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MAXIMAL AND MINIMAL SOLUTION OF NONLOCAL FRACTIONAL DIFFERENTIAL EQUATION

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Abstract. In this paper we consider a nonlinear fractional differential equation with weighted initial and nonlocal condition and prove the existence and approximation of the solution. We also extend to prove the existence of Maximal and Minimal solutions for a nonlinear fractional differential equation with weighted initial and nonlocal conditions, and these maximal and minimal solution will serve as bounds for the nonlinear fractional differential equation with weighted initial and nonlocal conditions.

Keywords: weighted nonlocal problem; nonlinear fractional differential equation; approximate solution.

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1. Introduction

In many engineering and scientific disciplines such as physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, etc the fractional differential and integral equations represents the processes in a more effective manner than by integer order. Because

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of this the subject of fractional order differential and integral equations became the interest of mathematicians and researchers.

Nonlinear fractional differential equation with weighted initial data has been studied by several authors. The weighted Cauchy-type problem

$$\begin{aligned} &D^{\alpha}(u(t)) = f(t,u(t)) \\ &t^{1-\alpha}u(t)|_{t=0} = b \end{aligned} \tag{1.1}$$

Studied by Khaled et al [5].

The solution of the periodic boundary value problem for a fractional differential equation involving a RiemannLiouville fractional derivative

$$\begin{aligned} D^{\alpha}(u(t)) &= f(t,u(t)) \\ t^{1-\alpha}u(t)|_{t=0} &= t^{1-\alpha}u(t)|_{t=T} \end{aligned} \tag{1.2}$$

Studied by Weia et al [6], Also the existence of solutions of fractional equations of Volterra type with the RiemannLiouville derivative,

$$\begin{aligned} D^{\alpha}(u(t)) &= f(t,u(t),\int_0^t k(t,s)u(s)ds) \\ t^{1-\alpha}u(t)|_{t=0} &= r \end{aligned} \tag{1.3}$$

Studied by Jankowski [7]. The weighted nonlocal fractional differential equation

$$\begin{aligned} {}^{c}D^{\alpha}(u(t)) &= f(t,u(t)) \\ \lim_{t\to 0^+} t^{1-\alpha}u(t) &= \sum_{i=1}^{m} a_i u(\tau_i) \end{aligned} \tag{1.4}$$

studied by Holambe et al[3, 4] etc. , and the references therein. Problems in nonlinear fractional differential equation were studied by various researchers.

The immportance of non-local problems appears to have been first noted in the literature by Bitsadze-Samarski[15]. By Byszewski[8, 9], the nonlocal condition can be more useful than the standard initial contion to describe some physical phenomena.

Now here we consider the weighted nonlocal fractional differential equation

$$\begin{aligned} &D^{\alpha}(u(t))+au(t)=f(t,u(t)) \\ &\lim_{t\to 0^+} t^{1-\alpha}u(t)=u_0 \end{aligned} \tag{1.5}$$

where D^{α} is Riemann-Liouville fractional derivatives of order $0<\alpha\leq 1$ and $0<t\leq T<\infty$.

2. **Auxiliary Results**

let E denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm $\|\cdot\|$. It is known that E is regular if $\{x_n\}_{n\in\mathbb{N}}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n\to x^*$ as $n\to\infty$, then $x_n\preceq x^*$ (resp. $x_n\succeq x^*$) for all $n\in\mathbb{N}$. Clearly, the partially ordered Banach space $C(J,\mathbb{R})$ is regular and the conditions guaranteeing the regularity of any partially ordered normed linear space E may be found in Heikkilä and Lakshmikantham [**?**] and the references therein.

We need the following definitions.

Definition 2.1. *A mapping $\mathscr{T}:E\to E$ is called isotone or nondecreasing if it preserves the order relation $\preceq$, that is, if $x\preceq y$ implies $\mathscr{T}x\preceq\mathscr{T}y$ for all $x,y\in E$.*

Definition 2.2 ([13])**.** *A mapping $\mathscr{T}:E\to E$ is called partially continuous at a point $a\in E$ if for $\varepsilon>0$ there exists a $\delta>0$ such that $\|\mathscr{T}x-\mathscr{T}a\|<\varepsilon$ whenever x is comparable to a and $\|x-a\|<\delta$. $\mathscr{T}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathscr{T}$ is partially continuous on E, then it is continuous on every chain C contained in E.*

Definition 2.3. *A mapping $\mathscr{T}:E\to E$ is called partially bounded if $\mathscr{T}(C)$ is bounded for every chain C in E. $\mathscr{T}$ is called uniformly partially bounded if all chains $\mathscr{T}(C)$ in E are bounded by a unique constant. $\mathscr{T}$ is called bounded if $\mathscr{T}(E)$ is a bounded subset of E.*

Definition 2.4. *A mapping $\mathscr{T}:E\to E$ is called partially compact if $\mathscr{T}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E. $\mathscr{T}$ is called uniformly partially compact if $\mathscr{T}(C)$ is a uniformly partially bounded and partially compact on E. $\mathscr{T}$ is* *called partially totally bounded if for any totally ordered and bounded subset C of E, $\mathscr{T}(C)$ is a relatively compact subset of E. If $\mathscr{T}$ is partially continuous and partially totally bounded, then it is called partially completely continuous on E.*

Definition 2.5 ([13]). *The order relation $\preceq$ and the metric d on a non-empty set E are said to be **compatible** if $\{x_n\}_{n\in\mathbb{N}}$ is a monotone, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ converges to x^* implies that the original sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E,\preceq,\|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible.*

Definition 2.6 ([10]). *An upper semi-continuous and monotone nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathscr{D}$-function provided $\psi(r) = 0$ iff $r = 0$. Let $(E,\preceq,\|\cdot\|)$ be a partially ordered normed linear space. A mapping $\mathscr{T} : E \to E$ is called partially nonlinear $\mathscr{D}$-Lipschitz if there exists a $\mathscr{D}$-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|\mathscr{T}x - \mathscr{T}y\| \le \psi(\|x-y\|) \tag{2.1}$$

for all comparable elements $x,y \in E$. If $\psi(r) = k\,r$, $k > 0$, then $\mathscr{T}$ is called a partially Lipschitz with a Lipschitz constant k.

