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# COMMON FIXED POINT THEOREM USING COMPATIBLE MAPPINGS OF TYPE (E)

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**Abstract:** The purpose of this paper is to present a common fixed point theorem in a metric space which extends the result of Bijendra Singh and M.S.Chauhan using the weaker conditions such as compatible mappings of type (E) and associated sequence in place of compatibility and completeness of the metric.

**Keywords:** fixed point; self maps; compatible mappings; compatible mappings of type (E) and associated sequence.

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## 1. Introduction

Fixed point theory is an important area of functional analysis. The study of common fixed point of mappings satisfying contractive type condition has been a very active field of research. G.Jungck [1] introduced the concept of compatible maps which is weaker than weakly commuting maps. After wards Jungck and Rhoades [4] defined weaker class of maps known as weakly compatible maps. This concept has been frequently used to prove existence theorem in common fixed point theory.

M.R.Singh and Y.M.Singh [6] introduced the concept of compatible mappings of type (E).In this paper we prove a common fixed point theorem for four self maps in which two pairs are compatible mappings of type (E).

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## 2. Definitions and Preliminaries

### 2.1 Compatible mappings

Two self maps S and T of a metric space (X,d) are said to be compatible mappings if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $< x_n >$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

### 2.2 Compatible mappings of type (A)

Two self maps S and T of a metric space (X,d) are said to be compatible mappings of type (A) if $\lim_{n\to\infty} d(STx_n, TTx_n) = 0$ and $\lim_{n\to\infty} d(TSx_n, SSx_n) = 0$ whenever $< x_n >$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$.

### 2.3 Compatible mappings of type (B)

Two self maps S and T of a metric space (X,d) are said to be compatible mappings of type (B) if

$\lim_{n\to\infty} d(STx_n, TTx_n) \le \frac{1}{2}\left[\lim_{n\to\infty} d(STx_n, St) + \lim_{n\to\infty} d(St, SSx_n)\right]$ and

$\lim_{n\to\infty} d(TSx_n, SSx_n) \le \frac{1}{2}\left[\lim_{n\to\infty} d(TSx_n, Tt) + \lim_{n\to\infty} d(Tt, TTx_n)\right]$ whenever $< x_n >$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$.

### 2.4 Compatible mappings of type (P)

Two self maps S and T of a metric space (X,d) are said to be compatible mappings of type (P) if $\lim_{n\to\infty} d(SSx_n, TTx_n) = 0$, whenever $< x_n >$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$.

### 2.5 Compatible mappings of type (E)

Two self maps S and T of metric space (X,d) are said to be compatible mappings of type (E) if $\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} STx_n = Tt$ and $\lim_{n\to\infty} TTx_n = \lim_{n\to\infty} TSx_n = St$ ,whenever $< x_n >$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Bijenrda Singh and M.S.Chauhan[5] proved the following theorem.

**2.6 Theorem:** Let A,B,S and T be self mappings from a complete metric space (X,d) into itself satisfying the following conditions

(2.6.1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$

(2.6.2) one of A, B, S and T is continuous

(2.6.3) $[d(Ax,By)]^2 \le k_1[d(Ax,Sx)d(By,Ty)+d(By,Sx)d(Ax,Ty)]$

$$+k_2[d(Ax,Sx)d(Ax,Ty)+d(By,Ty)d(By,Sx)]$$

where $0 \le k_1 + 2k_2 < 1, k_1, k_2 \ge 0$

(2.6.4) the pairs (A, S) and (B, T) are compatible on X

further, if X is a complete metric space then A,B,S and T have a unique common fixed point in X.

Now we generalize the theorem using compatible mappings of type (E) and associated sequence.

**2.7 Associated sequence [7]:** Suppose A, B, S and T are self maps of a metric space (X,d) satisfying the condition (2.6.1).Then for an arbitrary $x_0 \in X$ such that $Ax_0 = Tx_1$ and for this point $x_1$, there exist a point $x_2$ in X such that $Bx_1 = Sx_2$ and so on. Proceeding in the similar manner, we can define a sequence $< x_n >$ in X such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$ for $n \ge 0$. We shall call this sequence as an "Associated sequence of $x_0$" relative to four self maps A, B, S and T.

Now we prove a lemma which plays an important role in our main theorem.

**2.8 Lemma:** Let A, B, S and T be self mappings from a complete metric space (X, d) into itself satisfying the conditions (2.6.1) and (2.6.3). Then the associated sequence $\{y_n\}$ relative to four self maps is a Cauchy sequence in X.

