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BEST PROXIMITY RESULTS FOR CYCLIC α-IMPLICIT CONTRACTIONS IN QUASI-METRIC SPACES AND ITS CONSEQUENCES

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Abstract. Beforehand we set up the new concept of cyclic α-implicit contractions and subsequently find best proximity points for such type of mappings on quasi-metric spaces and partially ordered quasi-metric spaces. Besides, we construct an example and deduce several related results to demonstrate the usability of our derived theorem. As a consequence of this study, we deliver a fixed point result and also a result on the existence of solution of a type of integral equation. Additionally, we point out some mistakes in a recent paper by Aydi et al. [Nonlinear Anal. Model. Control 21 (2016), no. 1, 40-56].

Keywords: best proximity point; fixed point; cyclic α-implicit contractions; quasi-metric spaces; integral equations.

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1. Introduction

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Many problems in pure and applied mathematics such as differential and integral equations, discrete and continuous dynamical systems, variational analysis can be conceived as the problem of finding fixed points. Due to its numerous applications, metric fixed point theory has become a topic of immense interest to the mathematicians (see [4, 6, 5, 8, 10, 9, 12]). In this context, one of the most appealing research subjects is the investigation of existence of fixed points for non-self mappings. Precisely, for given non-empty closed subsets A and B of a complete metric space (X,d), a non-self contraction $T : A \to B$ does not necessarily have a fixed point. In this direction, it is absolutely natural to inspect an element $x \in X$ such that $d(x,Tx)$ is minimum. This point is called a best proximity point.

Popa [13, 14] originated the study of fixed point for mappings satisfying an implicit relation. It heads to some fascinating known fixed points results. In this sequel, Aydi et al. [3] introduced the notion of α-implicit contractions in quasi-metric spaces. They also deduced some fixed point results involving these contractions in the same spaces.

Inspired and encouraged by the above facts, in this manuscript, we generalize the concept of $\alpha-$implicit contractions. We also introduce the notion of best proximity points in a quasi-metric space and subsequently explore a best proximity point theorem for such type of mappings in the said spaces. Moreover we deliver consequences related to this theorem. As an application of our result, we derive a fixed point result and it is summarized by employing this to ensure the existence of a solution of a type of integral equations. Unexpectedly, we notice that the example provided by Aydi et al. [3, p.47, Example 4] is not correct. Also, the function d defined by them while providing an application [3, p.54, Theorem 8] is not a quasi-metric at all. As a result, the application becomes unjustified.

2. Preliminaries

For the sake of completeness, we recall some basic notions, definitions and fundamental results. Throughout the paper $\mathbb{N}_0$ and $\mathbb{R}_+$ will stand for the set of all non-negative integers and non-negative real numbers respectively.

Definition 2.1. [7] *Let (X,d) be a metric space and let A and B be two non-empty subsets of X. A mapping $T : A\cup B \to A\cup B$ is said to be a cyclic mapping if $T(A) \subset B$ and $T(B) \subset A$.*

Let us recall the definition of a quasi-metric space.

Definition 2.2. [3] *Let X be a non-empty set and let $d : X \times X \to [0,\infty)$ be a function which satisfies:*

(d_1) $d(x,y) = 0$ *if and only if* $x = y$;

(d_2) $d(x,y) \leq d(x,z) + d(z,y)$.

Then d is called a quasi-metric and the pair (X,d) is called a quasi-metric space.

It is obvious that a metric space is a quasi-metric space, but the converse is not true in general.

Let (X,d) be a quasi-metric space and let A and B be two non-empty subsets of X. Then the distance between the sets A and B is denoted by $D(A,B)$ and defined as

$$D(A,B) = inf\{d(x,y) : x \in A, y \in B\}.$$

In a similar fashion we can define $D(B,A)$. However it can be noted from the following example that the distance between two non-empty sets A and B, i.e., $D(A,B)$ and $D(B,A)$ may not be same in this setting.

Example 2.1. *Let $X = \mathbb{R}$ and $d : X \times X \to \mathbb{R}$ be the quasi-metric defined by:*

$$d(x,y) = \begin{cases} x - y, & \text{if } x \geq y; \\ \frac{1}{2}(y - x), & \text{if } y > x. \end{cases}$$

Also let $A = [0,1]$ and $B = \{2\}$. Then it is easy to check that $D(A,B) = \frac{1}{2}$ and $D(B,A) = 1$.

