Available online at http://scik.org

Adv. Fixed Point Theory, 7 (2017), No. 2, 304-314

ISSN: 1927-6303

COMMON FIXED POINT THEOREMS OF INTEGRAL TYPE CONTRACTION ON METRIC SPACES

MANOJ KUMAR[1,*], ANITA DAHIYA[2], ASHA RANI[2]

[1]Department of Mathematics, Lovely Professional University, Punjab, India

[2]Department of Mathematics, SRM University, Sonipat, India

Abstract: In this paper, using the (CLR) property, common fixed point results for two pairs of weakly compatible mappings satisfying contractive condition of integral type on metric spaces are established.

Keywords: contractive mappings of integral type; weakly compatible mappings; common fixed point; common (E.A) property; common (CLR) property.

2010 AMS Subject Classification: 47H10, 54H25.

1. Introduction and Preliminaries

Fixed point theory is one of the most fruitful and applicable topics of nonlinear analysis, which is widely used not only in other mathematical theories, but also in many practical problems of natural sciences and engineering. The Banach contraction mapping principle [1] is indeed the most popular result of metric fixed point theory. This principle has many application in several domains, such as differential equations, functional equations, integral equations, economics, wild life, and several others.

Branciari [2] gave an integral version of the Banach contraction principles and proved fixed point theorem for a single-valued contractive mapping of integral type in metric space. Afterwards many researchers [3–18] extended the result of Branciari and obtained fixed point and common fixed point theorems for various contractive conditions of integral type on different spaces.

---

[*]Corresponding author

Received January 12, 2017

Now, we recollect some known definitions and results from the literature which are helpful in the proof of our main results.

Definition 1.1. A coincidence point of a pair of self-mapping A, B : X→X is a point x ∈ X for which Ax = Bx.

A common fixed point of a pair of self-mapping A, B: X→X is a point x ∈ X for which Ax = Bx = x. Jungck [19] initiated the concept of weakly compatible maps to study common fixed point theorems.

Definition 1.2. [19] A pair of self-mapping A, B: X → X is weakly compatible if they commute at their coincidence points, that is, if there exists a point x ∈ X such that ABx = BAx whenever Ax = Bx.

In the study of common fixed points of weakly compatible mappings, we often require the assumption of completeness of the space or subspace or continuity of mappings involved besides some contractive condition. Aamri and El Moutawakil [20] introduced the notion of (E.A) property, which requires only the closedness of the subspace and Liu et al. [21] extended the (E.A) property to common the (E.A) property as follows.

Definition 1.3. Let (X, d) be a metric space and A, B, P, Q: X → X be four self-maps. The pairs (A, Q) and (B, P) satisfy the common (E.A) property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Qx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Py_n = s \in X.$$

Sintunavarat and Kumam [22] introduced the notion of the (CLR) property,which never requires any condition on closedness of the space or subspace and Imdad et al. [23] introduced the common (CLR) property which is an extension of the (CLR) property.

Definition 1.4. Let (X, d) be a metric space and A, B, P, Q : X → X be four self- maps. The pairs (A, Q) and (B, P) satisfy the common limit range property with respect to mappings Q and P, denoted by (CLR_{PQ}) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Qx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Py_n = s \in QX \cap PX.$$

Definition 1.5. Let Φ be the family of functions ϕ : [0,∞)→[0,∞) satisfying the following :

(1) ϕ is lower semi continuous.

(2) ϕ (t) > 0 for all t > 0 and φ(0) = 0.

(3) ϕ is discontinuous at t=0.

Definition 1.6. Let Ψ be the family of functions $\psi : [0,\infty)\rightarrow[0,\infty)$ satisfying the following :

(1) ψ is monotonically increasing and continuous .

(2) $\psi(t) < t$ for all $t > 0$ and $\psi(0) =0$.

Finally, we will need the following results.

Lemma 1.7. [9] Let $\varphi \in \Phi$ and $\{r_n\}_{n\in N}$ be a non negative sequence with $\lim_{n\to\infty} r_n = a$.Then

$$\lim_{n\to\infty}\int_0^{r_n}\varphi(t)dt = \int_0^{a}\varphi(t)dt.$$

2. Common Fixed Point Theorems

In this section, we study common fixed point theorems for weakly compatible mappings using (CLR) property and E.A. property.