Let $(E,\preceq,\|\cdot\|)$ be a partially ordered normed linear algebra. Denote

$$E^+ = \big\{x \in E \mid x \succeq \theta, \text{ where } \theta \text{ is the zero element of } E\big\}$$

and

$$\mathscr{K} = \{E^+ \subset E \mid uv \in E^+ \text{ for all } u,v \in E^+\}. \tag{2.2}$$

The elements of $\mathscr{K}$ are called the positive vectors of the normed linear algebra E. The following lemma follows immediately from the definition of the set $\mathscr{K}$ and which is often times used in the applications of hybrid fixed point theory in Banach algebras.

Lemma 2.7 ([11]). *If $u_1,u_2,v_1,v_2 \in \mathscr{K}$ are such that $u_1 \preceq v_1$ and $u_2 \preceq v_2$, then $u_1u_2 \preceq v_1v_2$.*

Definition 2.8. *An operator $\mathscr{T} : E \to E$ is said to be positive if the range $R(\mathscr{T})$ of $\mathscr{T}$ is such that $R(\mathscr{T}) \subseteq \mathscr{K}$.*

The method may be stated as *"the monotonic convergence of the sequence of successive approximations to the solutions of a nonlinear equation beginning with a lower or an upper solution of the equation as its initial or first approximation"* which is a powerful tool in the existence theory of nonlinear analysis. A few other hybrid fixed point theorems involving the method may be found in [13, 14].

Theorem 2.9 ([14]). *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear algebra such that the order relation $\preceq$ and the norm $\|\cdot\|$ in E are compatible in every compact chain of E. Let $\mathscr{A}, \mathscr{B} : E \to \mathscr{K}$ be nondecreasing operators such that*

(a) *$\mathscr{A}$ is partially bounded and partially nonlinear $\mathscr{D}$-Lipschitz with $\mathscr{D}$-function $\psi_{\mathscr{A}}$.*

(b) *$\mathscr{B}$ is partially continuous and uniformly partially compact, and*

(c) *$M\psi_{\mathscr{A}}(r) < r$, $r > 0$, where $M = \sup\{\|\mathscr{B}(C)\| : C$ is a chain in $E\}$, and*

(d) *there exists an element $x_0 \in X$ such that $x_0 \preceq \mathscr{A}x_0 + \mathscr{B}x_0$ or $x_0 \succeq \mathscr{A}x_0 + \mathscr{B}x_0$.*

Then the operator equation

$$\mathscr{A}x + \mathscr{B}x = x \tag{2.3}$$

has a solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathscr{A}x_n + \mathscr{B}x_n$, $n = 0, 1, \ldots$, converges monotonically to x^*.

Remark 2.10. The compatibility of the order relation $\preceq$ and the norm $\|\cdot\|$ in every compact chain of E holds if every partially compact subset of E possesses the compatibility property with respect to $\preceq$ and $\|\cdot\|$. Note that a subset S of the partially ordered Banach space $C(J, \mathbb{R})$ is called partially compact if every chain C in S is compact. This simple fact has been utilized to prove the main results of this paper.

3. Main Results

The equivalent integral form of the problem 1.5 is considered in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$

in $C(J,\mathbb{R})$ by

$$\|x\| = \sup_{t\in J}|x(t)| \tag{3.1}$$

and

$$x \le y \iff x(t) \le y(t) \tag{3.2}$$

for all $t \in J$ respectively. Clearly, $C(J,\mathbb{R})$ is a Banach algebra with respect to above supremum norm and is also partially ordered w. r. t. the above partially order relation $\le$. It is known that the partially ordered Banach algebra $C(J,\mathbb{R})$ has some nice properties concerning the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\le$ in certain subsets of of it. The following lemma in this connection follows by an application of Arzelá-Ascoli theorem.

Lemma 3.1. *Let $\big(C(J,\mathbb{R}),\le,\|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\le$ defined by (3.1) and (3.2) respectively. Then $\|\cdot\|$ and $\le$ are compatible in every partially compact subset of $C(J,\mathbb{R})$.*

Proof. The lemma mentioned in Dhage [14], but the proof appears in Dhage [**?**]. □

We need the following definition in what follows.

Definition 3.2. *A function $u_l \in C(J,\mathbb{R})$ is said to be a lower solution of the problem (1.5) if it satisfies*

$$D^{\alpha}(u_l(t)) + au_l(t) \le f(t,u_l(t))$$

$$\lim_{t\to 0^+} t^{1-\alpha}u_l(t) \le u_{l0}$$

for all $t \in J$. Similarly, a function $u_u \in C(J,\mathbb{R})$ is said to be an upper solution of the problem (1.5) if it satisfies the above inequalities with reverse sign.

Definition 3.3. *A function $f(t,u)$ is called Carathéodory if*

(i) *the map $t \mapsto f(t,u)$ is measurable for each $u \in \mathbb{R}$ and*

(ii) *the map $u \mapsto f(t,u)$ is continuous for each $t \in J$.*

A Caratheódory function f is called L^2-Carathéodory if

(iii) *there exists a function* $h \in L^2(J,\mathbb{R})$ *such that*

$$|f(t,u)| \leq h(t) \text{ a. e. } t \in J$$

We consider the following set of assumptions in what follows:

(A_1) The functions $f : J \times \mathbb{R} \to \mathbb{R}_+, \alpha : J \to \mathbb{R}_+$ where α is continuous function.

(A_2) There exist constants $M, M_f > 0$ such that $0 \leq t^{\alpha-1} \leq M$ and $0 \leq f(t,x) \leq M_f$ for all $t \in J$ and $x \in \mathbb{R}$.