Now we say the pair $(x,y) \in (A,B)$ is proximal in (A,B) if $d(x,y) = D(A,B)$. We also define the sets A_0 and B_0 in the following way:

$$A_0 = \{x \in A : d(x,y') = D(A,B) \text{ for some } y' \in B\}$$

and

$$B_0 = \{y \in B : d(x',y) = D(A,B) \text{ for some } x' \in A\}.$$

We define the concept of a best proximity point in quasi-metric spaces as follows:

Definition 2.3. *Let (X,d) be a quasi-metric space and A,B be two non-empty subsets of X. Suppose $T : A \to B$ be a mapping. Now $x \in A$ is said to be a best proximity point of T if either $d(x,Tx) = D(A,B)$ or $d(Tx,x) = D(B,A)$ holds.*

Consequently we can define best proximity points in case of a cyclic map too.

Now we adhere the concepts of convergence and continuity on quasi-metric spaces from [3].

Definition 2.4. [3] *Let (X,d) be a quasi-metric space. Suppose that (x_n) is a sequence in X and $x \in X$. The sequence (x_n) converges to x if and only if*

$$\lim_{n\to\infty} d(x_n,x) = \lim_{n\to\infty} d(x,x_n) = 0.$$

It can be noted that the limit of a sequence in a quasi-metric space is unique, if exists, since this is a Hausdorff space.

Definition 2.5. [3] *Let (X,d) be a quasi-metric space. The map $f : X \to X$ is said to be continuous if for each sequence (x_n) in X converging to $x \in X$, the sequence (fx_n) converges to fx, i.e.,*

$$\lim_{n\to\infty} d(fx_n, fx) = \lim_{n\to\infty} d(fx, fx_n) = 0.$$

In the existing literature, there are various classes of implicit contractions where several beautiful consequences of fixed point theorems can be deduced. We represent Ψ as the set of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying:

($\psi 1$) ψ is non-decreasing;

($\psi 2$) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t \in \mathbb{R}_+$, where ψ^n is the n-th iterate of ψ.

From the above definition of ψ, it is clear that $\psi(t) < t$ for all $t > 0$.

Definition 2.6. [3] *Let Γ be the set of all continuous functions $F(t_1,...,t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ such that:*

($F1$) *F is non-decreasing in variable t_1 and non-increasing in variable t_5;*

($F2$) *there exists $h_1 \in \Psi$ such that for all $u,v \geq 0$, $F(u,v,v,u,u+v,0) \leq 0$ implies $u \leq h_1(v)$;*

($F3$) *there exists $h_2 \in \Psi$ such that for all $t,s > 0$, $F(t,t,0,0,t,s) \leq 0$ implies $t \leq h_2(s)$.*

Example 2.2. [15] *The following examples highlight the previous definition.*

(1) $F(t_1,t_2,...,t_6) = t_1 - at_2 - bt_2 - ct_3 - dt_4 - et_6$, *where* $a,b,c,d,e \geq 0$, $a+b+c+2d+e < 1$.

(2) $F(t_1,...,t_6) = t_1 - k\, max\{t_2,....,t_6\}$, *where* $k \in [0,\frac{1}{2})$.

Of late, Samet et al. [16] coined the idea of $\alpha-$admissible maps and proposed an interesting class of maps, namely $\alpha-\psi$ contraction mappings, to analyze the existence and uniqueness of a fixed point.

Definition 2.7. [16] *Let X be a non-empty set. Also assume that $T: X \to X$ and $\alpha : X \times X \to [0,\infty)$ be two mappings. We say that T is α-admissible if for all $x,y \in X$*

$$\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1.$$

In the paper [3], the authors introduced the notion of α-implicit contractive mappings in quasi-metric spaces.

Definition 2.8. *Let (X,d) be a quasi-metric space and $f: X \to X$ be a given mapping. We say that f is an α-implicit contraction if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $F \in \Gamma$ such that*

$$F(\alpha(x,y)d(fx,fy), d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)) \leq 0$$

for all $x,y \in X$.

3. Best Proximity Point Result

In this section, we derive a best proximity point theorem for a cyclic α-implicit contraction in the framework of a quasi-metric space.

