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ON COMMON FIXED POINT OF A FUNCTION AND ITS DERIVATIVE

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Abstract. In this article, we present a common fixed point of a function with its derivative function. The existence of common fixed point is shown through a pair of commutative, weakly commutative, and compatible mapping. This result is relatively new for the pair raised was between functions with their derivatives.

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1. Introduction

The concept fixed point theorem for composition of derivatives have been investigated by Elekes [1], who gave an affirmative answer to a question of K. Ciesielski, which was asked whether the composition $f \circ g$ of two derivatives $f, g : [0,1] \longrightarrow [0,1]$ must always have a fixed point [2]. To be precise is theorem as follows

Theorem 1.1. *Let $f, g : [0,1] \longrightarrow [0,1]$ be derivatives. Then $f \circ g$ has a fixed point.*

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The results in [4] depicts interdependence between commuting mapping and the fixed point concept. The result in [1] and [4] are using the composition function concept to study the existence of fixed point or common fixed point. In this paper, we study common fixed point for two functions mutually dependent. For example, the function f with its derivatives.

Throughout this paper $\mathbb{R}$ is the real number system, $\mathbb{N}$ is natural number system, $\mathbb{Q}$ is rational number system, and X is metric space with respective metric d. The composition function between f and g at a point x denotes $f(g(x))$, while f' is notation for derivative of function f.

Definition 1.1. The function $f : [a,b] \longrightarrow \mathbb{R}$ is called differentiable at point $c \in (a,b)$ if there exists real number $f'(c)$ with property for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in [a,b]$ with $|x-c| < \delta$ we have

$$\left| \frac{f(x)-f(c)}{(x-c)} - f'(c) \right| < \varepsilon. \tag{1}$$

Typically we write

$$\lim_{x \to c} \frac{f(x)-f(c)}{x-c} = f'(c). \tag{2}$$

Definition 1.2. Let $f,g : \mathbb{R} \longrightarrow \mathbb{R}$ be mappings. A point $x_0 \in \mathbb{R}$ is called coincidence point of f and g if $f(x_0) = g(x_0)$.

Definition 1.3. Two mappings $f,g : \mathbb{R} \longrightarrow \mathbb{R}$ are said to be commuting if

$$|f(g(x)) - g(f(x))| = 0,$$

for all $x \in \mathbb{R}$.

2. Preliminaries

In [6] introduces the concept of contraction mapping as extension of fixed point theorem that introduced by Meir and Keller [5], such as:

Definition 2.1. Let f and g be self maps of a metric space (X,d). Map g is said (ε,δ)-f-contraction if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

i. $\varepsilon \le d(f(x), f(y)) < \varepsilon + \delta$ implies $d(g(x), g(y)) < \varepsilon$, and
ii. $g(x) = g(y)$ when $f(x) = f(y)$.

for all $x, y \in X$.

Relation commuting mappings have been the result in [6] the following.

Theorem 2.1.[6] *If f is continuous self map of a complete metric space X and g is (ε, δ)-f-contraction which commutes with f, then f and g have a unique common fixed point in X.*

Theorem 2.2.[3] *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous. If there is $g : \mathbb{R} \longrightarrow \mathbb{R}$ and $k \in (0,1)$ such that*

i. $f(g(x)) = g(f(x))$, *for all* $x \in \mathbb{R}$

ii. $g(\mathbb{R}) \subset f(\mathbb{R})$

iii $|g(x) - g(y)| \leq k|f(x) - f(y)|$,*for all* $x, y \in \mathbb{R}$,

then f and g have a unique common fixed point in $\mathbb{R}$.

Furthermore, Sessa in [8] generalized the commuting mapping as in the following.

Definition 2.2. Two mappings $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are said to be weakly commuting if

$$|f(g(x)) - g(f(x))| \leq |f(x) - g(x)|$$

for all $x \in \mathbb{R}$.

Note that the commuting mappings are weakly commuting but the converse is not necessarily true as is shown in [8].

