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FIXED POINT THEOREMS IN A SPACE WITH THREE METRICS

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Abstract. The purpose of this paper is to present some fixed point results for Banach, Kannan and Chatterjea contraction in a space with three metrics supported by some examples.

Keywords: fixed point; complete metric space; Kannan contraction; Chatterjea contraction.

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1. Introduction

Since its appearance in 1922, the Banach fixed point theorem [1] solved several problems of the existence of solutions of nonlinear problems arising in physical, biological, and social sciences.

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Theorem 1.1. *Let (X,d) be a complete metric space. Let T be a contraction on X, i.e., there exists $r \in [0,1[$ satisfying*

$$d(Tx,Ty) \leq rd(x,y), \text{ for all } x,y \in X.$$

Then T has a unique fixed point.

The generalization of this theorem have been established in various setting by many authors. The purpose of this article is to get a generalization of the Banach contraction fixed point theorem in a space with three metrics.

2. Preliminaries

In 1968, Kannan presented the following related fixed point theorem [2].

Theorem 2.1. *Let (X,d) be a complete metric space. Let T be a Kannan mapping on X, i.e., there exists $r \in [0,\frac{1}{2}[$ satisfying*

$$d(Tx,Ty) \leq r\left(d(x,Tx)+d(y,Ty)\right), \text{ for all } x,y \in X.$$

Then T has a unique fixed point.

In 1972, Chatterjea presented the following related fixed point theorem [3].

Theorem 2.2. *Let (X,d) be a complete metric space. Let T be a Chatterjea mapping on X, i.e., there exists $r \in [0,\frac{1}{2}[$ satisfying*

$$d(Tx,Ty) \leq r\left(d(x,Ty)+d(y,Tx)\right), \text{ for all } x,y \in X.$$

Then T has a unique fixed point.

The following results is due to Mizoguchi and Takahashi [5].

Theorem 2.3. *Let (X,d) be a complete metric space. Let T be a mapping satisfying*

$$d(Tx,Ty) \leq \alpha(d(x,y))d(x,y), \text{ for all } x,y \in X,$$

where $\alpha : [0,+\infty[\to [0,1[$ is a function such that $\limsup_{s\to r^+} \alpha(s) < 1$, for all $r \geq 0$.
Then T has a unique fixed point $x^* \in X$.

Recently, EL. Marhrani and K. Chaira proved a generalization of the Banach contraction fixed point theorem in a space with two metrics [4].

Definition 2.4. Let X be a nonempty set and let d, δ be two metrics on X. (X,d,δ) is called an (M)-space if for all Cauchy sequence $(x_n)_n$ in (X,d) and (X,δ), there exist $x^*,y^* \in X$ such that

$$\lim_n d(x_n,x^*) = \lim_n \delta(x_n,y^*) = 0.$$

Theorem 2.5. *Let X be non-empty set, d and δ two metrics on X and $T : X \to X$ a mapping such that:*

(1) (X,d,δ) is a (M)-space.

(2) For all $x,y \in X$, one of the following two conditions:

i. $d(x,Ty) \leq \delta(x,y)$,

ii. $\delta(x,Ty) \leq d(x,y)$,

implies

$$\begin{cases} d(Tx,Ty) \leq \alpha(\delta(x,y))\delta(x,y), \\ \delta(Tx,Ty) \leq \alpha(d(x,y))d(x,y), \end{cases}$$

where $\alpha : [0,+\infty[\to [0,1[$ is a function such that $\limsup_{s\to r^+} \alpha(s) < 1$, for all $r \geq 0$.
Then T has a unique fixed point $x^* \in X$.

3. Main results

Let X be a non-empty set and let d , δ and γ be three metrics on X.

Definition 3.1. (X,d,δ,γ) is called an (M)-space if for all Cauchy sequence $(x_n)_n$ in (X,d), (X,δ) and (X,γ), there exist $x^*,y^*,z^* \in X$ such that

$$\lim_n d(x_n,x^*) = \lim_n \delta(x_n,y^*) = \lim_n \gamma(x_n,z^*) = 0.$$

Example 3.2. if (X,d), (X,δ) and (X,γ) are complete metrics space, then (X,d,δ,γ) is an (M)-space.

Example 3.3. Let X be the set of all C^2 function u from $[0,1]$ into $\mathbb{R}$ with $u(0)=0$ and $u^{'}(0)=0$; we define three metrics on X by:

$$\begin{aligned} d(u,v) &= \sup_{x\in[0,1]} |u(x)-v(x)|, \\ \delta(u,v) &= \sup_{x\in[0,1]} |u^{'}(x)-v^{'}(x)|, \\ \gamma(u,v) &= \sup_{x\in[0,1]} |u^{''}(x)-v^{''}(x)|, \end{aligned}$$

for all $u,v\in X$. It is well know that the sequence of the polynomial function defined by:

$$\begin{aligned} u_1(x) &= 0, \\ u_{n+1}(x) &= u_n(x)+\frac{1}{2}(1-x-u_n^2(x)), \end{aligned}$$

are in X and converge uniformly to $x\mapsto\sqrt{1-x}$ which is not in X. Hence, (X,d) is non complete. We define the subsequence $(v_n)_n$ by:

$$v_n(x)=\int_0^x u_n(t)\,dt,\;\; x\in[0,1].$$

$(v_n)_n$ converge uniformly to $x\mapsto\int_0^x\sqrt{1-t}\,dt=\frac{2}{3}(1-(1-x)^{\frac{3}{2}})$, wich is not in X. Hence, (X,δ) is non complete.

If $(w_n)_n$ is a Cauchy sequence in (X,d), (X,δ) and (X,γ), there exist three continuous functions u,v,w such that $(w_n)_n$, $(w^{'}_n)_n$ and $(w^{''}_n)_n$ converge uniformly to u,v and w, respectively. Then u is of class C^2 and $u^{'}=v$, $u^{''}=w$ on X. Hence

$$\lim_n d(w_n,u)=\lim_n\delta(w_n,u)=\lim_n\gamma(w_n,u)=0.$$

It follows that (X,d,δ,γ) is an (M)-space.

Theorem 3.4. *Let X be non-empty set, d, δ and γ three metrics on X and $T:X\to X$ a mapping such that:*

(1) (X,d,δ,γ) is a (M)-space.

(2) For all $x,y\in X$, one of the following three conditions:

i. $d(x,Ty)\le\delta(x,y)$,

ii. $\delta(x,Ty)\le\gamma(x,y)$,

iii. $\gamma(x,Ty)\le d(x,y)$,

implies

$$\begin{cases} d(Tx,Ty) \leq \alpha(\delta(x,y))\delta(x,y), \\ \delta(Tx,Ty) \leq \alpha(\gamma(x,y))\gamma(x,y), \\ \gamma(Tx,Ty) \leq \alpha(d(x,y))d(x,y), \end{cases}$$

where $\alpha : [0,+\infty[\to [0,1[$ *is a function such that* $\limsup_{s\to r^+} \alpha(s) < 1$, *for all* $r \geq 0$.
Then T *has a unique fixed point* $x^* \in X$.