**Proof:** From the conditions (2.6.1), (2.6.3) and from the definition of associated sequence, we have

$$\left[d(y_{2n+1}, y_{2n})\right]^2 = \left[d(Ax_{2n}, Bx_{2n-1})\right]^2$$
$$\le k_1\left[d(Ax_{2n}, Sx_{2n})\ d(Bx_{2n-1}, Tx_{2n-1}) + d(Bx_{2n-1}, Sx_{2n})\ d(Ax_{2n}, Tx_{2n-1}\ \right]$$
$$+ k_2\left[d(Ax_{2n}, Sx_{2n})\ d(Ax_{2n}, Tx_{2n-1}) + d(Bx_{2n-1}, Tx_{2n-1})\ d(Bx_{2n-1}\ Sx_{2n})\ \right]$$

$$= k_1\left[d(y_{2n+1}, y_{2n})\ d(y_{2n}, y_{2n-1}) + 0\ \right]$$
$$+ k_2\left[d(y_{2n+1}, y_{2n})\ d(y_{2n+1}, y_{2n-1}) + 0\right]$$

This implies

$$d(y_{2n+1}, y_{2n}) \le k_1 \; d(y_{2n}, y_{2n-1}) + k_2 \left[ d(y_{2n+1}, y_{2n}) + d(y_{2n}, y_{2n-1}) \right]$$
$$d(y_{2n+1}, y_{2n}) \le h \; d(y_{2n}, y_{2n-1})$$
$$where \; h = \frac{k_1 + k_2}{1 - k_2} < 1$$

$for\ every\ \text{int}\,eger\ p > 0, we\ get$

$$\begin{aligned} d(y_n, y_{n+p}) &\le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots\ldots\ldots\ldots + d(y_{n+p-1}, y_{n+p}) \\ &\le h^n d(y_0, y_1) + h^{n+1} d(y_0, y_1) + \ldots\ldots\ldots\ldots + h^{n+p-1} d(y_0, y_1) \\ &\le \left(h^n + h^{n+1} + \ldots\ldots\ldots\ldots + h^{n+p-1}\right) d(y_0, y_1) \\ &\le h^n \left(1 + h + h^2 + \ldots\ldots\ldots\ldots + h^{p-1}\right) d(y_0, y_1) \end{aligned}$$

Since h<1, $h^n \to 0$ as $n \to \infty$, so that $d(y_n, y_{n+p}) \to 0$. This shows that the sequence $\{y_n\}$ is a Cauchy sequence in X and since X is a complete metric space; it converges to a limit, say $z \in X$.

The converse of the Lemma is not true, that is A,B,S and T are self maps of a metric space (X,d) satisfying (2.6.1) and (2.6.3), even if for $x_0 \in X$ and for associated sequence of $x_0$ converges, the metric space (X,d) need not be complete.

**2.9 Example:** Let $X = (0, \frac{1}{2}]$ with $d(x, y) = |x - y|$. Define self maps of A, B, S and T of X by

$$Ax = Bx = \begin{cases} \dfrac{1-3x}{2} & if \; x \in \left(0, \dfrac{1}{4}\right) - \left\{\dfrac{1}{8}\right\} \\ \dfrac{1}{16} & if \; x = \dfrac{1}{8} \\ \dfrac{5x-1}{2} & if \; x \in \left[\dfrac{1}{4}, \dfrac{1}{2}\right] \end{cases} \qquad Sx = \begin{cases} \dfrac{x}{2} & if \; x \in \left(0, \dfrac{1}{4}\right) - \left\{\dfrac{1}{8}\right\} \\ \dfrac{5}{16} & if \; x = \dfrac{1}{8} \\ \dfrac{5x-1}{2} & if \; x \in \left[\dfrac{1}{4}, \dfrac{1}{2}\right] \end{cases}$$

$$Tx = \begin{cases} \dfrac{x}{2} & if \; x \in \left(0, \dfrac{1}{4}\right) - \left\{\dfrac{1}{8}\right\} \\ \dfrac{5}{16} & if \; x = \dfrac{1}{8} \\ 4x - 1 & if \; x \in \left[\dfrac{1}{4}, \dfrac{1}{2}\right] \end{cases}$$