Example 2.1. Let $I = [0,1]$ be with usual metric. Define $f(x) = \frac{x}{2}$ and $g(x) = \frac{x}{2+x}$ for each $x \in I$. We have

$$\begin{aligned} |f(g(x)) - g(f(x))| &= \left| \frac{x}{4+x} - \frac{x}{4+2x} \right| = \left| \frac{x}{(4+x)(4+2x)} \right| \\ &\leq \left| \frac{x^2}{4+2x} \right| = \frac{x}{2} - \frac{x}{2+x} = |f(x) - g(x)| \end{aligned}$$

for each $x \in I$. Pairs (f, g) is weakly commuting but not commuting, namely $|f(g(x)) - g(f(x))| \neq 0$ for all $x \in I$.

The following Theorem highlight the role of weakly commutative in producing common fixed point, which the result by Sessa.

Theorem 2.2. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous mapping and $g : \mathbb{R} \longrightarrow \mathbb{R}$ satisfy*

i. $|f(g(x)) - g(f(x))| \leq |f(x) - g(x)|$, *for all* $x, y \in \mathbb{R}$

ii. $g(\mathbb{R}) \subset f(\mathbb{R})$

iii $|g(x)-g(y)| \leq \psi(M(x,y)$,*for all* $x,y \in \mathbb{R}$,

If there exists a $x_0 \in \mathbb{R}$ *such that* $\delta(O(x_0,\infty) < \infty$ *then* f *and* g *have a unique common fixed point in* $\mathbb{R}$.

More detail the notation $\psi(M(x,y)$ and $\delta(O(x_0,\infty))$ can be seen in [8]. The next generalization was introduced by Jungck [4] as follows.

Definition 2.3.[4] Two mappings $f,g:\mathbb{R}\longrightarrow\mathbb{R}$ are said to be compatible if $\langle x_n\rangle$ is a sequence in $\mathbb{R}$ such that $\lim_n f(x_n) = \lim_n g(x_n)$,

$$\lim_n |f(g(x_n)-g(f(x_n))| = 0.$$

Note that weakly commuting mappings are compatible but the converse is not necessarily true as is shown in [4]

Example 2.2. Let $f,g:\mathbb{R}\longrightarrow\mathbb{R}$ with $f(x)=x^3$ and $g(x)=2x^3$ Since $\lim_{x\to 0}|x^3-2x^3|=0$ we have

$$\lim_{x\to 0}|f(g(x))-g(f(x))| = \lim_{x\to 0} 6|x|^9 = 0.$$

The functions f,g is compatible but they are not weakly commuting pairs because $|f(g(x)) - g(f(x))| = 6|x|^9 > |x|^3 = |f(x)-g(x)|$.

The following Theorem highlights the role of compatibility in producing common fixed point, which is the result by Junck.

Theorem 2.3. [4] *Let* $f:\mathbb{R}\longrightarrow\mathbb{R}$ *be continuous, and* $g:\mathbb{R}\longrightarrow\mathbb{R}$ *satisfy*

i. (ε,δ)-f-*contraction*

ii. f *and* g *are compatible*

If δ *is lower semi continuous, then* f *and* g *have a unique common fixed point.*

The results in this article we need a definition of additive mapping.

Definition 2.4. A function $f:\mathbb{R}\longrightarrow\mathbb{R}$ is additive if it satisfies the Cauchy equation

$$f(x+y) = f(x)+f(y)$$

for all $x\in\mathbb{R}$.

Lemma 2.1. *If an additive function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$, then f is continuous for all point in $\mathbb{R}$.*

Lemma 2.2. *If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is additive continuous at a point, then for all $r \in \mathbb{Q}$, $f(rx) = rf(x)$ or f linear under rational number.*

3. Main results

Commutative nature of the composition function described in the introduction is for a pair of functions which are not mutually dependent. If a pair of interdependence functions, then problem would be different, such as composition functions with its derivatives.

In the main result of this, one of which will be shown is commutative properties function of composition $f \circ f'$.