Primitively, we introduce a new class of functions Ψ'. Let Ψ' be the set of all functions $\psi : [0,\infty) \to [0,\infty)$ satisfying:

(1) ψ is non-decreasing,

(2) $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t \in \mathbb{R}^+$.

One can easily check that Ψ is a proper subset of Ψ'. The following example presents the simplest instance of it.

Example 3.1. *Let us consider the function $f : [0,1] \to \mathbb{R}$ defined by $f(t) = \frac{t}{1+1}, f(\frac{t}{n}) = \frac{t}{n+1}$ where $n \in \mathbb{N}$. Here f is non-decreasing and $\lim_{n\to\infty} f^n(t) = 0$ for all $x \in [0,1]$. Hence $f \in \Psi'$, but $f \notin \Psi$.*

Here we conceive the notion of another interesting class of maps.

Definition 3.1. *Let Γ' be the set of all continuous functions $F(t_1-c,t_2,...,t_6):\mathbb{R}_+^6\to\mathbb{R}$ such that:*

(1) F is non-decreasing in variable t_1 and non-increasing in the variable t_5;

(2) there exists $h\in\Psi'$ such that for all $u,v\geq 0$,

$$F(u-c,v,v,u,u+v,0)\leq 0 \implies u-c\leq h(v-c)$$

for some $c\in\mathbb{R}$.

We can yield a non-trivial example for the previous definition.

Example 3.2. *Let $F:\mathbb{R}_+^6\to\mathbb{R}$ is defined by $F(t_1-l,t_2,....,t_6)=(t_1-l)-a(t_2-l)-b(t_3-l)+c(t_4-l)-c(t_5-2l)-dt_6$, where a,b,c,d are positive real numbers satisfying $a+b+c+d<1$ and $a+b>c$ and $l\in\mathbb{R}$.*

Now we generalize the concept of cyclic α-implicit contractive mappings [3].

Definition 3.2. *Let (X,d) be a quasi-metric space and A,B be two non-empty subsets of X. Let $T:A\cup B\to A\cup B$ be a mapping such that $T(A)\subset B,\ T(B)\subset A$. We say that T is a cyclic α-implicit contractive mapping if there exists two functions*

$$\alpha:A\times B\to[0,\infty)$$

and $F\in\Gamma'$ such that

$$F(\alpha(x,y)(d(Tx,Ty)-D(A,B)),d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx))\leq 0$$

for all $x,y\in A\cup B$.

Now we deliver our first contribution involving cyclic α-implicit contractive mapping defined in quasi-metric spaces.

Theorem 3.1. *Let (X,d) be a quasi-metric space and A,B be two non-empty subsets of X, where A is sequentially compact and B is closed. Let $T:A\cup B\to A\cup B$ be a cyclic α-implicit contractive mapping. Suppose that:*

(1) T is α admissible;

(2) there exists $x_0\in A$ such that $\alpha(x_0,Tx_0)\geq 1$ and $\alpha(Tx_0,x_0)\geq 1$;

(3) T is continuous.

Then there exists a best proximity point $x \in A \cup B$ such that $d(x,Tx) = D(A,B)$ and $d(Tx,x) = D(B,A)$.

Proof. Since A is sequentially compact and B is closed, so A_0 and B_0 are non-empty.

Let $x_0 \in A$. We define a sequence in $A \cup B$ by setting $x_{n+1} = T(x_n)$ for all $n \in \mathbb{N}_0$.

Since T is an α-admissible contractive mapping, we have

$$\begin{aligned} F(\alpha(x_{n-1},x_n)(d(Tx_{n-1},Tx_n)-D(A,B)), d(x_{n-1},x_n), d(x_{n-1},Tx_{n-1}),& \\ d(x_n,Tx_n), d(x_{n-1},Tx_n), d(x_n,Tx_{n-1})) &\leq 0. \end{aligned}$$

This leads to

$$\begin{aligned} F(\alpha(x_{n-1},x_n)(d(x_n,x_{n+1})-D(A,B)), d(x_{n-1},x_n), d(x_{n-1},x_n),& \\ d(x_n,x_{n+1}), d(x_{n-1},x_{n+1}), 0) &\leq 0. \end{aligned}$$

Since F is non-decreasing in the first variable and non-increasing in the fifth variable, we obtain

$$\begin{aligned} F((d(x_n,x_{n+1})-D(A,B)), d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1}),& \\ d(x_{n-1},x_n)+d(x_n,x_{n+1}), 0) &\leq 0. \end{aligned}$$