Proof. *step 1:*

Letting $x_0 \in X$, *we define the sequence* $(x_n)_n$ *by* $x_{n+1} = Tx_n$, *for each* $n \in \mathbb{N}$, *we have*

$$d(x_{n+1},Tx_n) = 0 \leq \delta(x_{n+1},x_n),$$

so, we obtain that

$$\begin{cases} d(Tx_{n+1},Tx_n) = d(x_{n+2},x_{n+1}) \leq \alpha(\delta(x_{n+1},x_n))\delta(x_{n+1},x_n), \\ \delta(Tx_{n+1},Tx_n) = \delta(x_{n+2},x_{n+1}) \leq \alpha(\gamma(x_{n+1},x_n))\gamma(x_{n+1},x_n), \\ \gamma(Tx_{n+1},Tx_n) = \gamma(x_{n+2},x_{n+1}) \leq \alpha(d(x_{n+1},x_n))d(x_{n+1},x_n), \end{cases}$$

then:

$$\begin{aligned} d(x_{n+1},x_{n+2}) &\leq \alpha(\delta(x_{n+1},x_n))\delta(x_{n+1},x_n) \\ &\leq \alpha(\delta(x_{n+1},x_n))\alpha(\gamma(x_{n-1},x_n))\gamma(x_{n-1},x_n) \\ &\leq \alpha(\delta(x_{n+1},x_n))\alpha(\gamma(x_{n-1},x_n))\alpha(d(x_{n-2},x_{n-1}))d(x_{n-2},x_{n-1}) \\ &\leq d(x_{n-2},x_{n-1}), \ \textit{for all} \ n \geq 2. \end{aligned}$$

Analogously, we obtain $\delta(x_{n+1},x_{n+2}) \leq \delta(x_{n-2},x_{n-1})$ *and* $\gamma(x_{n+1},x_{n+2}) \leq \delta(x_{n-2},x_{n-1})$.
It follows that $(d(x_{3p},x_{3p+1}))_p$, $(d(x_{3p+1},x_{3p+2}))_p$ *and* $(d(x_{3p+2},x_{3p+3}))_p$ *converges to* d_1, d_2, *and* d_3, *respectively. And* $(\delta(x_{3p},x_{3p+1}))_p$, $(\delta(x_{3p+1},x_{3p+2}))_p$ *and* $(\delta(x_{3p+2},x_{3p+3}))_p$ *converges to* δ_1, δ_2, *and* δ_3, *respectively. And* $(\gamma(x_{3p},x_{3p+1}))_p$, $(\gamma(x_{3p+1},x_{3p+2}))_p$ *and* $(\gamma(x_{3p+2},x_{3p+3}))_p$ *converges to* γ_1, γ_2, *and* γ_3, *respectively.*
Since $\limsup_{t\to\delta_1^+} \alpha(t) < 1$, $\limsup_{t\to\gamma_3^+} \alpha(t) < 1$ *and* $\limsup_{t\to d_2^+} \alpha(t) < 1$ *there exist* $p_1 \in \mathbb{N}$ *and* $r_1 \in [0,1[$ *such that for any integer* $p \geq p_1$

$$d(x_{3p+1},x_{3p+2}) \leq r_1 d(x_{3p-2},x_{3p-1}).$$

And $\limsup_{t\to\delta_2^+}\alpha(t)<1$, $\limsup_{t\to\gamma_1^+}\alpha(t)<1$ *and* $\limsup_{t\to d_3^+}\alpha(t)<1$ *there exist* $p_2\in\mathbb{N}$ *and* $r_2\in[0,1[$, *such that for any integer* $p\geq p_2$

$$d(x_{3p+2},x_{3p+3})\leq r_2 d(x_{3p-1},x_{3p}).$$

And $\limsup_{t\to\delta_3^+}\alpha(t)<1$, $\limsup_{t\to\gamma_2^+}\alpha(t)<1$ *and* $\limsup_{t\to d_1^+}\alpha(t)<1$, *there exist* $p_3\in\mathbb{N}$ *and* $r_3\in[0,1[$, *such that for any integer* $p\geq p_3$

$$d(x_{3p+3},x_{3p+4})\leq r_3 d(x_{3p},x_{3p+1}).$$

It follow that $\Sigma_{p\geq1}d(x_{3p-1},x_{3p})$, $\Sigma_{p\geq1}d(x_{3p-2},x_{3p-1})$ *and* $\Sigma_{p\geq0}d(x_{3p},x_{3p+1})$ *are convergent. Then*

$$\Sigma_{n\geq0}d(x_n,x_{n+1})=\Sigma_{p\geq0}d(x_{3p},x_{3p+1})+\Sigma_{p\geq1}d(x_{3p},x_{3p-1})+\Sigma_{p\geq1}d(x_{3p-1},x_{3p-2})$$

is convergent. In the same way; we find $\Sigma_{n\geq0}\delta(x_n,x_{n+1})$ *and* $\Sigma_{n\geq0}\gamma(x_n,x_{n+1})$ *are convergent. Hence* $(x_n)_n$ *is a Cauchy sequence in* $(X,d),(X,\delta)$ *and* (X,γ)*; Since* (X,d,δ,γ) *is an (M)-space, there, exist* $x^*,y^*,z^*\in X$ *such that*

$$\lim_n d(x_n,x^*)=\lim_n\delta(x_n,y^*)=\lim_n\gamma(x_n,z^*)=0.$$

Step 2:

Case 1: If $x^*\neq y^*$ *and* $y^*\neq z^*$.

Since $\lim_n d(Tx_n,x^*)=0$ *and* $\lim_n\delta(x_n,x^*)=\delta(y^*,x^*)>0$, *we obtain* $d(x^*,Tx_n)\leq\delta(x^*,x_n)$ *for large integers, which gives*

$$\begin{cases} d(Tx^*,x_{n+1})=d(Tx^*,Tx_n)\leq\alpha(\delta(x^*,x_n))\delta(x^*,x_n), & (1)\\ \delta(Tx^*,x_{n+1})=\delta(Tx^*,Tx_n)\leq\alpha(\gamma(x^*,x_n))\gamma(x^*,x_n), & (2)\\ \gamma(Tx^*,x_{n+1})=\gamma(Tx^*,Tx_n)\leq\alpha(d(x^*,x_n))d(x^*,x_n). & (3)\end{cases}$$

Therefor we have

$$Tx^*=z^*.$$

Since $\lim_n \delta(Tx_n, y^*) = 0$ *and* $\lim_n \gamma(x_n, y^*) = \gamma(z^*, y^*) > 0$, *we obtain* $\delta(y^*, Tx_n) \leq \gamma(y^*, x_n)$ *for large integers, which gives*

$$\begin{cases} d(Ty^*, x_{n+1}) = d(Ty^*, Tx_n) \leq \alpha(\delta(y^*, x_n))\delta(y^*, x_n), & (4) \\ \delta(Ty^*, x_{n+1}) = \delta(Ty^*, Tx_n) \leq \alpha(\gamma(y^*, x_n))\gamma(y^*, x_n), & (5) \\ \gamma(Ty^*, x_{n+1}) = \gamma(Ty^*, Tx_n) \leq \alpha(d(y^*, x_n))d(y^*, x_n). & (6) \end{cases}$$

Wherefrom

$$Ty^* = x^*.$$

if $x^* \neq z^*$.

Since $\lim_n \gamma(Tx_n, z^*) = 0$ *and* $\lim_n d(x_n, z^*) = d(x^*, z^*) > 0$, *we obtain* $\gamma(z^*, Tx_n) \leq d(x^*, x_n)$ *for large integers, which gives*

$$\begin{cases} d(Tz^*, x_{n+1}) = d(Tz^*, Tx_n) \leq \alpha(\delta(z^*, x_n))\delta(z^*, x_n), & (7) \\ \delta(Tz^*, x_{n+1}) = \delta(Tz^*, Tx_n) \leq \alpha(\gamma(z^*, x_n))\gamma(z^*, x_n), & (8) \\ \gamma(Tz^*, x_{n+1}) = \gamma(Tz^*, Tx_n) \leq \alpha(d(z^*, x_n))d(z^*, x_n). & (9) \end{cases}$$

So, we have

$$Tz^* = y^*.$$

Further, from (2), (6) and (7) we get for $k_1, k_2, k_3 \in [0, 1[$*:* $\delta(y^*, Tx^*) \leq k_1\gamma(z^*, x^*)$, $\gamma(z^*, Ty^*) \leq k_2 d(x^*, y^*)$ *and* $d(x^*, Tz^*) \leq \delta(y^*, z^*)$, *this yields* $\delta(y^*, z^*) \leq k_1\gamma(x^*, z^*)$, $\gamma(z^*, x^*) \leq k_2 d(x^*, y^*)$ *and* $d(x^*, y^*) \leq k_3\delta(y^*, z^*)$. *So, we have*

$$\begin{aligned} d(x^*, y^*) &\leq k_3\delta(y^*, z^*) \\ &\leq k_3 k_1\gamma(x^*, z^*) \\ &\leq k_3 k_1 k_2 d(x^*, y^*), \end{aligned}$$

therefore $x^* = y^* = z^*$, *wich is contraction. Thus* $x^* = z^*$.