Proposition 3.1. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable on $\mathbb{R}$. If f is additive and $f(f(x)) = f(x)$ for all $x \in \mathbb{R}$, then function f' commutes with f.*

Proof. Since f additive, By Lemma 2.2, $f(hx) = hf(x)$ for all $h \neq 0 \in \mathbb{Q} \subset \mathbb{R}$. Thus for all $x \in \mathbb{R}$ and $h \in \mathbb{Q}$ we obtain

$$\begin{aligned} f(f'(x)) &= f\left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right) \\ &= \lim_{h\to 0} f\left(\frac{f(x+h) - f(x)}{h}\right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(f(f(x)) + f(f(h)) - f(f(x))\right) \\ &= \lim_{h\to 0} \frac{1}{h} f(f(h)) \\ &= \lim_{h\to 0} \frac{1}{h} f(h). \end{aligned}$$

and

$$\begin{aligned} f'(f(x)) &= \lim_{h\to 0} \frac{f(f(x)+h) - f(f(x))}{h} \\ &= \lim_{h\to 0} \frac{f(f(x)) + f(h) - f(f(x))}{h} \\ &= \lim_{h\to 0} \frac{1}{h} f(h). \end{aligned}$$

Thus $f(f'(x)) = f'(f(x))$.

Example 3.1. The additive function $f:\mathbb{R}\longrightarrow\mathbb{R}$ that satisfies $f(f(x))=f(x)$ there are only two functions, namely $f(x)=x$ and $f(x)=-x$. In this case, the role of commuting maps $(f(f'(x))=f'(f(x))$ in producing common fixed point is very simple. The common fixed point $x=1$ satisfies $f'(1)=f(1)=1$ for function $f(x)=x$.

Uniqueness of the common fixed point shown in this the theorem.

Theorem 3.1. *Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be a differentiable on $\mathbb{R}$ satisfies*

A) f is additive, and

B) $f(f(x))=f(x)$ for all $x\in\mathbb{R}$

If there exists a $x_0\in\mathbb{R}$ such that $f'(x_0)=x_0$, then f and f' have a unique common fixed point $x_0\in\mathbb{R}$.

Proof. According to Proposition 3.1 we have $f(f'(x))=f'(f(x))$ for all $x\in\mathbb{R}$. Since $f'(x_0)=x_0$ for some $x_0\in\mathbb{R}$, we obtain

$$\begin{aligned}
f(x_0) &= f(f'(x_0)) = f'(f(x_0))\\
&= \lim_{x\to f(x_0)} \frac{f(x)-f(f(x_0))}{x-f(x_0)}\\
&= \lim_{x\to f(x_0)} \frac{f(x)-f(x_0)}{x-f(x_0)}\\
&= \lim_{x\to x_0} \frac{f(x-x_0+f(x_0))-f(x_0)}{x-x_0+f(x_0)-f(x_0)}\\
&= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}\\
&= f'(x_0) = x_0.
\end{aligned}$$

Thus x_0 is common fixed point. We show x_0 is unique. Suppose $x_0=f(x_0)=f'(x_0)$ and $y_0=f(y_0)=f'(y_0)$, $x_0\neq y_0$. Then with to use a derivative concept of (1) we obtained

$$\begin{aligned}
|x_0-y_0|^2 &= (x_0-y_0)^2 = x_0^2-x_0y_0-x_0y_0+y_0^2 = (x_0-y_0)x_0+(y_0-x_0)y_0\\
&= x_0-y_0-(x_0-y_0)y_0+y_0-x_0-(y_0-x_0)x_0
\end{aligned}$$

$$= f(x_0) - f(y_0) - (x_0 - y_0)f'(y_0) + f(y_0) - f(x_0) - (y_0 - x_0)f'(x_0)$$
$$\leq |f(x_0) - f(y_0) - (x_0 - y_0)f'(y_0) + f(y_0) - f(x_0) - (y_0 - x_0)f'(x_0)|$$
$$\leq |f(x_0) - f(y_0) - (x_0 - y_0)f'(y_0)| + |f(y_0) - f(x_0) - (y_0 - x_0)f'(x_0)|$$
$$< \frac{\varepsilon}{2}|x_0 - y_0| + \frac{\varepsilon}{2}|y_0 - x_0| = \varepsilon|x_0 - y_0|.$$

Since $\varepsilon > 0$ is arbitrary, $x_0 = y_0$.

The necessary condition of weakly commutative composition function is described by the following proposition.