(A_3) There exists a $\mathscr{D}$-function ψ_f such that

$$0 \leq f(t,x) - f(t,y) \leq \psi_f(x-y)$$

for all $t \in J$ and $x, y \in \mathbb{R}$, $x \leq y$.

(A_4) $f(t,x)$ is nondecreasing in x for all $t \in J$.

(A_5) The problem (1.5) has a lower solution $u_l \in C(J,\mathbb{R})$.

The following lemma is useful in what follows.

Lemma 3.4. *For any* $f \in C(J \times \mathbb{R}, \mathbb{R}))$, *if* u *is a solution of the problem*

$$D^{\alpha}(u(t)) + au(t) = f(t,u(t))$$

$$\lim_{t \to 0^+} t^{1-\alpha} u(t) = u_0$$

$$0 < \alpha \leq 1, \quad 0 < t \leq T < \infty$$

then

$$u(t) = u_0 t^{\alpha-1} E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t-s)^{\alpha}) f(s,u(s))ds \tag{3.3}$$

where $E_{\alpha,\alpha}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha(k+1))}$ *is the classical Mittag-Leffler function and vice-versa.*

Proof. The solution followed by [1, 2] □

Theorem 3.5. *Assume that hypotheses (A_1)-(A_5) hold. Furthermore, assume that*

$$\left(M_f T^{\alpha-1}\{1 - E_{\alpha,\alpha}(a(t)^{\alpha})\}\right) \psi_f(r) < r,\ r > 0, \tag{3.4}$$

then the FDE(1.5) has a solution x^* defined on J and the sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ of successive approximations defined by

$$x_{n+1}(t) = x_0 t^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})f(s,x_n(s))ds \tag{3.5}$$

for all $t \in J$, where $x_0 = c$, converges monotonically to x^*.

Proof. Set $E = C(J,\mathbb{R})$. Then, from Lemma 3.1 it follows that every compact chain in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ in E.

Define three operators $\mathscr{A}$ and $\mathscr{B}$ on E by

$$\mathscr{A}x(t) = x_0 t^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha)\ t \in J, \tag{3.6}$$

and

$$\mathscr{B}x(t) = \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})f(s,x(s))ds,\ t \in J. \tag{3.7}$$

From the continuity of the integral and the hypotheses (A_1)-(A_5), it follows that $\mathscr{A}$ and $\mathscr{B}$ define the maps $\mathscr{A},\mathscr{B} : E \to \mathscr{K}$. Now by definitions of the operators $\mathscr{A}$ and $\mathscr{B}$, the FDE (1.5) is equivalent to the operator equation

$$\mathscr{A}x(t) + \mathscr{B}x(t) = x(t),\ \ t \in J. \tag{3.8}$$

We shall show that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Theorem 2.9. This is achieved in the series of following steps.

Step I: *$\mathscr{A}$ and $\mathscr{B}$ are nondecreasing on E.*

Let $x,y \in E$ be such that $x \geq y$. Then by hypothesis (A_2), (A_3) and (A_4), we obtain

$$\begin{aligned}
\mathscr{A}x(t) &= x_0 t^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha),\ t \in J,\\
&\geq y_0 t^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha),\ t \in J,\\
&= \mathscr{A}y(t)
\end{aligned}$$

for all $t \in J$. This shows that $\mathscr{A}$ is nondecreasing operators on E into E. Similarly, using hypothesis (A_4),

$$\begin{aligned}
\mathscr{B}x(t) &= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t-s)^\alpha) f(s,x(s))ds \\
&\geq \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t-s)^\alpha) f(s,y(s))ds \\
&= \mathscr{B}y(t)
\end{aligned}$$

for all $t \in J$. Hence, it is follows that the operator $\mathscr{B}$ is also a nondecreasing operator on E into itself. Thus, $\mathscr{A}$ and $\mathscr{B}$ are nondecreasing positive operators on E into itself.

Step II: *$\mathscr{A}$ is partially bounded and partially $\mathscr{D}$-Lipschitz on E.*

Let $x \in E$ be arbitrary. Then by (A_2),

$$\begin{aligned}
|\mathscr{A}x(t)| &= \left|x_0 t^{\alpha-1} E_{\alpha,\alpha}(-at^\alpha)\Gamma(\alpha)\right| \\
&\leq \left|t^{\alpha-1}\right| |x_0 E_{\alpha,\alpha}(-at^\alpha)\Gamma(\alpha)| \\
&\leq \left|t^{\alpha-1}\right| |E_{\alpha,\alpha}(-at^\alpha)| |x_0 \Gamma(\alpha)| \\
&\leq M |E_{\alpha,\alpha}(-at^\alpha)| K_0
\end{aligned}$$

taking supremum over t, we get $\|\mathscr{A}x\| \leq M|E_{\alpha,\alpha}(-at^\alpha)|K_0$ and consequently $\mathscr{A}$ is partially bounded on E.

Next, let $x, y \in E$ be such that $x \leq y$. Then, by hypothesis (A_3),

$$\begin{aligned}
|\mathscr{A}x(t) - \mathscr{A}y(t)| &= \left|x_0 t^{\alpha-1} E_{\alpha,\alpha}(-at^\alpha)\Gamma(\alpha) - y_0 t^{\alpha-1} E_{\alpha,\alpha}(-at^\alpha)\Gamma(\alpha)\right| \\
&\leq \left|t^{\alpha-1} E_{\alpha,\alpha}(-at^\alpha)\Gamma(\alpha)\right| |x_0 - y_0| \\
&\leq \psi |x(t) - y(t)| \\
&\leq \psi(|x-y|),
\end{aligned}$$

for all $t \in J$. Taking supremum over t, we obtain

$$\|\mathscr{A}x - \mathscr{A}y\| \leq \psi(\|x-y\|)$$

for all $x, y \in E$ with $x \leq y$. Hence $\mathscr{A}$ is partially nonlinear $\mathscr{D}$-Lipschitz operators on E which further implies it is also a partially continuous on E into itself.