Using (2), we obtain $\delta(y^*, z^*) \leq k_4\gamma(x^*, z^*)$ *for* $k_4 \in [0, 1[$, *then* $y^* = z^*$, *which is absurd.*

Case 2: if $x^* \neq y^*$ *and* $y^* = z^*$.

Then $x^* \neq z^*$. *Moreover, from (7) we get* $d(y^*, x^*) \leq k_5\delta(z^*, y^*)$ *for* $k_5 \in [0, 1[$, *therefore* $x^* = y^*$, *which is contraction.*

Similarly if $x^* = y^*$ and $y^* \neq z^*$, we get a contraction.

We can conclude that

$$x^* = y^* = z^*.$$

step 3:

To prove that $Tx^* = x^*$, we consider the sets A,B and C defined by:

$$\begin{aligned} A &= \{n \in \mathbb{N} / d(Tx_n, x^*) \leq \delta(x_n, x^*)\}, \\ B &= \{n \in \mathbb{N} / \delta(Tx_n, x^*) \leq \gamma(x_n, x^*)\}, \\ C &= \{n \in \mathbb{N} / \gamma(Tx_n, x^*) \leq d(x_n, x^*)\}. \end{aligned}$$

We asserts that A or B or C is infinite; if A, B and C are finite, there exist as integer N such that, for all integers $n \geq N$,

$$\begin{aligned} d(Tx_n, x^*) &> \delta(x_n, x^*), \\ \delta(Tx_n, x^*) &> \gamma(x_n, x^*), \\ \gamma(Tx_n, x^*) &> d(x_n, x^*). \end{aligned}$$

Hence we have,

$$\begin{aligned} d(x_n, x^*) &< \gamma(x_{n+1}, x^*) \\ &< \delta(x_{n+2}, x^*) \\ &< d(x_{n+3}, x^*), \ \textit{for all } n \geq N. \end{aligned}$$

Therefor we have $d(x_n, x^*) < d(x_{n+3}, x^*)$, for all integers $n \geq N$ thus, the sequence $(d(x_{3n}, x^*))_n$ is strictly increasing to 0; which is a false assertion. If we assume that A is infinite, there exists some subsequence $(x_{\sigma(n)})_n$ such that $d(Tx_{\sigma(n)}, x^*) \leq \delta(x_{\sigma(n)}, x^*)$,

this yields

$$\begin{cases} d(Tx^*, Tx_{\sigma(n)}) \leq \alpha(\delta(x^*, x_{\sigma(n)}))\delta(x^*, x_{\sigma(n)}), \\ \delta(Tx^*, Tx_{\sigma(n)}) \leq \alpha(\gamma(x^*, x_{\sigma(n)}))\gamma(x^*, x_{\sigma(n)}), \\ \gamma(Tx^*, Tx_{\sigma(n)}) \leq \alpha(d(x^*, x_{\sigma(n)}))d(x^*, x_{\sigma(n)}), \end{cases}$$

which implies that

$$\gamma(Tx^*, x_{\sigma(n)+1}) \leq \alpha(d(x^*, x_{\sigma(n)}))d(x^*, x_{\sigma(n)}).$$

Thus $\gamma(Tx^*,x^*)=0$*, hence* x^* *is a fixed point of* T*. We have the same results if* B *or* C *are infinite.*

step 4:

For the uniqueness of the point, we assume that $\overline{x}$ *and* $\overline{y}$ *are two different fixed points of* T*. We have* $d(\overline{x},\overline{y})\leq\delta(\overline{x},\overline{y})$ *or* $\delta(\overline{x},\overline{y})\leq d(\overline{x},\overline{y})$*. For the first case, we obtain:* $d(\overline{x},T\overline{y})=d(\overline{x},\overline{y})\leq\delta(\overline{x},\overline{y})$ *and then*

$$\begin{cases} d(\overline{x},\overline{y})=d(T\overline{x},T\overline{y})\leq\alpha(\delta(\overline{x},\overline{y}))\delta(\overline{x},\overline{y})<\delta(\overline{x},\overline{y}),\\ \delta(\overline{x},\overline{y})=\delta(T\overline{x},T\overline{y})\leq\alpha(\gamma(\overline{x},\overline{y}))\gamma(\overline{x},\overline{y})<\gamma(\overline{x},\overline{y}),\\ \gamma(\overline{x},\overline{y})=\gamma(T\overline{x},T\overline{y})\leq\alpha(d(\overline{x},\overline{y}))d(\overline{x},\overline{y})<d(\overline{x},\overline{y}),\end{cases}$$

which is a contraction.

Thus, T *has a unique fixed point in* X*. This completes the proof.*

If $\delta=\gamma$*, we obtain the following result proved by EL. Marhrani and K. Chaira* [4].

Corollary 3.5. *Let* X *be non-empty set,* d *and* δ *two metrics on* X *and* $T:X\to X$ *a mapping such that:*

(1) (X,d,δ) *is a (M)-space.*
(2) *For all* $x,y\in X$*, one of the following two conditions:*
    i. $d(x,Ty)\leq\delta(x,y)$,
    ii. $\delta(x,Ty)\leq d(x,y)$,

implies

$$\begin{cases} d(Tx,Ty)\leq\alpha(\delta(x,y))\delta(x,y),\\ \delta(Tx,Ty)\leq\alpha(d(x,y))d(x,y),\end{cases}$$

where $\alpha:[0,+\infty[\to[0,1[$ *is a function such that* $\limsup_{s\to r^+}\alpha(s)<1$*, for all* $r\geq0$*.*
Then T *has a unique fixed point* $x^*\in X$*.*

Example 3.6. *Let* $X=[0,1]\cup\{2,3\}$ *endowed with the usual distance* d *and the distance* δ *and* γ *defined by*

$$\delta(x,y)=\begin{cases} |x-y| & \text{if } x,y\in[0,1],\\ x+y & \text{if } x \text{ or } y \text{ is not in } [0,1] \text{ and } x\neq y,\\ 0 & \text{if } x=y.\end{cases}$$

$$\gamma(x,y) = 2|x-y|.$$

(X,d), (X,δ) and (X,γ) are complete metric spaces. We define α from $[0,+\infty[$ into $[0,1[$ by $\alpha(t) = \frac{1}{3}e^{-t}$, and consider the mapping defined on X by

$$Tx = \begin{cases} \frac{1}{2e^7}x & \text{if } x \in [0,1[, \\ 0 & \text{if } x \geq 1. \end{cases}$$

We asserts that

$$\begin{cases} d(Tx,Ty) \leq \alpha(\delta(x,y))\delta(x,y), \\ \delta(Tx,Ty) \leq \alpha(\gamma(x,y))\gamma(x,y), \\ \gamma(Tx,Ty) \leq \alpha(d(x,y))d(x,y), \end{cases}$$

is obviously satisfied if $x,y \in [0,1[$ or $x,y \in \{2,3\}$, or $x \in [0,1[$ and $y \in \{2,3\}$.