Theorem 2.1 *Let (X, d) be a metric space and A, B, P, Q be four self maps on X satisfying the following:*

(1) *The pairs (A, P) and (B, Q) share (CLR_{PQ}) property;*

(2) $\psi \int_0^{d(Ax,By)} \varphi(t)dt \le \psi \int_0^{d(Ax,Px)+d(By,Qy)} \varphi(t)dt - \phi \int_0^{d(Ax,Px)+d(By,Qy)} \varphi(t)dt\,.$

(2.1)

IF the pairs (A,P) and (B,Q) are weakly compatible, then A,B,P and Q have a unique common fixed point in X.

Proof. Suppose that the pairs (A, P) and (B, Q) share the (CLR_{PQ}) property, then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Qy_n = z \text{ for some } z \in PX\cap QX. \qquad (2.2)$$

Since $z \in PX$, there exists a point $s \in X$ such that $Ps = z$ From (2.2), we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Qy_n = z = Ps \qquad (2.3)$$

Now we claim that As = Ps. Let if possible, As ≠ Ps. Putting x = s and y = y_n in (2.1), we get

$$\psi\int_0^{d(As,By_n)}\varphi(t)dt \leq \psi\int_0^{d(As,Ps)+d(By_n,Qy_n)}\varphi(t)dt - \phi\int_0^{d(As,Ps)+d(By_n,Qy_n)}\varphi(t)dt.$$

$$\leq \psi\int_0^{d(As,Ps)+d(By_n,Qy_n)}\varphi(t)dt \tag{2.4}$$

Making limit as $n\to\infty$, we get

$$\psi\int_0^{d(As,z)}\varphi(t)dt \leq \psi\int_0^{d(As,Ps)}\varphi(t)dt - \phi\int_0^{d(As,Ps)}\varphi(t)dt.$$

$$\leq \psi\int_0^{d(As,Ps)}\varphi(t)dt$$

Using (2.3)

$$\psi\int_0^{d(As,z)}\varphi(t)dt \leq \psi\int_0^{d(As,z)}\varphi(t)dt - \phi\int_0^{d(As,z)}\varphi(t)dt.$$

$$\leq \psi\int_0^{d(As,z)}\varphi(t)dt, \tag{2.5}$$

which is a contradiction, therefore

$$Ps = As = z. \tag{2.6}$$

Similarly, since z ∈ QX, so there exists a point v ∈ X such that Qv = z. Thus (2.2) becomes

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Qy_n = z = Qv \tag{2.7}$$

Now, we claim that Bv = Qv. To support the claim, let Bv ≠ Qv. Then on putting x = x_n and y = v in (2.1), one can get

$$Bv = Qv = z. \tag{2.8}$$

Therefore, from (2.6) and (2.8), one can write

$$As = Ps = Bv = Qv = z. \tag{2.9}$$

Next, we show that z is a common fixed point of A, B, P, and Q. To this aim, since the pairs (A, P) and (B, Q) are weakly compatible, then using (2.9) we have

$$As = Ps \Rightarrow PAs = APs \Rightarrow Az = Pz, \tag{2.10}$$

and

$$Bv = Qv \Rightarrow QBv = BQv \Rightarrow Bz = Qz. \tag{2.11}$$

We will show next that Az = z. Otherwise, if Az ≠ z, using (2.1) of Theorem 2.1 with x = z and y = v, we have

$$\psi\int_0^{d(Az,Bv)}\varphi(t)dt \leq \psi\int_0^{d(Az,Pz)+d(Bv,Qv)}\varphi(t)dt - \phi\int_0^{d(Az,Pz)+d(Bv,Qv)}\varphi(t)dt. \tag{2.12}$$

= 0, a contradiction.

Hence Az = z. From (2.10) ,we can write

$$Az = Pz = z \tag{2.13}$$

Similarly, setting x = u, y = z in 2.1 and using (2.9), (2.11), we can have

$$Bz = Qz = z. \tag{2.14}$$

Therefore from (2.13) and (2.14), it follows that

$$Az = Bz = Qz = Pz = z, \tag{2.15}$$

that is, z is a common fixed point of A, B, Q, and P.