Step III: *$\mathscr{B}$ is a partially continuous operator on E.*

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a chain C of E such that $x_n \to x$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \mathscr{B}x_n(t) &= \lim_{n\to\infty} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t-s)^{\alpha}) f(s, x_n(s))ds \\
&= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t-s)^{\alpha}) \left[\lim_{n\to\infty} f(s, x_n(s))\right] ds \\
&= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t-s)^{\alpha}) f(s, x(s))ds \\
&= \mathscr{B}x(t),
\end{aligned}$$

for all $t \in J$. This shows that $\mathscr{B}x_n$ converges monotonically to $\mathscr{B}x$ pointwise on J.

Next, we will show that $\{\mathscr{B}x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then

$$\begin{aligned}
&\left|Bx_n(t_2) - Bx_n(t_1)\right| \\
&\quad = \left|\int_0^{t_2} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t_2-s)^{\alpha}) f(s, x_n(s))ds \right. \\
&\qquad \left. - \int_0^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t_1-s)^{\alpha}) f(s, x_n(s))ds \right| \\
&\quad \leq \left|\int_0^{t_2} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t_2-s)^{\alpha}) f(s, x_n(s))ds \right. \\
&\qquad \left. - \int_0^{t_2} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t_1-s)^{\alpha}) f(s, x_n(s))ds \right| \\
&\quad + \left|\int_0^{t_2} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t_1-s)^{\alpha}) f(s, x_n(s))ds \right. \\
&\qquad \left. - \int_0^{t_1} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t_1-s)^{\alpha}) f(s, x_n(s))ds \right| \\
&\quad + \left|\int_0^{t_1} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t_1-s)^{\alpha}) f(s, x_n(s))ds \right.
\end{aligned}$$

(3.9)

$$
\begin{aligned}
&- \int_0^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t_1-s)^{\alpha}) f(s,x_n(s))ds \Bigg| \\
&\leq \int_0^{t_2} (t_2-s)^{\alpha-1} \left| E_{\alpha,\alpha}(-a(t_2-s)^{\alpha}) - E_{\alpha,\alpha}(-a(t_1-s)^{\alpha}) \right| \left| f(s,x_n(s)) \right| ds \\
&+ \left| \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t_1-s)^{\alpha}) f(s,x_n(s))ds \right| \\
&+ \int_0^{t_1} \left| (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right| E_{\alpha,\alpha}(-a(t_1-s)^{\alpha}) \left| f(s,x_n(s)) \right| ds \\
&\leq \int_0^{T} (t_2-s)^{\alpha-1} \left| E_{\alpha,\alpha}(-a(t_2-s)^{\alpha}) - E_{\alpha,\alpha}(-a(t_1-s)^{\alpha}) \right| M_f ds \\
&+ \int_0^{T} \left| (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right| E_{\alpha,\alpha}(-a(t_1-s)^{\alpha}) M_f ds \\
&+ \int_0^{T} \left| (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right| E_{\alpha,\alpha}(-a(t_1-s)^{\alpha}) M_f ds \\
&\leq M_f \left(\int_0^{T} \left| (t_2-s)^{\alpha-1} \right|^2 ds \right)^{1/2} \left(\int_0^{T} \left| E_{\alpha,\alpha}(-a(t_2-s)^{\alpha}) - E_{\alpha,\alpha}(-a(t_1-s)^{\alpha}) \right|^2 ds \right)^{1/2} \\
&+ 2 \left(\int_0^{T} \left| (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right|^2 ds \right)^{1/2} \left(\int_0^{T} \left| E_{\alpha,\alpha}(-a(t_1-s)^{\alpha}) \right|^2 ds \right)^{1/2} M_f
\end{aligned}
$$

Since the functions $E_{\alpha,\alpha}$ and α are continuous on compact interval J so uniformly continuous there. Therefore, from the above inequality (3.9) it follows that

$$
|\mathscr{B}x_n(t_2) - \mathscr{B}x_n(t_1)| \to 0 \quad \text{as} \quad n \to \infty
$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathscr{B}x_n \to \mathscr{B}x$ is uniform and hence $\mathscr{B}$ is partially continuous on E.

Step IV: *$\mathscr{B}$ is uniformly partially compact operator on E.*

Let C be an arbitrary chain in E. We show that $\mathscr{B}(C)$ is a uniformly bounded and equicontinuous set in E. First we show that $\mathscr{B}(C)$ is uniformly bounded. Let $y \in \mathscr{B}(C)$ be any element.

Then there is an element $x \in C$ be such that $y = \mathscr{B}x$. Now, by hypothesis ,

$$\begin{aligned} |y(t)| &= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t-s)^{\alpha}) f(s,x(s))ds \\ &\leq M_f T^{\alpha-1}\{1 - E_{\alpha,\alpha}(a(t)^{\alpha})\} \\ &= r \end{aligned}$$

for all $t \in J$. Taking the supremum over t, we obtain $\|y\| \leq \|\mathscr{B}x\| \leq r$ for all $y \in \mathscr{B}(C)$. Hence, $\mathscr{B}(C)$ is a uniformly bounded subset of E. Moreover, $\|\mathscr{B}(C)\| \leq r$ for all chains C in E. Hence, $\mathscr{B}$ is a uniformly partially bounded operator on E.