If $x \in [0,1[$ and $y = 1$, we have

$$(d(x,T1) \leq \delta(x,1) \text{ or } \delta(x,T1) \leq \gamma(x,1) \text{ or } \gamma(x,T1) \leq d(x,1)) \Leftrightarrow x \in [0,\frac{2}{3}].$$

And consequently

$$\begin{aligned} d(Tx,T1) &= \frac{1}{2e^7}x \leq \frac{2}{9e^6} \leq \frac{1}{3}e^{-(1-x)}(1-x) = \alpha(\delta(x,1))\delta(x,1), \\ \delta(Tx,T1) &= \frac{1}{2e^7}x \leq \frac{2}{9e^6} \leq \frac{2}{3}e^{-2(1-x)}(1-x) = \alpha(\gamma(x,1))\gamma(x,1), \\ \gamma(Tx,T1) &= 2\frac{1}{2e^7}x \leq \frac{4}{9e^6} \leq \frac{1}{3}e^{-(1-x)}(1-x) = \alpha(d(x,1))d(x,1). \end{aligned}$$

If $x = 0.999 \notin [0,\frac{2}{3}]$, we have

$$\begin{cases} (d(x,T1) > \delta(x,1), \\ \delta(x,T1) > \gamma(x,1), \\ \gamma(x,T1) > d(x,1)), \end{cases} \quad \text{and } d(Tx,T1) > \alpha(\delta(x,1))\delta(x,1).$$

Thus the assertion is satisfied. Then T has a unique fixed point in X, $T0 = 0$.

The following result generalizes theorem 2.2.

Theorem 3.7. *Let X be non-empty set, d, δ and γ three metrics on X and $T : X \to X$ a mapping such that:*

(1) (X,d,δ,γ) is a (M)-space.

(2) For all $x,y \in X$, one of the following three conditions:

i. $d(x,Ty) \leq \delta(x,y)$,

ii. $\delta(x,Ty) \leq \gamma(x,y)$,

iii. $\gamma(x,Ty) \leq d(x,y)$,

implies

$$\begin{cases} d(Tx,Ty) \leq \alpha(\delta(x,y))(d(y,Tx)+\delta(x,Ty)), \\ \delta(Tx,Ty) \leq \alpha(\gamma(x,y))(\delta(y,Tx)+\gamma(x,Ty)), \\ \gamma(Tx,Ty) \leq \alpha(d(x,y))(\gamma(y,Tx)+d(x,Ty)), \end{cases}$$

where $\alpha : [0,+\infty[\to [0,\frac{1}{2}[$ *is a function such that* $\limsup_{s\to r^+} \alpha(s) < \dfrac{1}{2}$, *for all* $r \geq 0$.
Then T *has a unique fixed point* $x^* \in X$.

Proof. *step 1:*

Letting $x_0 \in X$, *we define the sequence* $(x_n)_n$ *by* $x_{n+1} = Tx_n$ *for each* $n \in \mathbb{N}$, *we have*

$$d(x_{n+1},Tx_n) = 0 \leq \delta(x_{n+1},x_n),$$

so, we obtain that

$$\begin{cases} d(Tx_{n+1},Tx_n) \leq \alpha(\delta(x_{n+1},x_n))(d(x_n,Tx_{n+1})+\delta(x_{n+1},Tx_n)), \\ \delta(Tx_{n+1},Tx_n) \leq \alpha(\gamma(x_{n+1},x_n))(\delta(x_n,Tx_{n+1})+\gamma(x_{n+1},Tx_n)), \\ \gamma(Tx_{n+1},Tx_n) \leq \alpha(d(x_{n+1},x_n))(\gamma(x_n,Tx_{n+1})+d(x_{n+1},Tx_n)), \end{cases}$$

therefore

$$\begin{cases} d(Tx_{n+1},Tx_n) \leq \alpha(\delta(x_{n+1},x_n))d(x_n,x_{n+2}), \\ \delta(Tx_{n+1},Tx_n) \leq \alpha(\gamma(x_{n+1},x_n))\delta(x_n,x_{n+2}), \\ \gamma(Tx_{n+1},Tx_n) \leq \alpha(d(x_{n+1},x_n))\gamma(x_n,x_{n+2}), \end{cases}$$

wherefrom

$$\begin{cases} d(Tx_{n+1},Tx_n) \leq \alpha(\delta(x_{n+1},x_n))(d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})), \\ \delta(Tx_{n+1},Tx_n) \leq \alpha(\gamma(x_{n+1},x_n))(\delta(x_n,x_{n+1})+\delta(x_{n+1},x_{n+2})), \\ \gamma(Tx_{n+1},Tx_n) \leq \alpha(d(x_{n+1},x_n))(\gamma(x_n,x_{n+1})+\gamma(x_{n+1},x_{n+2})), \end{cases}$$

so, we have

$$\begin{cases} d(x_{n+2},x_{n+1}) \leq \frac{\alpha(\delta(x_{n+1},x_n))}{1-\alpha(\delta(x_{n+1},x_n))} d(x_n,x_{n+1}), \\ \delta(x_{n+2},x_{n+1}) \leq \frac{\alpha(\gamma(x_{n+1},x_n))}{1-\alpha(\gamma(x_{n+1},x_n))} \delta(x_n,x_{n+1}), \\ \gamma(x_{n+2},x_{n+1}) \leq \frac{\alpha(d(x_{n+1},x_n))}{1-\alpha(d(x_{n+1},x_n))} \gamma(x_n,x_{n+1}). \end{cases}$$

By hypothesis $\frac{\alpha(t)}{1-\alpha(t)} \leq 1$, *for all* $t \in [0,+\infty[$ *then*

$$\begin{cases} d(x_{n+2},x_{n+1}) \leq d(x_n,x_{n+1}), \\ \delta(x_{n+2},x_{n+1}) \leq \delta(x_n,x_{n+1}), \\ \gamma(x_{n+2},x_{n+1}) \leq \gamma(x_n,x_{n+1}). \end{cases}$$

It follows that $(d(x_n,x_{n+1}))_p$, $(\delta(x_n,x_{n+1}))_p$ *and* $(\gamma(x_n,x_{n+1}))_p$ *converges to* l_1, l_2 *and* l_1 , *respectively.*

Since $\limsup_{t\to l_1^+} \alpha(t) < \dfrac{1}{2}$, $\limsup_{t\to l_2^+} \alpha(t) < \dfrac{1}{2}$ *and* $\limsup_{t\to l_3^+} \alpha(t) < \dfrac{1}{2}$,
there exist $p_1,p_2,p_3 \in \mathbb{N}$ *and* $r_1,r_2,r_3 \in [0,\frac{1}{2}[$ *such that:*

$$\begin{cases} \alpha(d(x_{n+1},x_n)) \leq r_1, & \textit{for all } n \geq p_1, \\ \alpha(\delta(x_{n+1},x_n)) \leq r_2, & \textit{for all } n \geq p_2, \\ \alpha(\gamma(x_{n+1},x_n)) \leq r_3, & \textit{for all } n \geq p_3, \end{cases}$$

this yields

$$\begin{cases} \frac{\alpha(\delta(x_{n+1},x_n))}{1-\alpha(\delta(x_{n+1},x_n))} \leq \frac{r_1}{1-r_1}, & \textit{for all } n \geq p_1, \\ \frac{\alpha(\gamma(x_{n+1},x_n))}{1-\alpha(\gamma(x_{n+1},x_n))} \leq \frac{r_2}{1-r_2}, & \textit{for all } n \geq p_2, \\ \frac{\alpha(d(x_{n+1},x_n))}{1-\alpha(d(x_{n+1},x_n))} \leq \frac{r_3}{1-r_3}, & \textit{for all } n \geq p_3. \end{cases}$$

Then exist $R_1,R_2,R_3 \in [0,1[$ *such that*

$$\begin{cases} d(x_{n+2},x_{n+1}) \leq R_1 d(x_n,x_{n+1}), \\ \delta(x_{n+2},x_{n+1}) \leq R_2 \delta(x_n,x_{n+1}), \\ \gamma(x_{n+2},x_{n+1}) \leq R_3 \gamma(x_n,x_{n+1}). \end{cases}$$

Hence $(x_n)_n$ *is a Cauchy sequence in* (X,d),(X,δ) *and* (X,γ); *since* (X,d,δ,γ) *is an (M)-space, there exist* $x^*,y^*,z^* \in X$ *such that*

$$\lim_n d(x_n,x^*) = \lim_n \delta(x_n,y^*) = \lim_n \gamma(x_n,z^*) = 0.$$

Step 2:

Case 1: If $x^* \neq y^*$ *and* $y^* \neq z^*$.

