1}{y}-5| + |y+\frac{1}{x}-5|\right] \\ &= k_1|x-y| + k_2|x-fx| + k_3|y-fy| \\ &\quad + k_4\left[|x-fy| + |y-fx|\right].\end{aligned}$$

<u>*Case*</u> *(ii):* $x \in [4,5]\setminus\{4.25\},\ y = 4.25.$

In this case, $f(x) = 5-\frac{1}{x}$ *and* $f(y) = 5.$

$$\begin{aligned}|fx-fy| = |5-\frac{1}{x}-5| = \frac{1}{x} \leq \frac{1}{4} &\leq \frac{1}{16}|x-4.25| + \frac{1}{3}|5-\frac{x^2+1}{x}| + \frac{1}{3}|0.75| \\ &\quad + \frac{1}{48}\left[|x-5| + |0.75-\frac{1}{x}|\right] \\ &= k_1|x-y| + k_2|x-fx| + k_3|y-fy| \\ &\quad + k_4\left[|x-fy| + |y-fx|\right].\end{aligned}$$

<u>*Case*</u> *(iii):* $y \in [4,5]\setminus\{4.25\},\ x = 4.25.$

In this case, $f(y) = 5-\frac{1}{y}$ *and* $f(x) = 5.$

$$\begin{aligned}|fx-fy| = |5-\frac{1}{y}-5| = \frac{1}{y} \leq \frac{1}{4} &\leq \frac{1}{16}|y-4.25| + \frac{1}{3}|0.75| + \frac{1}{3}|5-\frac{y^2+1}{y}| \\ &\quad + \frac{1}{48}\left[|y-5| + |0.75-\frac{1}{y}|\right] \\ &= k_1|x-y| + k_2|x-fx| + k_3|y-fy| \\ &\quad + k_4\left[|x-fy| + |y-fx|\right].\end{aligned}$$

Hence from all the above cases f *satisfies the inequality* (7). *Hence* f *satisfies all the hypotheses of Theorem 2.1 and* $x = \frac{5+\sqrt{21}}{2}$ *is the unique fixed point of* f.

Here we observe that

$$fx = \begin{cases} 5-\frac{1}{x} & \text{if } x \in [4,5]\setminus\{4.25\}; \\ 5 & \text{if } x = 4.25 \end{cases}$$

is not continuous at $x = 4.25$, hence Theorem 1.2. is not applicable.

Remark 2.3. Banach fixed point theorem [6], Chatteirjea fixed point theorem [7], Kannan fixed point theorem [14] and Reich fixed point theorem [16] follows as a corollaries to Corollary 2.2.

3. Point of coincidence and common fixed point theorems

In the following we extend JS - Ψ_1 - contraction for a single selfmap to a pair of selfmaps.

Definition 3.1. Let (X,d) be a metric space. Let $f,g : X \to X$ be selfmaps. If there exist a function $\psi \in \Psi_1$ and nonnegative real numbers k_1,k_2,k_3 and k_4 with $0 \leq k_1+k_2+k_3+2k_4 < 1$ such that

$$\begin{aligned}\psi(d(fx,fy)) \leq \left[\psi(d(gx,gy))\right]^{k_1}\left[\psi(d(gx,fx))\right]^{k_2}\left[\psi(d(gy,fy))\right]^{k_3} \\ \times\left[\psi(d(gx,fy)+d(gy,fx))\right]^{k_4} \text{ for all } x,y \in X,\end{aligned} \tag{8}$$

then we say that f is a JS - Ψ_1 - contraction with respect to g.

Remark 3.2. If g is the identity map in (8) then f is a JS - Ψ_1 - contraction.

Theorem 3.3. Let (X,d) be a metric space and $f,g : X \to X$ be selfmaps of X, with $fX \subset gX$. If f is a JS -Ψ_1 - contraction with respect to g, then for any $x_0 \in X$, the Picared iterates $\{y_n\}$ defined by $y_n = fx_n = gx_{n+1}$ for $n = 0,1,2,\dots$ is Cauchy sequence in X.

Proof. Let $x_0 \in X$. Since $fX \subset gX$ there exists $x_1 \in X$ such that $fx_0 = gx_1 = y_0$ (say). Further corresponding to x_1, there exists $x_2 \in X$ such that $fx_1 = gx_2 = y_1$ (say). On continuing this process, inductively we obtain a sequence $\{y_n\}$ in X such that

$$y_n = fx_n = gx_{n+1},\ n = 0,1,2,.... \tag{9}$$

Now, we consider the following cases.