Finally, we prove the uniqueness of the common fixed point of A, B, Q, and P. Assume that z_1 and z_2 are two distinct common fixed points of A, B, Q, and P. Then replacing x by z_1 and y by z_2 in (2.1) of Theorem 2.1, we have

$$\psi \int_0^{d(Az_1,Bz_2)} \varphi(t)dt \leq \psi \int_0^{d(Az_1,Pz_1)+d(Bz_2,Qz_2)} \varphi(t)dt - \phi \int_0^{d(Az_1,Pz_1)+d(Bz_2,Qz_2)} \varphi(t)dt. \tag{2.16}$$

$= 0$, a contradiction

Hence $z_1 = z_2$. Therefore A, B, P and Q have a unique common fixed point in X.

From theorem 2.1, we can easily deduce the following corollaries.

Corollary 2.1. *Let (X, d) be a metric space and A, P, Q be three self maps on X satisfying the following:*

(1) *The pair (A, P) and (A, Q) share (CLR_{PQ}) property;*

(2) $\psi \int_0^{d(Ax,Ay)} \varphi(t)dt \leq \psi \int_0^{d(Ax,Px)+d(Ay,Qy)} \varphi(t)dt - \phi \int_0^{d(Ax,Px)+d(Ay,Qy)} \varphi(t)dt.$

IF the pairs (A, P) and (A, Q) are weakly compatible, then A, P and Q have a unique common fixed point in X.

Corollary 2.2. *Let (X, d) be a metric space and A, Q be two self maps on X satisfying the following:*

(1) *The pair (A, Q) share (CLR_Q) property;*

(2) $\psi \int_0^{d(Ax,Ay)} \varphi(t)dt \leq \psi \int_0^{d(Ax,Qx)+d(Ay,Qy)} \varphi(t)dt - \phi \int_0^{d(Ax,Qx)+d(Ay,Qy)} \varphi(t)dt.$

IF the pair (A, Q) is weakly compatible, then A and Q have a unique common fixed point in X.

Obviously, the (CLR_{MN}) property implies the common property (E.A) but the converse is not true in general. So replacing the (CLR_{MN}) property by the common property (E.A) in Theorem 2.1, we get the following results, the proofs of which can easily be done by following the lines of the proof of Theorem 2.1, because the (E.A) property together with the closedness property of a suitable subspace gives rise to the closed range property.

Corollary 2.3. *Let (X, d) be a metric space and A, B, P and Q be four self maps on X satisfying the following:*

(1) *The pair (A, P) and (B, Q) share common (E.A.) property such that QX (or NX) is closed subspace of X;*

$$(2)\ \psi\int_0^{d(Ax,By)}\varphi(t)dt \leq \psi\int_0^{d(Ax,Px)+d(By,Qy)}\varphi(t)dt - \phi\int_0^{d(Ax,Px)+d(By,Qy)}\varphi(t)dt.$$

IF the pairs (A, P) and (B, Q) are weakly compatible, then A, B, P and Q have a unique common fixed point in X.

Corollary 2.4. *Let (X, d) be a metric space and A be two self maps on X satisfying the following condition*

$$\psi\int_0^{d(Ax,Ay)}\varphi(t)dt \leq \psi\int_0^{d(Ax,x)+d(Ay,y)}\varphi(t)dt - \phi\int_0^{d(Ax,x)+d(Ay,y)}\varphi(t)dt.$$

For all x, y $\in$ X. Then A has a unique common fixed point in X.

To illustrate Theorem 2.1, we construct the following example.

Example 2.5. Let X = (0, 7) be a metric space with metric d(x, y) = |x – y|, where x, y ∈ X and A, B, P and Q be self-maps of X, defined by

$$\mathrm{Ax} = \begin{cases} 5 & \text{if } x \in (0,3] \\ 1 & \text{if } x \in (3,7) \end{cases}; \qquad \mathrm{Bx} = \begin{cases} 5 & \text{if } x \in (0,3] \\ \frac{1}{2} & \text{if } x \in (3,7) \end{cases}$$

$$\mathrm{Px} = \begin{cases} 5 & \text{if } x \in (0,3] \\ 2 & \text{if } x \in (3,7) \end{cases}; \qquad \mathrm{Qx} = \begin{cases} 5 & \text{if } x \in (0,3] \\ 4 & \text{if } x \in (3,7) \end{cases}$$

First we verify condition (1) of Theorem 2.1. To this aim,

let $\{x_n\} = \{\frac{3n}{n+1}\}_{n\geq 1}$ and $\{y_n\} = \{\frac{3}{n+1}\}_{n\geq 1}$ be two sequences in X. Then

$$\lim_{n\to\infty} \mathrm{A}x_n = \lim_{n\to\infty} \mathrm{A}\left(\frac{3n}{n+1}\right) = \mathrm{A}(3) = 5$$

$$\lim_{n\to\infty} \mathrm{P}x_n = \lim_{n\to\infty} \mathrm{P}\left(\frac{3n}{n+1}\right) = \mathrm{P}(3) = 5$$