Since $\lim_n d(Tx_n,x^*) = 0$ *and* $\lim_n \delta(x_n,x^*) = \delta(y^*,x^*) > 0$, *we obtain* $d(x^*,Tx_n) \leq \delta(x^*,x_n)$ *for large integers, which gives.*

$$\begin{cases} d(Tx^*,Tx_n) \leq \alpha(\delta(x^*,x_n))(d(x_n,Tx^*)+\delta(x^*,Tx_n)), & (10)\\ \delta(Tx^*,Tx_n) \leq \alpha(\gamma(x^*,x_n))(\delta(x_n,Tx^*)+\gamma(x^*,Tx_n)), & (11)\\ \gamma(Tx^*,Tx_n) \leq \alpha(d(x^*,x_n))(\gamma(x_n,Tx^*)+d(x^*,Tx_n)). & (12)\end{cases}$$

From (12), we have $Tx^* = z^*$.

Since $\lim_n \delta(Tx_n,y^*) = 0$ *and* $\lim_n \gamma(x_n,y^*) = \gamma(z^*,y^*) > 0$, *we obtain* $\delta(y^*,Tx_n) \leq \gamma(y^*,x_n)$ *for large integers, which gives*

$$\begin{cases} d(Ty^*,Tx_n) \leq \alpha(\delta(y^*,x_n))(d(x_n,Ty^*)+\delta(y^*,Tx_n)), & (13)\\ \delta(Ty^*,Tx_n) \leq \alpha(\gamma(y^*,x_n))(\delta(x_n,Ty^*)+\gamma(y^*,Tx_n)), & (14)\\ \gamma(Ty^*,Tx_n) \leq \alpha(d(y^*,x_n))(\gamma(x_n,Ty^*)+d(y^*,Tx_n)). & (15)\end{cases}$$

So, by (13) we get that $Ty^* = x^*$.

If $x^* \neq z^*$. *Then* $\lim_n \gamma(Tx_n,z^*) = 0$ *and* $\lim_n d(x_n,z^*) = d(x^*,z^*) > 0$, *we obtain* $\gamma(z^*,Tx_n) \leq d(z^*,x_n)$ *for large integers, which gives*

$$\begin{cases} d(Tz^*,Tx_n) \leq \alpha(\delta(z^*,x_n))(d(x_n,Tz^*)+\delta(z^*,Tx_n)), & (16)\\ \delta(Tz^*,Tx_n) \leq \alpha(\gamma(z^*,x_n))(\delta(x_n,Tz^*)+\gamma(z^*,Tx_n)), & (17)\\ \gamma(Tz^*,Tx_n) \leq \alpha(d(z^*,x_n))(\gamma(x_n,Tz^*)+d(z^*,Tx_n)). & (18)\end{cases}$$

Using (17), we obtain $Tz^* = y^*$ *and using (14) we get for* $k_1 \in [0,\frac{1}{2}[$

$$\delta(Ty^*,y^*) \leq k_1(\delta(y^*,Ty^*)+\gamma(y^*,z^*)),$$

then

$$\delta(x^*,y^*) \leq k_1(\delta(y^*,x^*)+\gamma(y^*,z^*)),$$

therefor we have

$$\delta(x^*,y^*) \leq \frac{k_1}{1-k_1}\gamma(y^*,z^*) < \gamma(y^*,z^*),$$

using (18) we obtain that there exists $k_2 \in [0, \frac{1}{2}[$ *such that*

$$\gamma(z^*, y^*) \leq \frac{k_2}{1-k_2} d(x^*, z^*) < d(x^*, z^*),$$

and using (10), we get for $k_3 \in [0, \frac{1}{2}[$

$$d(z^*, x^*) \leq \frac{k_3}{1-k_3} \delta(x^*, y^*) < \delta(x^*, y^*),$$

then $\delta(x^*, y^*) < \delta(x^*, y^*)$*, which is contraction.*

If $x^* = z^*$*.*

By (11) we conclude that there exists $k_4 \in [0, \frac{1}{2}[$ *such that*

$$\delta(Tx^*, y^*) \leq k_4(\delta(y^*, Tx^*) + \gamma(z^*, x^*)),$$

then

$$\delta(z^*, y^*) \leq k_4 \delta(y^*, z^*),$$

which is contraction.

case 2: if $x^* \neq y^*$ *and* $y^* = z^*$*. Then* $x^* \neq z^*$*.*

Using (17), we obtain $Tz^* = y^*$*, and using (16) we obtain that there exists* $k_5 \in [0, \frac{1}{2}[$ *such that:*

$$d(y^*, x^*) \leq k_5 d(y^*, x^*) + \delta(y^*, z^*),$$

it follows that $x^* = y^*$*, which is contraction.*

Similarly if $x^* = y^*$ *and* $y^* \neq z^*$*, we get a contraction.*

Thus

$$x^* = y^* = z^*.$$

step 3:

As in the step 3 the proof of theorem 3.4, we have a subsequence $(x_{\sigma(n)})_n$ *such that:*

$$\begin{cases} d(Tx^*, Tx_{\sigma(n)}) \leq \alpha(\delta(x^*, x_{\sigma(n)}))(d(x_{\sigma(n)}, Tx^*) + \delta(x^*, Tx_{\sigma(n)})), \\ \delta(Tx^*, Tx_{\sigma(n)}) \leq \alpha(\gamma(x^*, x_{\sigma(n)}))(\delta(x_{\sigma(n)}, Tx^*) + \gamma(x^*, Tx_{\sigma(n)})), \\ \gamma(Tx^*, Tx_{\sigma(n)}) \leq \alpha(d(x^*, x_{\sigma(n)}))(\gamma(x_{\sigma(n)}, Tx^*) + d(x^*, Tx_{\sigma(n)})). \end{cases}$$

Then there exists $k \in [0, \frac{1}{2}[$ *such that:*

$$d(Tx^*, x^*) \leq k(d(x^*, Tx^*) + \delta(x^*, y^*)).$$

Which implies $d(Tx^*,x^*) \leq kd(x^*,Tx^*)$ and hence $Tx^* = x^*$, thus x^* is a fixed point of T.

step 4:

For the uniqueness of the point, we assume that $\overline{x}$ and $\overline{y}$ are two different fixed points of T. We have $d(\overline{x},\overline{y}) \leq \delta(\overline{x},\overline{y})$ or $\delta(\overline{x},\overline{y}) \leq d(\overline{x},\overline{y})$. For the first case, we obtain: $d(\overline{x},T\overline{y}) = d(\overline{x},\overline{y}) \leq \delta(\overline{x},\overline{y})$ and then

$$\begin{cases} d(\overline{x},\overline{y}) = d(T\overline{x},T\overline{y}) \leq \alpha(\delta(\overline{x},\overline{y}))(d(\overline{y},T\overline{x}) + \delta(\overline{x},T\overline{y})), \\ \delta(\overline{x},\overline{y}) = \delta(T\overline{x},T\overline{y}) \leq \alpha(\gamma(\overline{x},\overline{y}))(\delta(\overline{y},T\overline{x}) + \gamma(\overline{x},T\overline{y})), \\ \gamma(\overline{x},\overline{y}) = \gamma(T\overline{x},T\overline{y}) \leq \alpha(d(\overline{x},\overline{y}))(\gamma(\overline{y},T\overline{x}) + d(\overline{x},T\overline{y})), \end{cases}$$

then

$$\begin{cases} d(\overline{x},\overline{y}) \leq \frac{\alpha(\delta(\overline{x},\overline{y}))}{1-\alpha(\delta(\overline{x},\overline{y}))}\delta(\overline{x},\overline{y}) < \delta(\overline{x},\overline{y}), \\ \delta(\overline{x},\overline{y}) \leq \frac{\alpha(\gamma(\overline{x},\overline{y}))}{1-\alpha(\gamma(\overline{x},\overline{y}))}\gamma(\overline{x},\overline{y}) < \gamma(\overline{x},\overline{y}), \\ \gamma(\overline{x},\overline{y}) \leq \frac{\alpha(d(\overline{x},\overline{y}))}{1-\alpha(d(\overline{x},\overline{y}))}d(\overline{x},\overline{y}) < d(\overline{x},\overline{y}), \end{cases}$$

which is contraction. Thus, T has a unique fixed point in X. This completes the proof.