<u>*Case*</u> (i): Suppose $y_n = y_{n+1}$ for some $n \in \mathbb{N}$.

From the inequality (8), we have

$$\begin{aligned}\psi(d(y_{n+1},y_{n+2})) &= \psi(d(fx_{n+1},fx_{n+2})) \\ &\leq \left[\psi(d(gx_{n+1},gx_{n+2}))\right]^{k_1}\left[\psi(d(gx_{n+1},fx_{n+1}))\right]^{k_2} \\ &\quad \times\left[\psi(d(gx_{n+2},fx_{n+2}))\right]^{k_3}\left[\psi(d(gx_{n+1},fx_{n+2})+d(gx_{n+2},fx_{n+1}))\right]^{k_4}.\end{aligned}$$

This implies that

$$\begin{aligned}\psi(d(y_{n+1},y_{n+2})) &\le [\psi(d(y_n,y_{n+1}))]^{k_1}[\psi(d(y_n,y_{n+1}))]^{k_2}[\psi(d(y_{n+1},y_{n+2}))]^{k_3}\\ &\quad\times[\psi(d(y_n,y_{n+2})+d(y_{n+1},y_{n+1}))]^{k_4}\\ &\le [\psi(d(yx_n,y_{n+1}))]^{k_1}[\psi(d(y_n,y_{n+1}))]^{k_2}[\psi(d(y_{n+1},y_{n+2}))]^{k_3}\\ &\quad\times[\psi(d(y_n,y_{n+1}))]^{k_4}[\psi(d(y_{n+1},y_{n+2}))]^{k_4}\\ &= [\psi(d(y_n,y_{n+1}))]^{k_1+k_2+k_4}[\psi(d(y_{n+1},y_{n+2}))]^{k_3+k_4}.\end{aligned}$$

Now, by using the property $\psi(0)=1$, we have

$\psi(d(y_{n+1},y_{n+2})) \le (\psi(d(y_{n+1},y_{n+2})))^{k_3+k_4} < \psi(d(y_{n+1},y_{n+2}))$,

a contradiction if $y_{n+1}\neq y_{n+2}$.

Therefore $y_{n+2}=y_{n+1}=y_n$.

Similarly, we can show that $y_{n+3}=y_{n+2}=y_{n+1}=y_n$.

This implies that $y_m=y_n$ for all $m\ge n$, so that $\{y_m\}_{m\ge n}$ is constant sequence.

Hence $\{y_m\}$ is a Cauchy sequence.

Case (ii): $y_n\neq y_{n+1}$ for all $n\in\mathbb{N}$.

First we show that $d(y_n,y_{n+1})\to 0$ as $n\to\infty$.

From the inequality (8) we have

$$\begin{aligned}\psi(d(y_n,y_{n+1})) &= \psi(d(fx_n,fx_{n+1}))\\ &\le \left[\psi(d(gx_n,gx_{n+1}))\right]^{k_1}\left[\psi(d(gx_n,fx_n))\right]^{k_2}\\ &\quad\times\left[\psi(d(gx_{n+1},fx_{n+1}))\right]^{k_3}\left[\psi(d(gx_{n+1},fx_n)+d(gx_n,fx_{n+1}))\right]^{k_4}.\end{aligned}$$

This implies that

$$\begin{aligned}\psi(d(y_n,y_{n+1})) &\le \left[\psi(d(y_{n-1},y_n))\right]^{k_1}\left[\psi(d(y_{n-1},y_n))\right]^{k_2}\left[\psi(d(y_n,y_{n+1}))\right]^{k_3}\\ &\quad\times\left[\psi(d(y_n,y_n)+d(y_{n-1},y_{n+1}))\right]^{k_4}\\ &\le \left[\psi(d(y_{n-1},y_n))\right]^{k_1+k_2}\left[\psi(d(y_n,y_{n+1}))\right]^{k_3}\\ &\quad\times\left[(d(y_{n-1},y_n)+d(y_n,y_{n+1}))\right]^{k_4}.\end{aligned}$$

Hence it follows that

$\psi(d(y_n,y_{n+1})) \le \left[\psi(d(y_{n-1},y_n))\right]^{k_1+k_2+k_4}\left[\psi(d(y_n,y_{n+1}))\right]^{k_3+k_4}$.