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} B\left(\frac{3}{n+1}\right) = B(0) = 5$$

$$\lim_{n\to\infty} Qy_n = \lim_{n\to\infty} Q\left(\frac{3}{n+1}\right) = Q(0) = 5$$

Thus

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Qy_n = 5 \in PX \cap QX.$$

That is, (A, P) and (B, Q) satisfies the common (CLR_{PQ})property.

Next, to verify condition (2) of Theorem 2.1 let us define ϕ (t) = $\frac{t}{3}$, $\varphi(t) = 2t$ and $\psi(t) = t$.

Let x,y ∈ (0,3] . Then Ax = Px = By = Qy = 5 and from equation (2.1)

L.H.S. = $\psi \int_0^{d(Ax,By)} \varphi(t)dt = \psi \int_0^{d(3,3)} 2t\, dt = \psi(0) = 0$

R.H.S. = $\psi \int_0^{d(Ax,Px)+d(By,Qy)} \varphi(t)dt - \phi \int_0^{d(Ax,Px)+d(By,Qy)} \varphi(t)dt.$

= $\psi \int_0^{d(3,3)+d(3,3)} \varphi(t)dt - \phi \int_0^{d(3,3)+d(3,3)} \varphi(t)dt.$

= $\psi(0) - \phi\,(0) = 0 – 0 = 0.$

Therefore L.H.S. = R.H.S.

Now let x, y ∈ (3,7)

L.H.S. = $\psi \int_0^{d(Ax,By)} \varphi(t)dt = \psi \int_0^{d\left(1,\frac{1}{2}\right)} 2t\, dt = \psi \int_0^{\left|1-\frac{1}{2}\right|} 2t\, dt$

$= \psi \int_0^{\frac{1}{2}} 2t\, dt = \psi\left(\frac{1}{2^2}\right) = \psi\left(\frac{1}{4}\right) = \frac{1}{4} = .25$

R.H.S. = $\psi \int_0^{d(Ax,Px)+d(By,Qy)} \varphi(t)dt - \phi \int_0^{d(Ax,Px)+d(By,Qy)} \varphi(t)dt$.

$= \psi \int_0^{d(1,2)+d(\frac{1}{2},4)} \varphi(t)dt - \phi \int_0^{d(1,2)+d(\frac{1}{2},4)} \varphi(t)dt$.

$= \psi \int_0^{1+\frac{7}{2}} \varphi(t)dt - \phi \int_0^{1+\frac{7}{2}} \varphi(t)dt$.

$= \psi \int_0^{\frac{9}{2}} 2tdt - \phi \int_0^{\frac{9}{2}} 2tdt. = \psi\left(\frac{81}{4}\right) - \phi\left(\frac{81}{4}\right) = \frac{81}{4} - \frac{81}{12} = \frac{27}{2} = 13.5$

Therefore L.H.S. < R.H.S.

Therefore from Theorem 2.1, the mappings A, B, P and Q have a unique common fixed point, which is x = 3.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] Banach, S: Sur les operations dans les ensembles abstraits et leurs applications aux equations integrales. Fundam. Math. 3 (1922), 133-181.

[2] Branciari, A: A fixed point theorem for mappings satisfying a general contractive condition of integral type. Int. J. Math. Math. Sci. 29 (9) (2002), 531-536.

[3] Aliouche, A: A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type. J. Math. Anal. Appl. 322 (2006), 796-802.

[4] Altun, I, Turkoglu, D: Some fixed point theorems for weakly compatible mappings satisfying an implicit relation. Taiwan. J. Math. 13 (2009), 1291-1304.

[5] Altun, I, Turkoglu, D, Rhoades, BE: Fixed points of weakly compatible maps satisfying a general contractive of integral type. Fixed Point Theory Appl. 2007 (2007), Article ID 17301.

[6] Djoudi, A, Aliouche, A: Common fixed point theorems of Greguš type for weakly compatible mappings satisfying contractive conditions of integral type. J. Math. Anal. Appl. 329 (2007), 31-45.

[7] Djoudi, A, Merghadi, F: Common fixed point theorems for maps under a contractive condition of integral type. J. Math. Anal. Appl. 341 (2008), 953-960.