By the definition of F, there exists a function $h \in \Psi'$ such that

$$\begin{aligned} d(x_n,x_{n+1}) - D(A,B) &\leq h(d(x_{n-1},x_n) - D(A,B)) \\ &\leq h^2(d(x_{n-2},x_{n-1}) - D(A,B)) \\ &\vdots \\ &\leq h^n(d(x_0,x_1) - D(A,B)). \end{aligned}$$

Since $h \in \Psi'$, we have

$$\lim_{n\to\infty} h^n(d(x_0,x_1) - D(A,B)) = 0.$$

Therefore,

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = D(A,B).$$

Here, we know $x_{2n} \in A$ for all $n \in \mathbb{N}_0$. Since A is sequentially compact, there exists a subsequence (x_{2n_k}) of (x_{2n}), which converges in A. Let (x_{2n_k}) converges to $x \in A$. Therefore we have

$$d(x, x_{2n_k+1}) \leq d(x, x_{2n_k}) + d(x_{2n_k}, x_{2n_k+1}).$$

Letting $k \to \infty$, we derive $d(x, x_{2n_k+1}) \to D(A,B)$.

Similarly, it can be shown that $d(x_{2n_k+1}, x) \to D(A,B)$ as $k \to \infty$.

We observe that,

$$\begin{aligned} d(x,Tx) &\leq d(x, x_{2n_k+1}) + d(x_{2n_k+1}, Tx) \\ &\leq d(x, x_{2n_k+1}) + d(Tx_{2n_k}, Tx). \end{aligned}$$

As $x_{2n_k} \to x$ and T is continuous, we obtain

$$d(x,Tx) = D(A,B).$$

Similarly we can deduce that,

$$d(Tx,x) = D(B,A).$$

Thus we have $d(x,Tx) = D(A,B)$ and $d(Tx,x) = D(B,A)$. Hence we can conclude that T has a best proximity point in X.

Here we put down the immediate fixed point result which can be obtained from the above theorem.

Corollary 3.1. *Let (X,d) be a compact quasi-metric space and $T : X \to X$ be an α-implicit contractive map. Suppose that:*

(i) T is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0) \geq 1$ and $\alpha(x_0, Tx_0) \geq 1$;

(iii) T is continuous.

Then there exists a fixed point $x \in X$ such that $Tx = x$.

Remark 3.1. *In* [3, p.47, Example 4], *the argument is completely wrong. So the deductions done there are not true.*

The authors considered the quasi-metric as

$$d(x,y) = \begin{cases} |x|, & \text{if } x \neq y; \\ 0, & \text{if } y = x, \end{cases}$$

where $x, y \in [0, \infty)$.

But according to the definition of a quasi-metric, it does not satisfy the condition:

$$d(x,y) = 0 \Leftrightarrow x = y.$$

For example, if we take $x = 0$ *and* $y = 1$*, we get* $d(x,y) = 0$*, although* $x \neq y$*. So this is not a quasi-metric.*

Similarly in [3, p.54, Theorem 8]*, the function defined as*

$$d(x,y) = \begin{cases} ||x||, & \text{if } x \neq y; \\ 0, & \text{if } y = x, \end{cases}$$

where $||x|| = \max_{t \in [a,b]} |x(t)|$ *and* $x, y \in C[a,b]$*, is also not a quasi-metric.*

We can prove it by taking $x(t) = 0$ *and* $y(t) = 1$*, where* $t \in [a,b]$*. Then* $d(x,y) = 0$ *but* $x \neq y$*, which contradicts the definition of a quasi-metric. Henceforth, this is not a suitable application for Corollary* 4 *in* [3].

4. Consequences

Consistent with [3], some basic definitions and notations required in this paper are recollected as follows. Furthermore, here we discuss some results which easily can be interpreted from our main result.

Definition 4.1. *Let* $(X, \preccurlyeq)$ *be a partially ordered set and* $f : X \to X$ *be a given mapping. We say that* f *is non-decreasing with respect to '*$\preccurlyeq$*' if*

$$x, y \in X,\ x \preccurlyeq y\ \Rightarrow fx \preccurlyeq fy.$$

Definition 4.2. *Let* $(X, \preccurlyeq)$ *be a partially ordered set. A sequence* (x_n) *is said to be non-decreasing with respect to '*$\preccurlyeq$*' if* $x_n \preccurlyeq x_{n+1}$ *for all* $n \in \mathbb{N}$.