This implies that

$$\psi(d(y_n,y_{n+1})) \leq \big[\psi(d(y_{n-1},y_n))\big]^{\frac{k_1+k_2+k_4}{1-k_3-k_4}} \leq \big[\psi(d(y_{n-2},y_{n-1}))\big]^{(\frac{k_1+k_2+k_4}{1-k_3-k_4})^2}$$
$$\leq \cdots \leq \big[\psi(d(y_0,y_1))\big]^{(\frac{k_1+k_2+k_4}{1-k_3-k_4})^n} \to 1 \text{ as } n \to \infty$$

so that $\psi(d(y_n,y_{n+1})) \to 1$ as $n \to \infty$.

Hence by the property (ii) and (iii) of ψ we have $d(y_n,y_{n+1}) \to 0$ as $n \to \infty$.

Now, we show that $\{y_n\}$ is Cauchy. Suppose $\{y_n\}$ is not a Cauchy sequences. Then by Lemma 1.1 there exist $\varepsilon > 0$ and sequence of positive integers $\{n_k\}$ and $\{m_k\}$ such that $n_k > m_k \geq k$ satisfying

$$d(y_{m_k},y_{n_k}) \;\geq\; \varepsilon. \tag{10}$$

Let us choose the smallest n_k satisfying (10), then we have $n_k > m_k \geq k$ with $d(y_{m_k},y_{n_k}) \;\geq\; \varepsilon$ and $d(y_{m_k},y_{n_k-1}) \; < \varepsilon$, satisfying (i) - (iv) of Lemma 1.1

Now we consider

$$\begin{aligned}\psi(d(y_{n_k},y_{m_k})) &= \psi(d(fx_{n_k},fx_{m_k}))\\ &\leq \big[\psi(d(gx_{n_k},gx_{m_k}))\big]^{k_1}\big[\psi(d(gx_{n_k},fx_{n_k}))\big]^{k_2}\\ &\quad\times\big[\psi(d(gx_{m_k},fx_{m_k}))\big]^{k_3}\big[\psi(d(gx_{m_k},fx_{n_k})+d(gx_{n_k},fx_{m_k}))\big]^{k_4}.\end{aligned}$$

This implies that

$$\begin{aligned}\psi(d(y_{n_k},y_{m_k})) &\leq \big[\psi(d(y_{n_k-1},y_{m_k-1}))\big]^{k_1}\big[\psi(d(y_{n_k-1},y_{n_k}))\big]^{k_2}\big[\psi(d(y_{m_k-1},y_{m_k}))\big]^{k_3}\\ &\quad\times\big[\psi(d(y_{m_k-1},y_{n_k})+d(y_{n_k-1},y_{m_k}))\big]^{k_4}\\ &\leq \big[\psi(d(y_{n_k-1},y_{m_k-1}))\big]^{k_1}\big[\psi(d(y_{n_k-1},y_{n_k}))\big]^{k_2}\big[\psi(d(y_{m_k-1},y_{m_k}))\big]^{k_3}\\ &\quad\times\big[\psi(d(y_{m_k-1},y_{n_k})+d(y_{n_k-1},y_{m_k}))\big]^{k_4}\\ &\leq \big[\psi(d(y_{n_k-1},y_{m_k-1}))\big]^{k_1}\big[\psi(d(y_{n_k-1},y_{n_k}))\big]^{k_2}\big[\psi(d(y_{m_k-1},y_{m_k}))\big]^{k_3}\\ &\quad\times\big[\psi(d(y_{m_k-1},y_{n_k}))\psi d(y_{n_k-1},y_{m_k}))\big]^{k_4}.\end{aligned} \tag{11}$$

on taking limits as $k \to \infty$ in (11), we have

$\psi(\varepsilon) \leq \big(\psi(\varepsilon)\big)^{k_1}\big(\psi(0)\big)^{k_2}\big(\psi(0)\big)^{k_3}\big(\psi(\varepsilon)\big)^{k_4}\big(\psi(\varepsilon)\big)^{k_4}$.

Now by using the property $\psi(0) = 1$ we have $\psi(\varepsilon) \leq \big(\psi(\varepsilon)\big)^{k_1+2k_4} < \psi(\varepsilon)$, a contradiction.

Hence $\{y_n\}$ is a Cauchy sequence in X.

Theorem 3.4. In addition to the hypothesis of Theorem 3.3 on f and g, if either gX or fX is complete, then for any $x_0 \in X$, the Picard iterates $\{y_n\}$ defined by (9) converges to z (say) in X and z is a unique point of coincidence of f and g.

Proof. By Theorem 3.3, the sequence $\{y_n\}$ defined by (9) is Cauchy in X. Since gX is complete and $\{y_n\} = \{gx_{n+1}\} \subset gX$, there exists a point $z \in gX$ such that $\lim_{n\to\infty} y_n = z$.

Hence there exists $u \in X$ such that $gu = z$.