[8] Jachymski, J: Remarks on contractive conditions of integral type. Nonlinear Anal. 71(2009), 1073-1081.

[9] Liu, Z, Li, X, Kang, SM, Cho, SY: Fixed point theorems for mappings satisfying contractive conditions of integral type and applications. Fixed Point Theory Appl. 2011 (2011), Article ID 64.

[10] Liu, Z, Zou, X, Kang, SM, Ume, JS: Common fixed points for a pair of mappings satisfying contractive conditions of integral type. J. Inequal. Appl. 2014 (2014), Article ID 394.

[11] Murthy, PP, Kumar, S, Tas, K: Common fixed points of self maps satisfying an integral type contractive condition in fuzzy metric spaces. Math. Commun. 15(2010), 521-537.

[12] Sintunavarat, W, Kumam, P: Gregus-type common fixed point theorems for tangential multi-valued mappings of integral type in metric spaces. Int. J. Math. Math. Sci. 2011 (2011), Article ID 923458.

[13] Alsulami, HH, Karapınar, E, O'Regan, D, Shahi, P: Fixed points of generalized contractive mappings of integral type. Fixed Point Theory Appl. 2014 (2014), Article ID 213.

[14] Karapınar, E, Shahi, P, Tas, K: Generalized α-ψ-contractive type mappings of integral type and related fixed point theorems. J. Inequal. Appl. 2014 (2014), Article ID 160.

[15] Chauhan, S, Karapınar, E: Some integral type common fixed point theorems satisfying (α-ψ)-contractive conditions. Bull. Belg. Math. Soc. Simon Stevin 21 (4) (2014), 593-612.

[16] Gulyaz, S, Karapınar, E, Rakocevic, V, Salimi, P: Existence of a solution of integral equations via fixed point theorem. J. Inequal. Appl. 2013 (2013), Article ID 529.

[17] Karapınar, E: Fixed points results for alpha-admissible mapping of integral type on generalized metric spaces. Abstr. Appl. Anal. 2014 (2014), Article ID 141409.

[18] Chauhan, S, Imdad, M, Karapınar, E, Fisher, B: An integral type fixed point theorem for multi-valued mappings employing strongly tangential property. J. Egypt. Math. Soc. 22 (2) (2014), 258-264.

[19] Jungck, G: Common fixed points for non-continuous non-self mappings on a non-numeric space, Far East J. Math. Sci. 4 (2) (1996), 199-212.

[20] Aamri, M, El Moutawakil, D: Some new common fixed point theorems under strict contractive conditions. J. Math. Anal. Appl. 270 (1) (2002), 181-188.

[21] Liu, W, Wu, J, Li, Z: Common fixed points of single-valued and multi-valued maps, Int. J. Math. Sci. 19(2005), 3045-3055.

[22] Sintunavarat, W, Kumam, P: Common fixed point theorem for a pair of weakly compatible mappings in fuzzy metric space. J. Appl. Math. 2011 (2011), Article ID 637958.

[23] Imdad, M, Pant, BD, Chauhan, S: Fixed point theorems in Menger spaces using the (CLRST) property and applications. J. Nonlinear Anal. Optim. 3 (2) (2012), 225-237.

[24] Liu, Z, Kang, SM: Existence and uniqueness of solutions for two classes of functional equations arising in dynamic programming. Acta Math. Appl. Sinica (Engl. Ser.) 23 (2007), 195-208.

[25] Hewitt, K, Stromberg, K: Real and Abstract Analysis. Springer, New York (1978).

[26] Manoj Kumar, Pankaj Kumar, Sanjay Kumar: Some Common Fixed Point Theorems Using (CLRg) Property in Cone Metric Spaces. Advances in Fixed Point Theory, 3 (2) (2012), 340-356.

[27] I. Beg and M. Abbas, Coincidence Point and Invariant Approximation for Mappings Satisfying Generalized Weak Contractive Condition, Fixed Point Theory and Applications 2006 (2006), Article Id 74503, 7 Pages.

[28] P. N. Dutta, B. S. Choudhury, A Generalisation Of Contraction Principle In Metric Spaces, Fixed Point Theory And Applications, 2008 (2008), Article ID 406368, 8 Pages.

[29] O. Dori´c, Common fixed point for generalized (ψ, ϕ) - weak contractions, Applied Mathematics Letters, 22 (2009), 1896-1900.