Definition 4.3. *Let $(X,\preccurlyeq)$ be a partially ordered set and d be a quasi-metric on X. We say that $(X,\preccurlyeq,d)$ is regular if for every non-decreasing sequence (x_n) in X such that $x_n \to x$ as $n \to \infty$, there exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \preccurlyeq x$ for all $k \in \mathbb{N}$.*

We now manage to prove the following best proximity point result.

Corollary 4.1. *Let $(X,\preccurlyeq,d)$ be a partially ordered quasi-metric space and A,B be two non-empty disjoint subsets of X. Let A is a sequentially compact set and B is closed. Let $T: A\cup B \to A\cup B$ be a non-decreasing map with respect to '$\preccurlyeq$'. Also suppose that there exists a function $F \in \Gamma'$ such that*

$$F((d(Tx,Ty)-D(A,B)),d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)) \leq 0,$$

for all $x,y \in A\cup B$. Also assume that the following conditions hold:

(1) there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$ or $Tx_0 \preccurlyeq x_0$;

(2) T is continuous.

Then T has a best proximity point.

Proof. Define the mapping $\alpha : A\times B \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x \preccurlyeq y \text{ or } x \succcurlyeq y; \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that with respect to this α, T is a cyclic α-implicit contractive mapping. Hence

$$F(\alpha(x,y)d(Tx,Ty),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx) \leq 0,$$

for all $x \in A$ and $y \in B$. From condition (1), we have $\alpha(x_0,Tx_0) \geq 1$ and $\alpha(Tx_0,x_0) \geq 1$. By the monotone property of T, we have

$$\begin{aligned} \alpha(x,y) \geq 1 \quad &\Rightarrow \quad x \succcurlyeq y \text{ or } x \preccurlyeq y \\ &\Rightarrow \quad Tx \succcurlyeq Ty \text{ or } Tx \preccurlyeq Ty \\ &\Rightarrow \quad \alpha(Tx,Ty) \geq 1. \end{aligned}$$

Thus T is α-admissible. Now since T is continuous, the existence of best proximity point follows from Theorem (3.1).

Corollary 4.2. *Let (X,d) be a quasi-metric space. Also assume that A and B are two non-empty subsets of X. Suppose A is sequentially compact and B is a closed subset. Let $T : A \cup B \to A \cup B$ be a continuous cyclic map. Suppose that for some $F \in \Gamma'$,*

$$F((d(Tx,Ty) - D(A,B)), d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)) \leq 0$$

for all $x, y \in X$. Then T has a best proximity point.

Proof. Taking $\alpha(x,y) = 1$ for all $x, y \in A \cup B$, the conclusion can be obtained from Theorem (3.1).

Corollary 4.3. *Let (X,d) be a quasi-metric space. Also assume that A and B are two non-empty subsets of X. Suppose A is sequentially compact and B is closed subset. Let $T : A \cup B \to A \cup B$ is a continuous cyclic map such that for all $x, y \in A \cup B$, the condition*

$$\begin{aligned} d(Tx,Ty) - D(A,B) \quad &\leq \quad k\, max\{d(x,y), d(x,Tx), d(y,Ty), \\ &\qquad d(x,Ty), d(y,Tx)\} - 2k\, D(A,B) \end{aligned}$$

holds, where $k \in [0, \frac{1}{2})$. Then T has a best proximity point.

Proof. Let

$$F(t_1 - D(A,B), t_2, ..., t_6) = (t_1 - D(A,B)) - k\, max\{t_2, ..., t_6\} + 2k\, D(A,B)$$

be a mapping in Γ' with $k \in [0, \frac{1}{2})$. Now we apply Corollary (4.2) to obtain a best proximity point.

Definition 4.4. [11] *Let (X,G) be a G-metric space and $d : X \times X \to [0,\infty)$ be a function defined by*

$$d(x,y) = G(x,y,y).$$

Then (X,d) is a quasi-metric space.