If $\delta = \gamma$, we obtain the following result.

Corollary 3.8.

Let X be non-empty set, d and δ two metrics on X and $T : X \to X$ a mapping such that:

(1) (X,d,δ) is a (M)-space.

(2) For all $x,y \in X$, one of the following two conditions:

i. $d(x,Ty) \leq \delta(x,y)$,

ii. $\delta(x,Ty) \leq d(x,y)$,

implies

$$\begin{cases} d(Tx,Ty) \leq \alpha(\delta(x,y))(d(y,Tx) + \delta(x,Ty)), \\ \delta(Tx,Ty) \leq \alpha(d(x,y))(\delta(y,Tx) + d(x,Ty)), \end{cases}$$

where $\alpha : [0,+\infty[\to [0,\frac{1}{2}[$ is a function such that $\limsup_{s\to r^+} \alpha(s) < \dfrac{1}{2}$, for all $r \geq 0$. Then T has a unique fixed point $x^* \in X$.

Example 3.9. *Let* $X = \{(0,0),(4,0),(0,4),(5,0),(4,5),(5,4)\}$ *endowed with the distance* d *and* δ *defined by*

$$d((x,x'),(y,y')) = |x-y| + |x'-y'| \quad \text{and} \quad \delta((x,x'),(y,y')) = \frac{\sqrt{5}}{2}(|x-y| + |x'-y'|),$$

for all $((x,x'),(y,y')) \in X^2$.

We put $r = \frac{2}{\sqrt{5}}$, *and consider the mapping defined on* X *by*

$$T(x,x') = \begin{cases} (x',0) \text{ if } x \le x' \text{ and } (x,x') \in X \setminus \{(0,4)\}, \\ (0,x') \text{ if } x > x' \text{ and } (x,x') \in X \setminus \{(0,4)\}, \\ (0,0) \text{ if } (x,x') = (0,4). \end{cases}$$

First case : $((x,x'),(y,y')) \notin \{((4,5),(5,4)),((5,4),(4,5))\}$, *we have*

$$\begin{cases} d(T(x,x'),T(y,y')) \le r\,(d((y,y'),T(x,x')) + \delta((x,x'),T(y,y'))), \\ \delta(T(x,x'),T(y,y')) \le r\,(\delta((y,y'),T(x,x')) + d((x,x'),T(y,y'))). \end{cases}$$

Second case : $(x,x') = (4,5)$ *and* $(y,y') = (5,4)$.

$$d((x,x'),T(y,y')) = 5 \text{ and } \delta((x,x'),T(y,y')) = \frac{5\sqrt{5}}{2},$$

$$d((y,y'),T(x,x')) = 4 \text{ and } \delta((y,y'),T(x,x')) = \frac{4\sqrt{5}}{2},$$

$$d((x,x'),(y,y')) = 2 \text{ and } \delta((x,x'),(y,y')) = \sqrt{5}.$$

Note that

$$d((x,x'),T(y,y')) > \delta((x,x'),(y,y')),$$

and

$$\delta((x,x'),T(y,y')) > d((x,x'),(y,y')).$$

Since $d(T(x,x'),T(y,y')) = 9$ *and* $\delta(T(x,x'),T(y,y')) = \frac{9\sqrt{5}}{2}$, *so*

$$\begin{cases} d(T(x,x'),T(y,y')) > r\ (d((y,y'),T(x,x')) + \delta((x,x'),T(y,y'))), \\ \delta(T(x,x'),T(y,y')) > r\ (\delta((y,y'),T(x,x')) + d((x,x'),T(y,y'))). \end{cases}$$

Similarly for $(x,x') = (5,4)$ *and* $(y,y') = (4,5)$.

Hence, T *satisfies the hypotheses of corollary 3.8 but we haven't*

$$\begin{cases} d(T(x,x'),T(y,y')) \le r\left(d((y,y'),T(x,x')) + \delta((x,x'),T(y,y'))\right), \\ \delta(T(x,x'),T(y,y')) \le r\left(\delta((y,y'),T(x,x')) + d((x,x'),T(y,y'))\right), \end{cases}$$

on the hole space. Note that T *have a unique fixed point* $x^* = (0,0)$.

If $d = \delta = \gamma$*, we obtain the following result.*

Corollary 3.10. *Let* (X,d) *a complete metric space and let* $T : X \to X$ *be a mapping such that, for all* $x,y \in X$,

$$d(x,Ty) \le d(x,y) \text{ implies } d(Tx,Ty) \le \alpha(d(x,y))(d(y,Tx) + d(x,Ty)),$$

where $\alpha : [0,+\infty[\to [0,\frac{1}{2}[$ *is a function such that* $\limsup_{s\to r^+} \alpha(s) < \frac{1}{2}$*, for all* $r \ge 0$.
Then, there exist a unique element $x^* \in X$ *such that* $Tx^* = x^*$.

Remarque 3.11. *In corollary 3.10 if the function* α *is replaced by a constant* $r \in [0,\frac{1}{2}[$ *we get the theorem 2.2*

The following result generalizes theorem 2.1.

Theorem 3.12. *Let* X *be non-empty set,* d*,* δ *and* γ *three metrics on* X *and* $T : X \to X$ *a mapping such that:*

(1) (X,d,δ,γ) *is a (M)-space.*
(2) *For all* $x,y \in X$*, one of the following three conditions:*
   i. $d(x,Ty) \le \delta(x,y)$,
   ii. $\delta(x,Ty) \le \gamma(x,y)$,
   iii. $\gamma(x,Ty) \le d(x,y)$,

implies

$$\begin{cases} d(Tx,Ty) \le \alpha(\delta(x,y))(d(x,Tx) + \delta(y,Ty)), \\ \delta(Tx,Ty) \le \alpha(\gamma(x,y))(\delta(x,Tx) + \gamma(y,Ty)), \\ \gamma(Tx,Ty) \le \alpha(d(x,y))(\gamma(x,Tx) + d(y,Ty)), \end{cases}$$

where $\alpha : [0,+\infty[\to [0,\frac{1}{2}[$ *is a function such that* $\limsup_{s\to r^+}\alpha(s) < \dfrac{1}{2}$, *for all* $r \geq 0$.
Then T *has a unique fixed point* $x^* \in X$.