Now, we show that $gu = fu$.

Suppose $gu \neq fu$. From the inequality (8), we have

$$\begin{aligned}\psi(d(gx_{n+1}, fu)) = \psi(d(fx_n, fu)) &\\ &\leq \big[\psi(d(gx_n, gu))\big]^{k_1}\big[\psi(d(gx_n, fx_n))\big]^{k_2}\big[\psi(d(gu, fu))\big]^{k_3}\\ &\quad \times \big[\psi(d(gx_n, fu) + d(gu, fx_n))\big]^{k_4}\\ &\leq \big[\psi(d(gx_n, gu))\big]^{k_1}\big[\psi(d(gx_n, fx_n))\big]^{k_2}\big[\psi(d(gu, fu))\big]^{k_3}\\ &\quad \times \big[\psi(d(gx_n, fu))\big]^{k_4}\big[\psi(d(gu, fx_n))\big]^{k_4}.\end{aligned}$$

On letting $n \to \infty$, we have

$$\begin{aligned}\psi(d(gu, fu)) &\leq \big[\psi(d(gu, gu))\big]^{k_1}\big[\psi(d(gu, gu)\big]^{k_2}\big[\psi(d(gu, fu))\big]^{k_3}\\ &\quad \times \big[\psi(d(gu, fu))\big]\big[d(gu, gu))\big]^{k_4},\\ &\leq \big[\psi(d(gu, fu))\big]^{k_3+k_4} < \psi(d(gu, fu)),\end{aligned}$$

a contradiction.

Hence $gu = fu = z$ so that z is a point of coincidence of f and g.

Now, we show that a point of coincidence of f and g is unique. Suppose for some $t \in X$, $ft = gt = v$ (say) with $v \neq z$. Then by the inequality (8), we have

$$\begin{aligned}\psi(d(z, v)) = \psi(d(fu, ft)) &\leq \big[\psi(d(gu, gt))\big]^{k_1}\big[\psi(d(gu, fu))\big]^{k_2}\big[\psi(d(gt, ft))\big]^{k_3}\\ &\quad \times \big[\psi(d(gu, ft) + d(gt, fu))\big]^{k_4}.\end{aligned}$$

This implies that

$\psi(d(z,v)) \leq \left[\psi(d(z,v))\right]^{k_1}\left[\psi(d(z,z))\right]^{k_2}\left[\psi(d(v,v))\right]^{k_3}\left[\psi(d(z,v)+d(v,z))\right]^{k_4}$.

Hence $\psi(d(z,v)) \leq \left[\psi(d(z,v))\right]^{k_1+2k_4} < \psi(d(z,v))$, a contradiction.

Therefore f and g have a unique point of coincidence in X.

Theorem 3.5. Under the assumptions of Theorem 3.4, if the pair (f,g) is weakly compatible, then f and g have a unique common fixed point.

Proof. By Theorem 3.4, z is a point of coincidence of f and g. Hence there exists $u \in X$ such that $fu = gu = z$. Since the pair (f,g) is weakly compatible, we have $fgu = gfu$. i.e, $fz = gz$. Now, we claim that z is a common fixed point of f and g.

Suppose that $fz \neq z$. Then by the inequality (8), we have

$$\begin{aligned}\psi(d(fz,z)) = \psi(d(fz,fu)) \ &\leq \left[\psi(d(gz,fu))\right]^{k_1}\left[\psi(d(gz,fz))\right]^{k_2}\left[\psi(d(gu,fu))\right]^{k_3} \\ &\quad \times \left[\psi(d(gz,fu)+d(gu,fz))\right]^{k_4}.\end{aligned}$$

Hence

$$\begin{aligned}\psi(d(fz,z)) \leq &\left[\psi(d(fz,z))\right]^{k_1}\left[\psi(d(fz,fz))\right]^{k_2}\left[\psi(d(z,z))\right]^{k_3} \\ &\times \left[\psi(d(fz,z)+d(z,fz))\right]^{k_4}, \\ \leq &\left[\psi(d(fz,z))\right]^{k_1+2k_4} < \psi(d(fz,z)),\end{aligned}$$

a contradiction.

Hence $gz = fz = z$ so that z is a common fixed point of f and g.

Now, we show that the common fixed point is unique.