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Then $A(X) = B(X) = \left(\frac{1}{8}, \frac{1}{2}\right) \cup \left\{\frac{1}{16}\right\} \cup \left[\frac{1}{8}, \frac{3}{4}\right]$ , $S(X) = \left(0, \frac{1}{8}\right) \cup \left\{\frac{5}{16}\right\} \cup \left[\frac{1}{8}, \frac{3}{4}\right]$ and $T(X) = \left(0, \frac{1}{8}\right) \cup \left\{\frac{5}{16}\right\} \cup [0,1]$ and so that the conditions $A(X) \subset T(X)$ and $B(X) \subset S(X)$ are satisfied. The associated sequence $Ax_0, Bx_1, Ax_2, Bx_3....Ax_{2n}, Bx_{2n+1},....$ converges to the point '$\frac{1}{3}$',

but X is not a complete metric space.

We need the following proposition for the proof of our main result.

**2.10 Proposition:** If A and S be compatible mappings of type (E) on a metric space (X,d) and if one of function is continuous. Then we have

a) A(x) =S(x) and $\lim_{n\to\infty} AAx_n = \lim_{n\to\infty} SSx_n = \lim_{n\to\infty} ASx_n = \lim_{n\to\infty} SAx_n$,

b) If there exist $u \in X$ such that $Au = Su = x$ then $ASu = SAu$.

Whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$ for some x in X.

**Proof:** Let $\{x_n\}$ be a sequence of X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$, for some x in X. Then by definition of compatible of type (E), we have $\lim_{n\to\infty} AAx_n = \lim_{n\to\infty} ASx_n = S(x)$. If A is a continuous mapping, then we get $\lim_{n\to\infty} AAx_n = A(\lim_{n\to\infty} Sx_n) = A(x)$. This implies $A(x) = S(x)$.

Also $\lim_{n\to\infty} AAx_n = \lim_{n\to\infty} SSx_n = \lim_{n\to\infty} ASx_n = \lim_{n\to\infty} SAx_n$. Similarly, if S is continuous then, we get the same result. This is the proof of part (a).

Again, suppose $Au = Su = x$ for some $u \in X$. Then, $ASu = A(Su) = Ax$ and $SAu = S(Au) = Sx$.

From (a), we have $Ax = Sx$. Hence $ASu = SAu$. This is the proof of part (b).

The following example establishes this.

## 3 Main Result

**3.1 Theorem:** Let A, B, S and T be self mappings from a metric space (X,d) into itself satisfying the following conditions

$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ (3.1.1)

$$[d(Ax,By)]^2 \le k_1[d(Ax,Sx)d(By,Ty) + d(By,Sx)d(Ax,Ty)] + k_2[d(Ax,Sx)d(Ax,Ty) + d(By,Ty)d(By,Sx)] \quad (3.1.2)$$

for all x,y in X where $0 \le k_1 + 2k_2 < 1, k_1, k_2 \ge 0$

one of the mapping s of (A,S) and (B,T) of X is continuous (3.1.3)

the pairs (A,S) and (B,T) compatible mappings of type(E) (3.1.4) the sequence $Ax_0, Bx_1, Ax_2, Bx_{3,} ...., Ax_{2n}, Bx_{2n+1}...$ converges to $z \in X$ . (3.1.5)

Then A, B, S and T have a unique common fixed point in z in X.

**Proof:** From the condition (3.1.5), we have

$Ax_{2n} \to z, Tx_{2n+1} \to z, Bx_{2n+1} \to z \text{ and } Sx_{2n} \to z$ as $n \to \infty$. (3.1.6)

Suppose A is continuous. Then $AAx_{2n} \to Az, ASx_{2n} \to Az$ as $n \to \infty$.

Since A and S are compatible of type (E) and one of the mapping of the pair (A, S) is continuous then by proposition 2.10, we have $Az = Sz$ .

Since $A(X) \subseteq T(X)$ implies that there exists $w \in X$ such that $Az = Tw$.

Put $x = z, y = w$ in condition (3.1.2), we have

$$[d(Az, Bw)]^2 \le k_1[d(Az, Sz)d(Bw, Tw) + d(Bw, Sz)d(Az, Tw)] \\ + k_2[d(Az, Sz)d(Az, Tw) + d(Bw, Tw)d(Bw, Sz)]$$

Using the conditions $Az = Tw$ and $Az = Sz$ , then we get

$$[d(Az, Bw)]^2 \le k_1[d(Az, Az)d(Bw, Az) + d(Bw, Az)d(Az, Az)] \\ + k_2[d(Az, Az)d(Az, Az) + d(Bw, Az)d(Bw, Az)]$$

$$[d(Az, Bw)]^2 \le k_2[d(Bw, Az)]^2$$

$(1 - k_2)[d(Az, Bw)]^2 \le 0$ , since $0 \le k_1 + 2k_2 < 1$

$d(Az, Bw) = 0$ implies that $Az = Bw$.