Proof. *step 1:*

Letting $x_0 \in X$, *we define the sequence* $(x_n)_n$ *by* $x_{n+1} = Tx_n$ *for each* $n \in \mathbb{N}$, *we have*

$$d(x_{n+1},Tx_n) = 0 \leq \delta(x_{n+1},x_n),$$

therefor we have

$$\begin{cases} d(Tx_{n+1},Tx_n) \leq \alpha(\delta(x_{n+1},x_n))(d(x_{n+1},Tx_{n+1}) + \delta(x_n,Tx_n)), \\ \delta(Tx_{n+1},Tx_n) \leq \alpha(\gamma(x_{n+1},x_n))(\delta(x_{n+1},Tx_{n+1}) + \gamma(x_n,Tx_n)), \\ \gamma(Tx_{n+1},Tx_n) \leq \alpha(d(x_{n+1},x_n))(\gamma(x_{n+1},Tx_{n+1}) + d(x_n,Tx_n)), \end{cases}$$

wherefrom

$$\begin{cases} d(x_{n+2},x_{n+1}) \leq \alpha(\delta(x_{n+1},x_n))(d(x_{n+1},x_{n+2}) + \delta(x_n,x_{n+1})), \\ \delta(x_{n+2},x_{n+1}) \leq \alpha(\gamma(x_{n+1},x_n))(\delta(x_{n+1},x_{n+2}) + \gamma(x_n,x_{n+1})), \\ \gamma(x_{n+2},x_{n+1}) \leq \alpha(d(x_{n+1},x_n))(\gamma(x_{n+1},x_{n+2}) + d(x_n,x_{n+1})), \end{cases}$$

this yields

$$\begin{cases} d(x_{n+2},x_{n+1}) \leq a(n)\delta(x_n,x_{n+1}), \\ \delta(x_{n+2},x_{n+1}) \leq b(n)\gamma(x_n,x_{n+1}), \\ \gamma(x_{n+2},x_{n+1}) \leq c(n)d(x_n,x_{n+1}), \end{cases}$$

with

$$\begin{cases} a(n) = \frac{\alpha(\delta(x_{n+1},x_n))}{1-\alpha(\delta(x_{n+1},x_n))}, \\ b(n) = \frac{\alpha(\gamma(x_{n+1},x_n))}{1-\alpha(\gamma(x_{n+1},x_n))}, \\ c(n) = \frac{\alpha(d(x_{n+1},x_n))}{1-\alpha(d(x_{n+1},x_n))}. \end{cases}$$

Thus, we have

$$\begin{aligned} d(x_{n+2},x_{n+1}) &\leq a(n)\delta(x_n,x_{n+1}) \\ &\leq a(n)b(n-1)\gamma(x_{n-1},x_n) \\ &\leq a(n)b(n-1)c(n-2)d(x_{n-2},x_{n-1}), \text{ for all } n \geq 2. \end{aligned}$$

Analogously, we obtain $\delta(x_{n+2},x_{n+1}) \leq b(n)c(n-1)a(n-2)\delta(x_{n-2},x_{n-1})$ *and* $\gamma(x_{n+2},x_{n+1}) \leq c(n)a(n-1)b(n-2)\gamma(x_{n-2},x_{n-1})$.

By hypothesis, $0 \leq \frac{\alpha(t)}{1-\alpha(t)} < 1$, $\forall t \in [0,+\infty[$, *then:*

$$\begin{cases} d(x_{n+2},x_{n+1}) \leq d(x_{n-2},x_{n-1}), \\ \delta(x_{n+2},x_{n+1}) \leq \delta(x_{n-2},x_{n-1}), \\ \gamma(x_{n+2},x_{n+1}) \leq \gamma(x_{n-2},x_{n-1}). \end{cases}$$

It follows that $(d(x_{3p},x_{3p+1}))_p$, $(d(x_{3p+1},x_{3p+2}))_p$ *and* $(d(x_{3p+2},x_{3p+3}))_p$ *converges to* d_1,d_2, *and* d_3, *respectively. And* $(\delta(x_{3p},x_{3p+1}))_p$, $(\delta(x_{3p+1},x_{3p+2}))_p$ *and* $(\delta(x_{3p+2},x_{3p+3}))_p$ *converges to* δ_1,δ_2, *and* δ_3, *respectively. And* $(\gamma(x_{3p},x_{3p+1}))_p$, $(\gamma(x_{3p+1},x_{3p+2}))_p$ *and* $(\gamma(x_{3p+2},x_{3p+3}))_p$ *converges to* γ_1,γ_2, *and* γ_3, *respectively.*

Since $\limsup_{t\to\delta_1^+} \alpha(t) < \dfrac{1}{2}$, $\limsup_{t\to\gamma_3^+} \alpha(t) < \dfrac{1}{2}$ *and* $\limsup_{t\to d_2^+} \alpha(t) < \dfrac{1}{2}$.

There exist $p_1 \in \mathbb{N}$ *and* $r_1 \in [0,\frac{1}{2}[$ *such that for any integer* $p \geq p_1$

$$\max\{\alpha(d(x_{3p+1},x_{3p+2}));\alpha(\delta(x_{3p},x_{3p+1}));\alpha(\gamma(x_{3p+2},x_{3p+3}))\} \leq r_1.$$

Hence

$$\begin{cases} \frac{\alpha(\delta(x_{3p+1},x_{3p}))}{1-\alpha(\delta(x_{3p+1},x_{3p}))} \leq \frac{r_1}{1-r_1}, \\ \frac{\alpha(\gamma(x_{3p},x_{3p-1}))}{1-\alpha(\gamma(x_{3p},x_{3p-1}))} \leq \frac{r_1}{1-r_1}, \\ \frac{\alpha(d(x_{3p-1},x_{3p-2}))}{1-\alpha(d(x_{3p-1},x_{3p-2}))} \leq \frac{r_1}{1-r_1}. \end{cases}$$

There exist $R_1 \in [0,1[$ *such that*

$$d(x_{3p+1},x_{3p+2}) \leq R_1 d(x_{3p-2},x_{3p-1}).$$

In the same way, we find that exist $p_2,p_3 \in \mathbb{N}$ *and* $R_2,R_3 \in [0,1[$ *such that*

$$d(x_{3p+2},x_{3p+3}) \leq R_2 d(x_{3p-1},x_{3p}) \text{ for } p \geq p_2,$$

$$d(x_{3p+4},x_{3p+3}) \leq R_3 d(x_{3p},x_{3p+1}) \text{ for } p \geq p_3.$$

It follow that $\Sigma_{p\geq 1} d(x_{3p-1},x_{3p})$, $\Sigma_{p\geq 1} d(x_{3p-2},x_{3p-1})$ *and* $\Sigma_{p\geq 0} d(x_{3p},x_{3p+1})$ *are convergent. Therefore*

$$\Sigma_{n\geq 0} d(x_n,x_{n+1}) = \Sigma_{p\geq 0} d(x_{3p},x_{3p+1}) + \Sigma_{p\geq 1} d(x_{3p},x_{3p-1}) + \Sigma_{p\geq 1} d(x_{3p-1},x_{3p-2}),$$

is convergent. In the same way; we find $\Sigma_{n\geq 0}\delta(x_n,x_{n+1})$ *and* $\Sigma_{n\geq 0}\gamma(x_n,x_{n+1})$ *are convergent. Hence* $(x_n)_n$ *is a Cauchy sequence in* (X,d), (X,δ) *and* (X,γ)*; since* (X,d,δ,γ) *is an (M)-space, there exist* $x^*,y^*,z^*\in X$ *such that*

$$\lim_n d(x_n,x^*)=\lim_n \delta(x_n,y^*)=\lim_n \gamma(x_n,z^*)=0.$$

Step 2:

If $x^*\neq y^*$. *And since* $\lim_n d(Tx_n,x^*)=0$ *and* $\lim_n \delta(x_n,x^*)=\delta(y^*,x^*)>0$, *we obtain* $d(x^*,Tx_n)\leq \delta(x^*,x_n)$ *for large integers, which gives*

$$\begin{cases} d(Tx^*,Tx_n)\leq \alpha(\delta(x^*,x_n))(d(x^*,Tx^*)+\delta(x_n,Tx_n)), & (19)\\ \delta(Tx^*,Tx_n)\leq \alpha(\gamma(x^*,x_n))(\delta(x^*,Tx^*)+\gamma(x_n,Tx_n)), & (20)\\ \gamma(Tx^*,Tx_n)\leq \alpha(d(x^*,x_n))(\gamma(x^*,Tx^*)+d(x_n,Tx_n)). & (21)\end{cases}$$