Definition 4.5. [2] *Let X be a non-empty set and $T : X \to X$ and $\beta : X^3 \to [0,\infty)$ are two mappings. We say that the mapping T is β-admissible if for all $x, y \in X$, we have*

$$\beta(x,y,y) \geq 1 \Rightarrow \beta(Tx,Ty,Ty) \geq 1.$$

Theorem 4.1. *Let (X,G) be a G-metric space. Assume that A and B are two non-empty subsets of X. Suppose A is sequentially compact and B is a closed subset. Also let $T: A\cup B \to A\cup B$ be a cyclic map satisfying the condition:*

$$\begin{aligned} F(\beta(x,y,y)(G(Tx,Ty,Ty)-G(A,B,B)),G(x,y,y),G(x,Tx,Tx),\\ G(y,Ty,Ty),G(x,Ty,Ty),G(y,Tx,Tx)) \quad \leq \quad 0 \end{aligned}$$

for all $x,y\in A\cup B$, where $\beta: D^3 \to [0,\infty)$, $D = A\cup B$ and $F\in\Gamma'$.

Suppose that:

(i) T is β-admissible;

(ii) there exists $x_0\in A\cup B$ such that $\beta(x_0,Tx_0,Tx_0)\geq 1$ and $\beta(Tx_0,Tx_0,x_0)\geq 1$;

(iii) T is continuous.

Then there exists $x\in A\cup B$ such that $G(x,Tx,Tx)=G(A,B,B)$ and $G(Tx,Tx,x)=G(B,A,A)$.

Proof. It is sufficient to take $d(x,y)=G(x,y,y)$ and $\alpha(x,y)=\beta(x,y,y)$. Then the proof of this theorem follows from Theorem (3.1).

Corollary 4.4. *Let (X,G) be a G metric-space. Assume that A and B are two non-empty subsets of X. Suppose A is sequentially compact and B is a closed subset. Let $T: A\cup B\to A\cup B$ be a cyclic map such that for all $x,y\in A\cup B$*

$$\begin{aligned} \beta(x,y,y)G(Tx,Ty,Ty)-G(A,B,B) \quad \leq \quad & k\ max\{G(x,y,y),G(x,Tx,Tx),\\ & G(y,Ty,Ty),G(x,Ty,Ty),\\ & G(y,Tx,Tx)\}-2k\ G(A,B,B) \end{aligned}$$

holds, where $k\in[0,\frac{1}{2})$ and $\beta: D^3\to[0,\infty)$, where $D=A\cup B$.

Suppose that:

(i) T is β admissible;

(ii) there exists $x_0\in A\cup B$ such that $\beta(x_0,Tx_0,Tx_0)\geq 1$ and $\beta(Tx_0,x_0,x_0)\geq 1$;

(iii) T is continuous.

Then there exists $x\in A\cup B$ such that $G(x,Tx,Tx)=G(A,B,B)$ and $G(Tx,Tx,x)=G(B,A,A)$.

Proof. The proof follows from Theorem (4.1).

5. Application

The main purpose of this section is to provide an application of the results discussed in this sequel. Here we offer a fixed point theorem which is a direct consequence of a derived corollary of our manuscript. Moreover, we employ it to ensure the existence of a solution of a type of non-linear integral equation. Also we construct an example to substantiate the attained result.

Now we consider the following non-linear integral equation:

$$f(t) = \int_0^t K(t,s,f(s))ds, \tag{1}$$

where $t \in I = [0,1]$ and $K : I \times I \times C[0,1] \to \mathbb{R}_+$ is a continuous map.

Let $X = C[0,1]$ and $d : X \times X \to [0,\infty)$ be a mapping defined by

$$d(f,g) = max\{\sup_{x\in I}(f(x)-g(x)), 2\sup_{x\in I}(g(x)-f(x))\}. \tag{2}$$

For $f,g \in X$, with $f \neq g$, we have either $(f-g)(x) > 0$ or $(g-f)(x) > 0$ for some $x \in X$.

Hence, $f \neq g$ implies the non-negativity, i.e., $d(f,g) > 0$.

Clearly, $d(f,g) = 0 \Longleftrightarrow f = g$.

Also one can simply verify that $d(f,g) \neq d(g,f)$, in general.

Again the triangle inequality follows from the fact

$$\sup_{x\in I}\{(f+g)(x)\} \leq \sup_{x\in I} f(x) + \sup_{x\in I} g(x).$$

Therefore d defines a quasi-metric on X.