Let z and w be two common fixed points of f and g with $z \neq w$. Then we have

$$\begin{aligned}\psi(d(z,w)) = \psi(d(fz,fw)) \leq &\left[\psi(d(gz,gw))\right]^{k_1}\left[\psi(d(gz,fz))\right]^{k_2}\left[\psi(d(gw,fw))\right]^{k_3} \\ &\times \left[\psi(d(gz,fw)+d(gw,fz))\right]^{k_4},\end{aligned}$$

hence we have, $\psi(d(z,w)) \leq \left[\psi(d(z,w))\right]^{k_1+2k_4} < \psi(d(z,w))$, a contradiction.

Therefore $z = w$. Hence f and g have a unique common fixed point.

Corollary 3.6. Let (X,d) be a metric space, and let $f,g : X \to X$ be weakly compatible selfmaps. Assume that f and g satisfy $fX \subset gX$ and either gX or fX is complete. If there exist nonnegative numbers k_1, k_2, k_3 and k_4 with $0 \leq k_1 + k_2 + k_3 + 2k_4 < 1$ such that

$$\begin{aligned} d(fx, fy) \leq & k_1 d(gx, gy) + k_2 d(gx, fx) + k_3 d(gy, fy) \\ & + k_4[d(gx, fy) + d(gy, fx)] \text{ for all } x, y \in X. \end{aligned} \tag{12}$$

Then f and g have a unique common fixed point.

Proof. From the inequality (12), we have

$$e^{d(fx,fy)} \leq e^{k_1 d(gx,gy)} e^{k_2 d(gx,fx)} e^{k_3 d(gy,fy)} e^{k_4[d(gx,fy)+d(gy,fx)]} \text{ for all } x, y \in X. \tag{13}$$

By choosing $\psi : [0, \infty) \to [1, \infty)$ defined by $\psi(t) = e^t$, $t \geq 0$, then $\psi \in \Psi_1$ and the inequality (13) is a special case of the inequality (8). Thus, by Theorem 3.5, f and g have a unique common fixed point in X.

Example 3. *Let $X = (0, 1]$ with the usual metric. We define mappings g and f on X by*

$$fx = \begin{cases} \frac{1}{10} & \text{if } x \in (0, \frac{2}{5}); \\ \frac{1}{2} - \frac{x}{4} & \text{if } x \in [\frac{2}{5}, 1] \end{cases} \quad gx = \begin{cases} 1 & \text{if } x \in (0, \frac{2}{5}); \\ \frac{3}{5} - \frac{x}{2} & \text{if } x \in [\frac{2}{5}, 1]. \end{cases}$$

Here $fX = \{\frac{1}{10}\} \cup [\frac{1}{4}, \frac{2}{5}]$, $gX = \{1\} \cup [\frac{1}{10}, \frac{2}{5}]$ so that $fX \subset gX$ and gX is complete. Since $x = \frac{2}{5}$ is the only coincidence point of f and g and $fg(\frac{2}{5}) = gf(\frac{2}{5})$, and hence f and g are weakly compatible.

We now define $\psi : [0, \infty) \to [1, \infty)$ by $\psi(t) = e^t$, Clearly $\psi(t) = e^t \in \Psi_1$.

We now verify the inequality (8) *with $k_1 = \frac{1}{2}$ and k_2, k_3 and k_4 are arbitrary nonnegative real numbers such that* $0 \leq k_1 + k_2 + k_3 + 2k_4 < 1$.

Since $\psi(t) = e^t$, we have

$$\begin{aligned} |fx - fy| \ \leq & k_1|gx - fy| + k_2|gx - fx| + k_3|gy - fy| \\ & + k_4\big[|gx - fy| + |gy - fx|\big]. \end{aligned} \tag{14}$$

Now, we verify the inequality (14).

For $x = y$ and $x, y \in (0, \frac{2}{5})$ the inequality (14) *holds trivially. In the following, we verify the inequality* (14) *for the remaining cases:*

<u>*Case*</u> *(i): $x, y \in [\frac{2}{5}, 1]$.*

In this case, $fx = \frac{1}{2} - \frac{x}{4},\ fy = \frac{1}{2} - \frac{y}{4},\ gx = \frac{3}{5} - \frac{x}{2}$ *and* $gy = \frac{3}{5} - \frac{y}{2}$.