Next, we will show that $\mathscr{B}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then, for any $y \in \mathscr{B}(C)$, one has

$$\begin{aligned} &\Big|Bx(t_2) - Bx(t_1)\Big| \\ &\quad = \Big|\int_0^{t_2} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t_2-s)^{\alpha}) f(s,x(s))ds \\ &\qquad - \int_0^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t_1-s)^{\alpha}) f(s,x(s))ds\Big| \\ &\quad \leq \Big|\int_0^{t_2} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t_2-s)^{\alpha}) f(s,x(s))ds \\ &\qquad - \int_0^{t_2} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t_1-s)^{\alpha}) f(s,x(s))ds\Big| \\ &\quad + \Big|\int_0^{t_2} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t_1-s)^{\alpha}) f(s,x(s))ds \\ &\qquad - \int_0^{t_1} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t_1-s)^{\alpha}) f(s,x(s))ds\Big| \\ &\quad + \Big|\int_0^{t_1} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t_1-s)^{\alpha}) f(s,x(s))ds \\ &\qquad - \int_0^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t_1-s)^{\alpha}) f(s,x(s))ds\Big| \end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^{t_2} (t_2-s)^{\alpha-1} \left|E_{\alpha,\alpha}(-a(t_2-s)^{\alpha}) - E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})\right| |f(s,x(s))|\, ds \\
&+ \left| \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t_1-s)^{\alpha}) f(s,x(s))ds \right| \\
&+ \int_0^{t_1} \left|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right| E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})\, |f(s,x(s))|\, ds \\
&\leq \int_0^{T} (t_2-s)^{\alpha-1} \left|E_{\alpha,\alpha}(-a(t_2-s)^{\alpha}) - E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})\right| M_f ds \\
&+ \int_0^{T} \left|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right| E_{\alpha,\alpha}(-a(t_1-s)^{\alpha}) M_f ds \\
&+ \int_0^{T} \left|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right| E_{\alpha,\alpha}(-a(t_1-s)^{\alpha}) M_f ds \\
&\leq M_f \left(\int_0^T \left|(t_2-s)^{\alpha-1}\right|^2 ds \right)^{1/2} \left(\int_0^T \left|E_{\alpha,\alpha}(-a(t_2-s)^{\alpha}) - E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})\right|^2 ds \right)^{1/2} \\
&+ 2 \left(\int_0^T \left|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right|^2 ds \right)^{1/2} \left(\int_0^T \left|E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})\right|^2 ds \right)^{1/2} M_f \\
&\longrightarrow 0 \quad \text{as} \quad t_1 \to t_2,
\end{aligned}
$$

uniformly for all $y \in \mathscr{B}(C)$. Hence $\mathscr{B}(C)$ is an equicontinuous subset of E. Now, $\mathscr{B}(C)$ is a uniformly bounded and equicontinuous set of functions in E, so it is compact. Consequently, $\mathscr{B}$ is a uniformly partially compact operator on E into itself.

Step V: *u_l satisfies the operator inequality $u_l \leq \mathscr{A} u_l + \mathscr{B} u_l$.*

By hypothesis (A_5), the FDE 1.5 has a lower solution u_l defined on J. Then, we have

$$
u_l(t) \leq u_l(0) t^{\alpha-1} E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t-s)^{\alpha}) f(s,u_l(s))ds \tag{3.10}
$$

for all $t \in J$. From the definitions of the operators $\mathscr{A}$ and $\mathscr{B}$ it follows that $u_l(t) \leq \mathscr{A} u_l(t) + \mathscr{B} u_l(t)$ for all $t \in J$. Hence $u_l \leq \mathscr{A} u_l + \mathscr{B} u_l$.

Step VI: *The $\mathscr{D}$-functions $\psi_{\mathscr{A}}$ satisfy the growth condition $M\psi_{\mathscr{A}}(r) < r$, for $r > 0$.*

Finally, the $\mathscr{D}$-function $\psi_{\mathscr{A}}$ of the operator $\mathscr{A}$ satisfy the inequality given in hypothesis of Theorem 2.9, viz. ,

$$
M\psi_{\mathscr{A}}(r) \leq r
$$

for all $r > 0$.

Thus $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Theorem 2.9 and we conclude that the operator equation $\mathscr{A}x + \mathscr{B}x = x$ has a solution. Consequently the FDE (1.5) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}_{n\in\mathbb{N}}$ of successive approximations defined by (3.5) converges monotonically to x^*. This completes the proof. □

The conclusion of Theorems 3.5 also remains true if we replace the hypothesis (A_5) with the following one:

(A'_5) The FDE (1.5) has an upper solution $u_u \in C(J,\mathbb{R})$.

The proof of Theorem 3.5 under this new hypothesis is similar and can be obtained by closely observing the same arguments with appropriate modifications. We need the following definition in what follows.

Definition 3.6. *A function $r \in C(J,\mathbb{R})$ is said be a maximal solution of the FDE (1.5)if for any other solution x of the FDE (1.5), one has $x(t) \le r(t)$ for all $t \in J$. Similarly, a minimal solution ρ of the FDE (1.5) can be defined in a similar way by reversing the above inequality.*

The following lemma is fundamental in the proof of maximal and minimal solutions for the FDE (1.5) on J.

Lemma 3.7. *Suppose that there exist two functions $y,z \in C(J,\mathbb{R})$ satisfying*

$$y(t) \le y(0)t^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})f(s,y(s))ds \tag{3.11}$$

and

$$z(t) \ge z(0)t^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})f(s,z(s))ds \tag{3.12}$$

for all $t \in J$. If one of the inequalities (3.11) *and* (3.12) *is strict, then*

$$y(t) < z(t) \tag{3.13}$$

for all $t \in J$.