Using (19), we obtain $Tx^*=x^*$ *and by (20) we conclude that* $\delta(Tx^*,y^*)\leq\delta(x^*,Tx^*)$ *so, we have* $Tx^*=y^*$, *also* $x^*=y^*$, *which is contraction.*
Similarly if $x^*\neq z^*$ *and* $y^*=z^*$, *we get a contraction.*
Thus

$$x^*=y^*=z^*.$$

step 3:

As in the step 3 the proof of theorem 3.1, we have a subsequence $(x_{\sigma(n)})_n$ *such that:*

$$\begin{cases} d(Tx^*,Tx_{\sigma(n)})\leq \alpha(\delta(x^*,x_{\sigma(n)}))(d(x^*,Tx^*)+\delta(x_{\sigma(n)},Tx_{\sigma(n)})),\\ \delta(Tx^*,Tx_{\sigma(n)})\leq \alpha(\gamma(x^*,x_{\sigma(n)}))(\delta(x^*,Tx^*)+\gamma(x_{\sigma(n)},Tx_{\sigma(n)}))\\ \gamma(Tx^*,Tx_{\sigma(n)})\leq \alpha(d(x^*,x_{\sigma(n)}))(\gamma(x^*,Tx^*)+d(x_{\sigma(n)},Tx_{\sigma(n)})).\end{cases}$$

Furthermore, $\limsup_n \alpha(\delta(x^*,x_{\sigma(n)})<\frac{1}{2}$ *implies that exists* $k\in[0,\frac{1}{2}[$ *such that*

$$d(Tx^*,Tx_{\sigma(n)})\leq k(d(x^*,Tx^*)+\delta(x_{\sigma(n)},Tx_{\sigma(n)})),$$

and consequently

$$d(Tx^*,x^*)\leq kd(x^*,Tx^*).$$

Thus $d(Tx^*,x^*)=0$, hence, x^* is a fixed point of T.

step 4:

For the uniqueness of the point, we assume that $\overline{x}$ and $\overline{y}$ are two different fixed points of T. We have $d(\overline{x},\overline{y}) \leq \delta(\overline{x},\overline{y})$ or $\delta(\overline{x},\overline{y}) \leq d(\overline{x},\overline{y})$. For the first case, we obtain: $d(\overline{x},T\overline{y}) = d(\overline{x},\overline{y}) \leq \delta(\overline{x},\overline{y})$ and then

$$\begin{cases} d(T\overline{x},T\overline{y}) \leq \alpha(\delta(\overline{x},\overline{y}))(d(\overline{x},T\overline{x})+\delta(\overline{y},T\overline{y})), \\ \delta(T\overline{x},T\overline{y}) \leq \alpha(\gamma(\overline{x},\overline{y}))(\delta(\overline{x},T\overline{x})+\gamma(\overline{y},T\overline{y})), \\ \gamma(T\overline{x},T\overline{y}) \leq \alpha(d(\overline{x},\overline{y}))(\gamma(\overline{x},T\overline{x})+d(\overline{y},T\overline{y})). \end{cases}$$

Then $d(\overline{x},\overline{y})=0$, thus, T has a unique fixed point in X. This completes the proof.

If $\delta=\gamma$, we obtain the following result.

Corollary 3.13. *Let X be non-empty set, d and δ two metrics on X and $T: X \to X$ a mapping such that:*

(1) (X,d,δ) is a (M)-space.

(2) For all $x,y \in X$, one of the following two conditions:

i. $d(x,Ty) \leq \delta(x,y)$,

ii. $\delta(x,Ty) \leq d(x,y)$,

implies

$$\begin{cases} d(Tx,Ty) \leq \alpha(\delta(x,y))(d(x,Tx)+\delta(y,Ty)), \\ \delta(Tx,Ty) \leq \alpha(d(x,y))(\delta(x,Tx)+d(y,Ty)), \end{cases}$$

where $\alpha: [0,+\infty[\to [0,\frac{1}{2}[$ is a function such that $\limsup_{s\to r^+}\alpha(s) < \dfrac{1}{2}$, for all $r \geq 0$. Then T has a unique fixed point $x^* \in X$.

Corollary 3.14. *Let (X,d,δ,γ) a (M)-space and $T: X \to X$ a mapping such that:*

$$\begin{cases} d(Tx,Ty) \leq \alpha(\delta(x,y))(d(x,Tx)+\delta(y,Ty)), \\ \delta(Tx,Ty) \leq \alpha(\gamma(x,y))(\delta(x,Tx)+\gamma(y,Ty)), \\ \gamma(Tx,Ty) \leq \alpha(d(x,y))(\gamma(x,Tx)+d(y,Ty)), \end{cases}$$

where $\alpha: [0,+\infty[\to [0,\frac{1}{2}[$ is a function such that $\limsup_{s\to r^+}\alpha(s) < \dfrac{1}{2}$, for all $r \geq 0$. Then T has a unique fixed point $x^* \in X$.

Example 3.15. *Let $X=[0,1]$ endowed with the usual distance d and the distance δ and γ defined by $\delta(x,y)=2|x-y|$ and $\gamma(x,y)=3|x-y|$.*
(X,d), (X,δ) and (X,γ) are complete metric spaces. We define α from $[0,+\infty[$ into $[0,1[$ by $\alpha(t)=\frac{5}{12}e^{-\frac{t}{6}}$, and consider the mapping defined on X by

$$Tx=\begin{cases}\frac{1}{10}x & \text{if } x\in[0,1[,\\ 0 & \text{if } x=1.\end{cases}$$

For $x,y\in[0,1[$, we have

$$\begin{aligned} d(Tx,Ty) &= \frac{1}{10}|x-y|\le\frac{5}{12}e^{-\frac{1}{3}|x-y|}(\frac{9}{10}x+\frac{9}{5}y)=\alpha(\delta(x,y))(d(x,Tx)+\delta(y,Ty)),\\ \delta(Tx,Ty) &= \frac{1}{5}|x-y|\le\frac{5}{12}e^{-\frac{1}{2}|x-y|}(\frac{9}{5}x+\frac{27}{10}y)=\gamma(d(x,y))(\delta(x,Tx)+\gamma(y,Ty)),\\ \gamma(Tx,Ty) &= \frac{3}{10}|x-y|\le\frac{5}{12}e^{-\frac{1}{6}|x-y|}(\frac{27}{10}x+\frac{9}{10}y)=\gamma(d(x,y))(\gamma(x,Tx)+d(y,Ty)). \end{aligned}$$

For $x\in[0,1[$ and $y=1$ T satisfy corollary 3.14, similarly for $y\in[0,1[$ and $x=1$. Then T has a unique fixed point in X, $T0=0$.

Corollary 3.16. *Let (X,d) a complete metric space and let $T:X\to X$ be a mapping such that, for all $x,y\in X$,*

$$d(x,Ty)\le d(x,y) \text{ implies } d(Tx,Ty)\le\alpha(d(x,y))(d(x,Tx)+d(y,Ty)),$$

where $\alpha:[0,+\infty[\to[0,\frac{1}{2}[$ is a function such that $\limsup_{s\to r^+}\alpha(s)<\dfrac{1}{2}$, for all $r\ge0$.
Then, there exist a unique element $x^*\in X$ such that $Tx^*=x^*$

Remarque 3.17. *In corollary 3.16 if the function α is replaced by a constant $r\in[0,\frac{1}{2}[$ we get the theorem 2.1.*

Conflict of Interests

The authors declare that there is no conflict of interests.

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