From the inequality (14)*, we have*

$$\begin{aligned}|fx - fy| = \frac{1}{4}|x-y| \leq \frac{1}{2}|x-y| \leq \frac{1}{2}|x-y| + k_2\left|\frac{x}{4} - \frac{1}{10}\right| + k_3\left|\frac{y}{4} - \frac{1}{10}\right| \\ + k_4\left[\left|\frac{x}{2} - \frac{y}{4} - \frac{1}{10}\right| + \left|\frac{y}{2} - \frac{x}{4} - \frac{1}{10}\right|\right] \\ = k_1|x-y| + k_2|gx - fx| + k_3|gy - fy| \\ + k_4\left[|gx - fy| + |gy - fx|.\right.\end{aligned}$$

<u>Case</u> (ii): $x \in (0, \frac{2}{5})$ *and* $y \in [\frac{2}{5}, 1]$

In this case, $fx = \frac{1}{10},\ gx = 1,\ fy = \frac{1}{2} - \frac{y}{4}$ *and* $gy = \frac{3}{5} - \frac{y}{2}$. *Using the inequality* (14)*, we have*

$$\begin{aligned}|fx - fy| = \left|\frac{4}{10} - \frac{y}{4}\right| \ \leq\ \frac{1}{2}\left|\frac{2}{5} + \frac{y}{2}\right| \leq \frac{1}{2}\left|\frac{2}{5} + \frac{y}{2}\right| + k_2\left|\frac{9}{10}\right| + k_3\left|\frac{y}{4} - \frac{1}{10}\right| \\ + k_4\left[\left|\frac{y}{2} - \frac{5}{10}\right| + \left|\frac{y}{4} + \frac{1}{2}\right|\right] \\ = k_1|gx - gy| + k_2|gx - fx| + k_3|gy - fy| \\ + k_4\big[|gx - fy| + |gy - fx|\big].\end{aligned}$$

<u>Case</u> (iii): $y \in (0, \frac{2}{5})$ *and* $x \in [\frac{2}{5}, 1]$

In this case, $fy = \frac{1}{10},\ gy = 1, fx = \frac{1}{2} - \frac{x}{4}$ *and* $gx = \frac{3}{5} - \frac{x}{2}$.

Using inequality (14)*, we have*

$$\begin{aligned}|fx - fy| = \left|\frac{4}{10} - \frac{x}{4}\right| \leq \frac{1}{2}\left|\frac{2}{5} + \frac{x}{2}\right| \leq \frac{1}{2}\left|\frac{2}{5} + \frac{x}{2}\right| + k_2\left|\frac{1}{10} - \frac{x}{4}\right| + k_3\left|\frac{9}{10}\right| \\ + k_4\left[\left|\frac{x}{4} + \frac{5}{10}\right| + \left|\frac{x}{2} - \frac{1}{2}\right|\right] \\ = k_1|gx - gy| + k_2|gx - fx| + k_3|gy - fy| \\ + k_4\big[|gx - fy| + |gy - fx|\big].\end{aligned}$$

Hence from all the above cases f and g satisfy the inequality (14)*. Therefore f and g satisfy all the hypotheses of Theorem 3.5 and $x = \frac{2}{5}$ is a unique common fixed point of f and g.*

4. Common fixed points of a pair of self maps satisfying property (E.A)

Theorem 4.1. Let (X,d) be a metric space, and let $f,g : X \to X$ be weakly compatible selfmaps satisfying property (E.A). Assume that there exist nonnegative real numbers k_1, k_2, k_3 and k_4 with $0 \leq k_1 + k_2 + k_3 + 2k_4 < 1$ and $\psi \in \Psi_1$ such that

$$\begin{aligned} \psi(d(fx,fy)) \leq & \big[\psi(d(gx,gy))\big]^{k_1}\big[\psi(d(gx,fx))\big]^{k_2}\big[\psi(d(gy,fy))\big]^{k_3} \\ & \times \ \big[\psi(d(gx,fy)+d(gy,fx))\big]^{k_4} \text{ for all } x,y \in X. \end{aligned} \tag{15}$$

If gX is closed, then f and g have a unique common fixed point.

Proof. Since the pair (f,g) satisfies property (E.A), there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$. Since gX is closed, $z \in gX$. This implies that $gu = z$ for some $u \in X$

We now claim that u is a coincidence point of f and g.

Suppose $fu \neq gu$. Replacing x by u and y by x_n in the inequality (15) we have

$$\begin{aligned} \psi(d(fx_n,fu)) \leq & \ \big[\psi(d(gx_n,gu))\big]^{k_1}\big[\psi(d(gx_n,fx_n))\big]^{k_2}\big[\psi(d(gu,fu))\big]^{k_3} \\ & \times \big[\psi(d(gx_n,fu)+d(gu,fx_n))\big]^{k_4}. \end{aligned}$$

On letting $n \to \infty$ we have

$$\begin{aligned} \psi(d(gu,fu)) \leq & \ \big[\psi(d(gu,gu))\big]^{k_1}\big[\psi(d(gu,gu))\big]^{k_2}\big[\psi(d(gu,fu))\big]^{k_3} \\ & \times \big[\psi(d(gu,fu)+d(gu,gu))\big]^{k_4} \\ \leq & \big[\psi(d(gu,fu))\big]^{k_1+k_4} < \psi(d(gu,fu)), \end{aligned}$$

a contradiction.