Proof. Suppose that the inequality (3.12) is strict and let the conclusion (3.13) be false. Then there exists $t_1 \in J$ such that

$$y(t_1) = z(t_1), t_1 > 0,$$

and

$$y(t) < z(t), 0 < t < t_1.$$

From the monotonicity of $f(t,x)$ in x , we get

$$\begin{aligned} y(t_1) &\leq y(0)t_1^{\alpha-1}E_{\alpha,\alpha}(-at_1^{\alpha})\Gamma(\alpha) + \int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})f(s,y(s))ds \\ &= z(0)t_1^{\alpha-1}E_{\alpha,\alpha}(-at_1^{\alpha})\Gamma(\alpha) + \int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})f(s,z(s))ds \\ &< z(t_1) \end{aligned} \tag{3.14}$$

which contradicts the fact that $y(t_1) = z(t_1)$. Hence, $y(t) < z(t)$ for all $t \in J$. □

Theorem 3.8. *Suppose that all the hypotheses of Theorem 3.5 hold. Then the FDE (1.5) has a maximal and a minimal solution on J.*

Proof. Let $\varepsilon > 0$ be given. Now consider the fractional integral equation

$$x_\varepsilon(t) = x_\varepsilon(0)t^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})f_\varepsilon(s,x_\varepsilon(s))ds \tag{3.15}$$

for all $t \in J$, where

$$f_\varepsilon(t,x_\varepsilon(t)) = f(t,x_\varepsilon(t)) + \varepsilon$$

Clearly the function $f_\varepsilon(t,x_\varepsilon(t))$, satisfy all the hypotheses (A_1)-(A_5)and therefore, by Theorem 3.5, FDE (1.5) has at least a solution $x_\varepsilon(t) \in C(J,\mathbb{R})$.

Let ε_1 and ε_2 be two real numbers such that $0 < \varepsilon_2 < \varepsilon_1 < \varepsilon$. Then, we have

$$\begin{aligned} x_{\varepsilon_2}(t) &= x_{\varepsilon_2}(0)t^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})f_{\varepsilon_2}(s,x_{\varepsilon_2}(s))ds \\ &= x_{\varepsilon_2}(0)t^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})[f(s,x_{\varepsilon_2}(s)) + \varepsilon_2]ds \end{aligned} \tag{3.16}$$

and

$$
\begin{aligned}
x_{\varepsilon_1}(t) &= x_{\varepsilon_1}(0)t^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})f_{\varepsilon_1}(s,x_{\varepsilon_1}(s))ds \\
&= x_{\varepsilon}(0)t^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha_1) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})[f(s,x_{\varepsilon_1}(s))+\varepsilon_1]ds \\
&> x_{\varepsilon_1}(0)t^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})[f(s,x_{\varepsilon_2}(s))+\varepsilon_2]ds \qquad (3.17)
\end{aligned}
$$

for all $t \in J$. Now, applying the Lemma 3.7 to the inequalities (3.16) and (3.17), we obtain

$$
x_{\varepsilon_2}(t) < x_{\varepsilon_1}(t) \tag{3.18}
$$

for all $t \in J$.

Let $\varepsilon_0 = \varepsilon$ and define a decreasing sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ of positive real numbers such that $\lim_{n\to\infty}\varepsilon_n = 0$. Then in view of the above facts $\{x_{\varepsilon_n}\}$ is a decreasing sequence of functions in $C(J,\mathbb{R})$. We show that is is uniformly bounded and equicontinuous. Now, by hypotheses,

$$
\begin{aligned}
|x_{\varepsilon_n}(t)| &\le |x_{\varepsilon_n}(0)t^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})f_{\varepsilon_n}(s,x_{\varepsilon_n}(s))ds| \\
&\le (KME_{\alpha,\alpha}(-aT^{\alpha})\Gamma(\alpha)) + \varepsilon + \left(M_f T^{\alpha-1}(1-E_{\alpha,\alpha}(aT^{\alpha}))\right) + \varepsilon \\
&\le r
\end{aligned}
$$

for all $t \in J$. Taking the supremum over t, we obtain $\|x_{\varepsilon_n}\| \le r$ for all $n \in \mathbb{N}$. This shows that the sequence $\{x_{\varepsilon_n}\}$ is uniformly bounded.

Next we show that $\{x_{\varepsilon_n}\}$ is an equicontinuous sequence of functions in $C(J,\mathbb{R})$. Let $t_1, t_2 \in J$ be arbitrary. Then,

$$
\begin{aligned}
|x_{\varepsilon_n}(t_1)-x_{\varepsilon_n}(t_2)| \le |x_{\varepsilon_n}(0)t_1^{\alpha-1}E_{\alpha,\alpha}(-at_1^{\alpha})\Gamma(\alpha) + \int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})f_{\varepsilon_n}(s,x_{\varepsilon_n}(s))ds \\
-x_{\varepsilon_n}(0)t_2^{\alpha-1}E_{\alpha,\alpha}(-at_2^{\alpha})\Gamma(\alpha) - \int_0^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_2-s)^{\alpha})f_{\varepsilon_n}(s,x_{\varepsilon_n}(s))ds|
\end{aligned}
$$

(3.19)

$$
\begin{aligned}
&= x_{\varepsilon_n}(0)\Gamma(\alpha)|t_1^{\alpha-1}E_{\alpha,\alpha}(-at_1^{\alpha}) - t_2^{\alpha-1}E_{\alpha,\alpha}(-at_2^{\alpha})| \\
&\quad + |\int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})f_{\varepsilon_n}(s,x_{\varepsilon_n}(s))ds \\
&\qquad - \int_0^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_2-s)^{\alpha})f_{\varepsilon_n}(s,x_{\varepsilon_n}(s))ds|
\end{aligned}
$$