Hence $Az = Bw = Tw = Sz$ .

Now put $x = z, y = x_{2n+1}$ in condition (3.1.2), we have

$$[d(Az, Bx_{2n+1})]^2 \le k_1[d(Az, Sz)d(Bx_{2n+1}, Tx_{2n+1}) + d(Bx_{2n+1}, Sz)d(Az, Tx_{2n+1})] \\ + k_2[d(Az, Sz)d(Az, Tx_{2n+1}) + d(Bx_{2n+1}, Tx_{2n+1})d(Bx_{2n+1}, Sz)]$$

Letting $n \to \infty$ and using the condition $Az = Sz$ , we have

$$[d(Az, z)]^2 \le k_1[d(Az, Az)d(z, z) + d(z, Az)d(Az, z)] \\ + k_2[d(Az, Az)d(Az, z) + d(z, z)d(z, Az)]$$

$$[d(Az, z)]^2 \le k_1[d(Az, z)]^2$$

$(1 - k_1)[d(Az, z)]^2 \le 0$ , since $0 \le k_1 + 2k_2 < 1$

$d(Az, z) = 0$ implies $Az = z$ .

Therefore $Az = Sz = z$ and hence $Az = Sz = Tw = Bw = z$ .

Again, if B and T are compatible of type (E) and one of mappings say B of the pair (B, T) is continuous, so we get $Bw = Tw = Az = z$. By using proposition 2.10, we get $BBw = BTw = TBw = TTw$. Thus, we get $Bz = Tz$.

Put $x = x_{2n}, y = z$ in condition (3.1.2), we have

$$[d(Ax_{2n}, Bz)]^2 \le k_1[d(Ax_{2n}, Sx_{2n})d(Bz, Tz) + d(Bz, Sx_{2n})d(Ax_{2n}, Tz)] \\ + k_2[d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tz) + d(Bz, Tz)d(Bz, Sx_{2n})]$$

Letting $n \to \infty$ and using the conditions (3.1.6) and $Bz = Tz$, we get

$$[d(z, Bz)]^2 \le k_1[d(z, z)d(Bz, Bz) + d(Bz, z)d(z, Bz)] \\ + k_2[d(z, z)d(z, Bz) + d(Bz, Bz)d(Bz, z)]$$

$$[d(z, Bz)]^2 \le k_1[d(z, Bz)]^2$$

$(1 - k_1)[d(Bz, z)]^2 \le 0$, since $0 \le k_1 + 2k_2 < 1$

$d(Bz, z) = 0$ implies $Bz = z$.

Therefore $Tz = Bz = z$.

Hence z is a common fixed point of B and T.

Since $Bz = Tz = Az = Sz = z$, we get z is a common fixed point of A, B, S and T. The uniqueness of the fixed point can be easily proved.

**3.2 Remark:** From the example given earlier, the pairs (A, S) and (B, T) are compatible mappings of type (E). But the pairs (A, S) and (B, T) are not any one of compatible, compatible mappings of type (A), compatible mappings of type (B), compatible mappings of type (P). For this, take a sequence $x_n = \frac{1}{4} - \frac{1}{n}$, for $n \ge 1$, then $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \frac{1}{8} = t$ (Say) and $\lim_{n\to\infty} AAx_n = \lim_{n\to\infty} ASx_n = S(t) = \frac{5}{16}$ also $\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} SAx_n = A(t) = \frac{1}{16}$. Further the condition (3.1.2) holds for the values of $0 \le k_1 + 2k_2 < 1$, where $k_1, k_2 \ge 0$. It is also clear that A is continuous in the pair (A,S) and B is continuous in the pair (B,T). We also note that X is not a complete metric space. Also from the example 2.9, we observe that $\frac{1}{3}$ is a common fixed point of A, B, S and T. In fact '$\frac{1}{3}$' is the unique common fixed point of A, B, S and T.

**Conflict of Interests**

The authors declare that there is no conflict of interests.

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