Therefore $gu = fu = z$ so that u is a coincidence point of f and g.

Since (f,g) is a pair of weakly compatible maps, we have $gfu = fgu$ and hence $fz = gz$.

Now, we show that z is a common fixed points of f and g.

Suppose that $fz \neq z$. From the inequality (15), we have

$$\begin{aligned} \psi(d(fz,z)) = \psi(d(fz,fu)) \leq & \big[\psi(d(gz,fu))\big]^{k_1}\big[\psi(d(gz,fz))\big]^{k_2}\big[\psi(d(gu,fu))\big]^{k_3} \\ & \times \big[\psi(d(gz,fu)+d(gu,fz))\big]^{k_4}. \end{aligned}$$

Now, we have

$$\psi(d(fz,z)) \leq \left[\psi(d(fz,z))\right]^{k_1}\left[\psi(d(fz,fz))\right]^{k_2}\left[\psi(d(z,z))\right]^{k_3} \times \left[\psi(d(fz,z)+d(z,fz))\right]^{k_4} = \left[\psi(d(fz,z))\right]^{k_1+2k_4} < \psi(d(fz,z)),$$

a contradiction.

Therefore $fz = z$ this implies $fz = gz = z$, z is a common fixed point of f and g.

We now show that the common fixed point of f and g is unique.

Let z and w be two common fixed points of f and g with $z \neq w$.

From the inequality (15) we have

$$\begin{aligned}\psi(d(z,w)) = \psi(d(fz,fw)) &\leq \left[\psi(d(gz,gw))\right]^{k_1}\left[\psi(d(gz,fz))\right]^{k_2}\left[\psi(d(gw,fw))\right]^{k_3} \\ &\quad \times \left[\psi(d(gz,fw)+d(gw,fz))\right]^{k_4} \\ &\leq \left[\psi(d(gz,gw))\right]^{k_1}\left[\psi(d(gz,fz))\right]^{k_2}\left[\psi(d(gw,fw))\right]^{k_3} \\ &\quad \times \left[\psi(d(gz,fw))\right]^{k_4}\left[\psi(d(gw,fz))\right]^{k_4}.\end{aligned}$$

Hence we have

$\psi(d(z,w)) \leq \left[\psi(d(z,w))\right]^{k_1+2k_4} < \psi(d(z,w))$, a contradiction. Therefore $z = w$.

Hence f and g have a unique common fixed point.

Since two non-compatible selfmaps of metric space satisfy property (E.A), we have the following corollary.

Corollary 4.2. Let (X,d) be a metric space, and let $f,g : X \to X$ be non-compatible and weakly compatible self maps. Assume that there exist non-negative real numbers k_1,k_2,k_3 and k_4 with $0 \leq k_1+k_2+k_3+2k_4 < 1$ and $\psi \in \Psi_1$ such that

$$\psi(d(fx,fy)) \leq \left[\psi(d(gx,gy))\right]^{k_1}\left[\psi(d(gx,fx))\right]^{k_2}\left[\psi(d(gy,fy))\right]^{k_3} \times \left[\psi(d(gx,fy)+d(gy,fx))\right]^{k_4} \text{ for all } x,y \in X.$$

If gX is closed, then f and g have a unique common fixed point in X.

Example 4. *Let $X = (0,1]$ with the usual metric and we define $g,f : X \to X$ by*

$$fx = \begin{cases} \frac{1}{2} & \text{if } x \in (0,\frac{2}{5}); \\ \frac{1}{2}-\frac{x}{4} & \text{if } x \in [\frac{2}{5},1], \end{cases} \quad gx = \begin{cases} 1 & \text{if } x \in (0,\frac{2}{5}); \\ \frac{3}{5}-\frac{x}{2} & \text{if } x \in [\frac{2}{5},1]. \end{cases}$$