$$
\begin{aligned}
&\le x_{\varepsilon_n}(0)\Gamma(\alpha)|t_1^{\alpha-1}E_{\alpha,\alpha}(-at_1^{\alpha}) - t_1^{\alpha-1}E_{\alpha,\alpha}(-at_2^{\alpha}) + t_1^{\alpha-1}E_{\alpha,\alpha}(-at_2^{\alpha}) - t_2^{\alpha-1}E_{\alpha,\alpha}(-at_2^{\alpha})| \\
&\quad + \left|\int_0^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_2-s)^{\alpha})f_{\varepsilon_n}(s,x_{\varepsilon_n}(s))ds \right. \\
&\quad \left. - \int_0^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})f_{\varepsilon_n}(s,x_{\varepsilon_n}(s))ds\right| \\
&+ \left|\int_0^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})f_{\varepsilon_n}(s,x_{\varepsilon_n}(s))ds \right. \\
&\quad \left. - \int_0^{t_1}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})f_{\varepsilon_n}(s,x_{\varepsilon_n}(s))ds\right| \\
&+ \left|\int_0^{t_1}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})f_{\varepsilon_n}(s,x_{\varepsilon_n}(s))ds \right. \\
&\quad \left. - \int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})f_{\varepsilon_n}(s,x_{\varepsilon_n}(s))ds\right| \\
&\le x_{\varepsilon_n}(0)\Gamma(\alpha)\left\{t_1^{\alpha-1}|E_{\alpha,\alpha}(-at_1^{\alpha}) - E_{\alpha,\alpha}(-at_2^{\alpha})| + E_{\alpha,\alpha}(-at_2^{\alpha})|t_1^{\alpha-1}-t_2^{\alpha-1}|\right\} \\
&\quad + \int_0^{t_2}(t_2-s)^{\alpha-1}\,|E_{\alpha,\alpha}(-a(t_2-s)^{\alpha}) - E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})|\,|f_{\varepsilon_n}(s,x_{\varepsilon_n}(s))|\,ds \\
&+ \left|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})f_{\varepsilon_n}(s,x_{\varepsilon_n}(s))ds\right| \\
&+ \int_0^{t_1}\left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right|E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})\,|f_{\varepsilon_n}(s,x_{\varepsilon_n}(s))|\,ds
\end{aligned}
$$

(3.20)

$$\leq x_{\varepsilon_n}(0)\Gamma(\alpha)\left\{t_1^{\alpha-1}|E_{\alpha,\alpha}(-at_1^{\alpha})-E_{\alpha,\alpha}(-at_2^{\alpha})|+E_{\alpha,\alpha}(-at_2^{\alpha})|t_1^{\alpha-1}-t_2^{\alpha-1}|\right\}$$
$$+\int_0^T(t_2-s)^{\alpha-1}\left|E_{\alpha,\alpha}(-a(t_2-s)^{\alpha})-E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})\right|[M_f+\varepsilon_n]ds$$
$$+\int_0^T\left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right|E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})[M_f+\varepsilon_n]ds$$
$$+\int_0^T\left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right|E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})[M_f+\varepsilon_n]ds$$
$$\leq x_{\varepsilon_n}(0)\Gamma(\alpha)\left\{t_1^{\alpha-1}|E_{\alpha,\alpha}(-at_1^{\alpha})-E_{\alpha,\alpha}(-at_2^{\alpha})|+E_{\alpha,\alpha}(-at_2^{\alpha})|t_1^{\alpha-1}-t_2^{\alpha-1}|\right\}$$
$$+[M_f+\varepsilon_n]\left(\int_0^T\left|(t_2-s)^{\alpha-1}\right|^2ds\right)^{1/2}\left(\int_0^T\left|E_{\alpha,\alpha}(-a(t_2-s)^{\alpha})-E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})\right|^2ds\right)^{1/2}$$
$$+2\left(\int_0^T\left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right|^2ds\right)^{1/2}\left(\int_0^T\left|E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})\right|^2ds\right)^{1/2}[M_f+\varepsilon_n]$$

Since the functions f and $E_{\alpha,\alpha}$ are continuous on compact $[0,T]\times[-r,r]\times[-r,r]$, $(t-s)^{1-\alpha}$ is continuous on compact $[0,T]\times[0,T]$, so uniformly continuous there. Hence, from (3.19) it follows that

$$|x_{\varepsilon_n}(t_1)-x_{\varepsilon_n}(t_2)|\to 0 \quad \text{as } t_1\to t_2$$

uniformly for all $n\in\mathbb{N}$. As a result $\{x_{\varepsilon_n}\}$ is an equicontinuous sequence of functions in $C(J,\mathbb{R})$. Now the sequence $\{x_{\varepsilon_n}\}$ is uniformly bounded and equicontinuous, so it is compact in view of Arzelá-Ascoli theorem. By Lemma 3.1, $\{x_{\varepsilon_n}\}$ converges uniformly to a function say $r\in C(J,\mathbb{R})$, i. e. $\lim_{n\to\infty}x_{\varepsilon_n}(t)=r(t)$ uniformly on J.

We show that the function r is a solution of the FDE (1.5) on J. Now, $\{x_{\varepsilon_n}\}$ is a solution of the FDE

$$x_{\varepsilon_n}(t)=x_{\varepsilon_n}(0)t^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha)+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})f_{\varepsilon_n}(s,x_{\varepsilon_n}(s))ds$$

(3.21)

$$= x_{\varepsilon_n}(0)t^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})[f(s,x_{\varepsilon_n}(s)) + \varepsilon_n]ds \tag{3.22}$$

for all $t \in J$. Now, taking the limit as by hypotheses $n \to \infty$ in the above inequality (3.21), we obtain

$$r(t) = r(0)t^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})f(s,r(s))ds$$

for all $t \in J$. This shows that r is a solution of the FDE (1.5) defined on J.

Finally, we shall show that $r(t)$ is the maximal solution of the FDE (1.5) defined on J. To do this, let $x(t)$ be any solution of the FDE (1.5) defined on J. Then, we have

$$x(t) = x(0)t^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})f(s,x(s))ds \tag{3.23}$$

for all $t \in J$. Similarly, if x_ε is any solution of the FDE

(3.24)

$$x_\varepsilon(t) = x_\varepsilon(0)t^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})[f(s,x_\varepsilon(s)) + \varepsilon]ds$$

then,

$$x_\varepsilon(t) > x_\varepsilon(0)t^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})f(s,x_\varepsilon(s))ds \tag{3.25}$$

for all $t \in J$. From the inequalities (3.23) and (3.25) it follows that $x(t) \leq x_\varepsilon(t),\ t\ J$. Taking the limit as $\varepsilon \to 0$, we obtain $x(t) \leq r(t)$ for all $t \in J$. Hence r is a maximal solution of the FDE (1.5) defined on J. In the same way Minimal solution of the FDE can be obtained □

Further we prove now that the maximal and minimal solutions serve as the bounds for the solutions of the related differential inequality to FDE (1.5) on $J = [0,T]$.