Now we deliver a fixed point result which follows from Corollary (4.3).

Theorem 5.1. *Let (X,d) be a quasi-metric space. Also suppose that A is a non-empty and sequentially compact subset of X. Let $T : A \to A$ is a continuous map such that for all $x,y \in A$, the condition*

$$\begin{aligned} d(Tx,Ty) &\leq k\, max\{d(x,y), d(x,Tx), d(y,Ty), \\ &\qquad d(x,Ty), d(y,Tx)\} \end{aligned}$$

holds, where $k \in [0,\frac{1}{2})$. Then T has a fixed point in A.

Now we yield an example to confirm our attained result.

Example 5.1. *Let $X = [0,1]$ and $d : X \times X \to [0,\infty)$ be defined by*

$$d(x,y) = \begin{cases} x-y, & \text{if } x \geq y; \\ \frac{1}{2}(y-x), & \text{otherwise.} \end{cases}$$

Clearly, (X,d) is a compact quasi-metric space.

We consider the mapping $T : X \to X$ defined by

$$T(x) = \tfrac{x}{4}, \text{ for all } x \in X$$

Now we have

$$d(Tx,Ty) = \begin{cases} \frac{x-y}{4}, & \text{if } x \geq y; \\ \frac{1}{2}\frac{(y-x)}{4}, & \text{otherwise.} \end{cases}$$

Therefore we obtain

$$d(Tx,Ty) \leq \frac{d(x,y)}{4}.$$

Hence,

$$d(Tx,Ty) \leq \frac{1}{4} max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$$

for all $x,y \in X$.

Let $\alpha : X \times X \to [0,\infty)$ be a mapping defined by

$$\alpha(x,y) = 1$$

for all $x,y \in X$.

With respect to this α, T is an α-implicit contractive map and moreover, T is continuous. So by Theorem (5.1), there exists a fixed point $x \in X$ of T. Here the point is $x = 0$.

Theorem 5.2. *Let us assume that the following conditions hold:*

(i) there exists a function $p : I \times I \to \mathbb{R}_+$ such that

$$K(t,s,f) \leq p(t,s)f$$

for each $t,s \in I$ and $f \in X$;

(ii) for $f,g \in X$ with $f \neq g$, $\int_0^t K(t,s,f(s))ds \neq \int_0^t K(t,s,g(s))ds$;

(iii) $\sup_{t \in I} p(t,s) < \dfrac{1}{3}$.

Then the integral equation (1) has a solution in X.

Proof. Let us consider the quasi-metric d defined by (2).

One can easily check that (X,d) is a compact quasi-metric space.

Consider the mapping $T : X \to X$ defined by

$$Tf(t) = \int_0^t K(t,s,f(s))ds$$

for all $f \in X$.

Now, we have

$$\begin{aligned} d(Tf,Tg) &= \max\Big\{ \sup_I \int_0^t \Big(K(t,s,f(s)) - K(t,s,g(s))\Big)ds, \\ &\quad 2\sup_I \int_0^t \Big(K(t,s,g(s)) - K(t,s,f(s))\Big)ds\Big\} \\ &\leq \max\Big\{ \int_0^t \sup_I \Big(K(t,s,f(s)) - K(t,s,g(s))\Big)ds, \\ &\quad 2\int_0^t \sup_I \Big(K(t,s,g(s)) - K(t,s,f(s))\Big)ds\Big\} \\ &\leq \max\Big\{ \int_0^t \sup_I \Big(p(t,s)f(s) - p(t,s)g(s)\Big)ds, \\ &\quad 2\int_0^t \sup_I \Big(p(t,s)g(s) - p(t,s)f(s)\Big)ds\Big\} \\ &\leq \max\Big\{ \int_0^t \frac{1}{3}\sup_I \Big(f(s) - g(s)\Big)ds, 2\int_0^t \frac{1}{3}\sup_I \Big(g(s) - f(s)\Big)ds\Big\} \\ &\leq \frac{1}{3}d(f,g) \\ &\leq \frac{1}{3}\max\{d(f,g), d(f,Tf), d(g,Tg), d(f,Tg), d(g,Tf)\}. \end{aligned}$$

Hence by Theorem (5.1), the integral equation has a solution in X.

Conflict of Interests

The authors declare that there is no conflict of interests.

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