We have $gfx = fgx$ *whenever* $fx = gx$, *hence* f *and* g *are weakly compatible. Since there is a sequence* $x_n = \frac{2}{5} + \frac{1}{n}$, $n \geq 1$ *with* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \frac{2}{5}$, *the pair* (f,g) *satisfy the property (E.A), and clearly* $gX = \{1\} \cup [\frac{1}{10}, \frac{2}{5}]$ *is closed.*
We define $\psi : [0,\infty) \to [1,\infty)$ *by* $\psi(t) = e^t$. *Clearly,* $\psi \in \Psi_1$.
Now, we verify the inequality (15) *with* $k_1 = \frac{1}{2}$ *and* k_2, k_3 *and* k_4 *are arbitrary non-negative real numbers such that* $0 \leq k_1 + k_2 + k_3 + 2k_4 < 1$.
Since $\psi(t) = e^t$, *We have*

$$\begin{aligned} |fx - fy| \leq & k_1|gx - gy| + k_2|gx - fx| + k_3|gy - fy| \\ & + k_4[|gx - fy| + |gy - fx|] \text{ for all } x, y \in X \end{aligned} \tag{16}$$

We now verify the inequality (16)
For $x = y$ *and* $x, y \in (0, \frac{2}{5})$ *the inequality* (16) *holds trivially. In the following, we verify the inequality* (16) *for the remaining cases:*
<u>Case</u> *(i) :* $x, y \in [\frac{2}{5}, 1]$.
In this case, $fx = \frac{1}{2} - \frac{x}{4}, fy = \frac{1}{2} - \frac{y}{4}, gx = \frac{3}{5} - \frac{x}{2}$ *and* $fx = \frac{3}{5} - \frac{y}{2}$.
We have

$$\begin{aligned} |fx - fy| = |\frac{x}{4} - \frac{y}{4}| &= \frac{1}{2}|\frac{x}{2} - \frac{y}{2}| \\ &\leq \frac{1}{2}|\frac{x}{2} - \frac{y}{2}| + k_2|\frac{1}{10} - \frac{x}{4}| + k_3|\frac{1}{10} - \frac{y}{4}| \\ &\quad + k_4[|\frac{x}{2} - \frac{y}{4} - \frac{1}{10}| + |\frac{y}{2} - \frac{x}{2} - \frac{1}{10}|] \\ &= k_1|gx - gy| + k_2|gx - fx| + k_3|gy - fy| \\ &\quad + k_4[|gx - fy| + |gy - fx|]. \end{aligned}$$

<u>Case</u> *(ii) :* $x \in (0, \frac{2}{5})$ *and* $y \in [\frac{2}{5}, 1]$.
In this case, $fx = \frac{1}{2}\ gx = 1,\ fy = \frac{1}{2} - \frac{y}{4}$ *and* $gy = \frac{3}{5} - \frac{y}{2}$.

Now, we have

$$
\begin{aligned}
|fx-fy| = \frac{y}{4} = \frac{1}{2}(\frac{y}{2}) \leq \frac{1}{2}(\frac{y}{2}+\frac{2}{5}) + \frac{k_2}{2} + k_3|\frac{1}{10}-\frac{y}{4}| \\
+k_4[(\frac{y}{2}+\frac{1}{2}) + |\frac{y}{2}-\frac{1}{10}|] \\
= k_1|gx-gy| + k_2|gx-fx| + k_3|gy-fy| \\
+k_4[|gx-fy| + |gy-fx|].
\end{aligned}
$$

Case (iii) : $y \in (0, \frac{2}{5})$ *and* $x \in [\frac{2}{5}, 1]$.

In this case, we have $fy = \frac{1}{2},\ gy = 1,\ fx = \frac{1}{2} - \frac{x}{4}$ *and* $gx = \frac{3}{5} - \frac{x}{2}$.

from the inequality (16) *We have*

$$
\begin{aligned}
|fx-fy| = \frac{x}{4} = \frac{1}{2}(\frac{x}{2}) \ \leq \frac{1}{2}(\frac{x}{2}+\frac{2}{5}) + k_2|\frac{1}{10}-\frac{x}{4}| + \frac{k_3}{2} \\
+k_4[(\frac{x}{2}+\frac{1}{2}) + |\frac{x}{2}-\frac{1}{10}|] \\
= k_1|gx-gy| + k_2|gx-fx| + k_3|gy-fy| \\
+k_4[|gx-fy| + |gy-fx|].
\end{aligned}
$$

Hence from all the above cases f and g satisfy the inequality (16). *Therefore f and g satisfy all the hypotheses of Theorem 4.1 and* $x = \frac{2}{5}$ *is the unique common fixed point of f and g.*

Here we observe that $fX = \{\frac{1}{2}\} \cup [\frac{1}{4}, \frac{2}{5}] \not\subseteq \{1\} \cup [\frac{1}{10}, \frac{2}{5}] = gX$ so that Theorem 3.5 is not applicable.

Conflict of Interests

The authors declare that there is no conflict of interests.

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