Theorem 3.9. *Suppose that all the hypotheses of Theorem 3.5 hold. Further, if there exists a function $u \in C(J,\mathbb{R})$ such that*

$$u(t) \leq u(0)t^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})f(s,u(s))ds \tag{3.26}$$

for all $t \in J$, then,

$$u(t) \leq r(t) \tag{3.27}$$

for all $t \in J$, where r is a maximal solution of the FDE (1.5) on J.

Proof. Let $\varepsilon > 0$ be arbitrary small. Then, by Theorem 3.5, $r_\varepsilon(t)$ is a solution of the FDE and that the limit

$$r(t) = \lim_{\varepsilon \to 0} r_\varepsilon(t) \tag{3.28}$$

is uniform on J and is a maximal solution of the FDE (1.5) on J. Hence, we obtain

(3.29)

$$r_\varepsilon(t) = r_\varepsilon(0)t^{\alpha-1}E_{\alpha,\alpha}(-at^\alpha)\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)[f(s,r_\varepsilon(s)) + \varepsilon]ds$$

for all $t \in J$. From the above inequality it follows that

$$r_\varepsilon(t) > r_\varepsilon(0)t^{\alpha-1}E_{\alpha,\alpha}(-at^\alpha)\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)f(s,r_\varepsilon(s))ds \tag{3.30}$$

Now we apply Lemma 3.7 to the inequalities (3.26) and (3.30) and conclude that

$$u(t) < r_\varepsilon(t) \tag{3.31}$$

for all $t \in J$. This further in view of limit (3.28) implies that the inequality (3.27) holds on J. This completes the proof. □

Similarly, we have the following result for the FDE (1.5) on J.

Theorem 3.10. *Suppose that all the hypotheses of Theorem 3.5 hold. Further, if there exists a function $v \in C(J,\mathbb{R})$ such that*

$$v(t) \geq v(0)t^{\alpha-1}E_{\alpha,\alpha}(-at^\alpha)\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)f(s,v(s))ds \tag{3.32}$$

for all $t \in J$, then,

$$v(t) \geq \rho(t) \tag{3.33}$$

for all $t \in J$, where ρ is a minimal solution of the FDE (1.5) on J.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] Ravi P. Agarwal, V. Lakshmikanthama, Juan J. Nieto, Nonlinear Analysis 72 (2010), 2859-2862.

[2] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier Science B. V, Amsterdam, 2006.

[3] Tarachand L . Holambe , Mohammed Mazhar - Ul-Haque, Govind P. Kamble ; Approximations to the Solution of Cauchy Type Weighted Nonlocal Fractional Differential Equation, Nonlinear Analysis and Differential Equations, 4 (15) (2016), 697 - 717.

[4] Mohammed Mazhar - Ul-Haque, Tarachand L . Holambe , Govind P. Kamble ; Solution to Weighted nonlocal fractional differential equation, International Journal of Pure and Applied Mathematics, 108 (1) (2016), 79-91.

[5] M. Khaled, M. Furati, N. E. Tatar, Power-type estimates for nonlinear fractional differential equation, Nonlinear Anal. 62 (2005), 1025-1036.

[6] Zhongli Weia, Wei Dongc, Junling Chea, Periodic boundary value problems for fractional differential equations involving a RiemannLiouville fractional derivative, Nonlinear Anal. 73 (2010), 3232-3238

[7] T. Jankowski, Fractional equations of Volterra type involving aRiemannLiouville derivative, Appl. Math. Lett. 26 (2013) 344-350.

[8] Butzer, P. L. , Kilbas, A. A. , and Trujillo, J. J. , Fractional calculus in the Mellin setting and Hadamard-type fractional integrals, Journal of Mathematical Analysis and Applications, 269 (2002), 1-27.

[9] Butzer, P. L. , Kilbas, A. A. , and Trujillo, J. J. , Mellin transform analysis and integration by parts for Hadamard-type fractional integrals, Journal of Mathematical Analysis and Applications, 270 (2002), 1-15.

[10] B. C. Dhage, A nonlinear alternative in Banach algebras with applications to functional differential equations, Nonlinear Funct. Anal. & Appl. 8 (2004), 563–575.

[11] B. C. Dhage, Fixed point theorems in ordered Banach algebras and applications, PanAmer. Math. J. 9 (4) (1999), 93-102.

[12] B. C. Dhage, Hybrid fixed point theory in partially ordered normed linear spaces and applications to fractional integral equations, Differ. Equ Appl. 5 (2013), 155-184.

[13] B. C. Dhage, Partially condensing mappings in ordered normed linear spaces and applications to functional integral equations, Tamkang J. Math. 45 (4) (2014), 397-426.

[14] B. C. Dhage, Nonlinear $\mathscr{D}$-set-contraction mappings in partially ordered normed linear spaces and applications to functional hybrid integral equations, Malaya J. Mat. 3 (1) (2015), 62-85.

[15] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[16] V. Lakshmikantham, S. Leela, Differential and integral inequalities, Vol I, New York, London, 1909.

[17] K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Differential Equations, John Wiley, New York, 1993.

[18] B. R. Sontakke, G. P. Kamble, Mohammed Mazhar-ul-Haque, Some integral transform of generalized Mittag-Lefller functions, International Journal of Pure and Applied Mathematics, 108 (2) (2016), 327-339.

[19] T. L. Holambe, Mohammed Mazhar-ul-Haque. A remark on semigroup property in fractional calculus. International Journal of Mathematics and computer Application Research. 4 (6) (2014), 27-32.

[20] Mohammed Mazhar-ul-Haque, T. L. Holambe. A Q Function in fractional calculus. Journal of Basic and Applied Research International 6 (4) (2015), 248-252.