Proposition 3.2. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable on $\mathbb{R}$. If f is additive and $f(f'(x)) = f'(x)$ for all $x \in \mathbb{R}$, then f and f' are weakly commutative.*

Proof. Since f is additive, we have

$$\begin{aligned} f'(f(x)) &= \lim_{h\to 0} \frac{f(f(x)+h) - f(f(x))}{h} \\ &= \lim_{h\to 0} \frac{f(f(x)) + f(h) - f(f(x))}{h} \\ &= \lim_{h\to 0} \frac{1}{h} f(h) \end{aligned}$$

and

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h} f(h), \end{aligned}$$

hence $f'(f(x)) = f'(x)$ for all $x \in \mathbb{R}$. By hypothesis and the above result, for all $x \in \mathbb{R}$, we obtain $|f(f'(x)) - f'(f(x))| = |f'(x) - f'(x)| = 0 \leq |f(x) - f'(x)|$. This completes the proof.

The requirement $f(f'(x)) = f'(x)$ in Proposition 3.2 can fulfilled only if $f(x) = x$. Therefore, the Proposition 3.2 depicts that any commuting maps $(fof') = (f'of)$ are special cases of weakly commutative.

A more general statement which depicts the existence an additive function where the composition function with its derivative has properties weakly commutative, such as the following propositions.

Proposition 3.3. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an additive with $f(x) = cx$ and $c \neq 0$. If $|c| \leq 1$, then for each $x \geq \frac{1}{c}$, $x \in \mathbb{R}$ the composition f and f' are weakly commutative.*

Proof. Clearly that $f'(x) = c$ so that $f'(f(x)) = f'(cx) = c$ and $f(f'(x)) = f(c) = c^2$. For each $x \in \mathbb{R}$ with $cx \geq 1$, we have $cx - c \geq 1 - c$. It follow that,

$$\begin{aligned} |f'(f(x)) - f(f'(x))| &= |c - c^2| = |c||1 - c| \\ &\leq |1 - c| = 1 - c \leq cx - c \\ &\leq |cx - c| = |f(x) - f'(x)|. \end{aligned}$$

for each $x \geq \dfrac{1}{c}$. Proved that the composition f and f' are weakly commutative.

Example 3.2. Let $I = [\frac{3}{2}, 3]$ be with usual metric. Define $f(x) = \dfrac{x}{2}$ for each $x \in I$. We obtain $f'(x) = \dfrac{1}{2}$ so $f(f'(x)) = f(\frac{1}{2}) = \dfrac{1}{4}$ and $f'(f(x)) = f'(\frac{x}{2}) = \dfrac{1}{2}$. Therefore, for each $x \in I$ we have

$$\begin{aligned} |f'(f(x)) - f(f'(x))| &= \left|\frac{1}{4} - \frac{1}{2}\right| = \frac{1}{4} \\ &\leq \frac{x-1}{2} \leq \left|\frac{x}{2} - \frac{1}{2}\right| = |f(x) - f'(x)|. \end{aligned}$$

Proposition 3.4. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an additive with $f(x) = cx$ and $c \neq 0$.*

A) If $c > 1$ then for each $x \geq c$, $x \in \mathbb{R}$ the composition f and f' are weakly commutative.

B) If $c < -1$ then for each $x \leq c$, $x \in \mathbb{R}$ the composition f and f' are weakly commutative.

Proof. Clearly that $f'(x) = c$ so $f'(f(x)) = f'(cx)) = c$ and $f(f'(x)) = f(c) = c^2$.

A) For each $x \geq c$ and $c > 1$ we have $cx - c \geq c^2 - c > 0$. It follow that

$$\begin{aligned} |f(f'(x)) - f'(f(x))| &= |c^2 - c| = c^2 - c \leq cx - c \\ &\leq |cx - c| = |f(x) - f'(x)|. \end{aligned}$$

B) For each $x \leq c$ and $c < -1$ we have $cx - c \geq c^2 - c > 0$. It follow that,

$$\begin{aligned} |f(f'(x)) - f'(f(x))| &= |c^2 - c| = c^2 - c \leq cx - c \\ &\leq |cx - c| = |f(x) - f'(x)|. \end{aligned}$$

This completes the proof.

Remark 3.1. For the additive function $f(x) = cx$ where $c \neq 0$ that its composition f and f' are weakly commuting depends on constant c and domain of function. This fact may can called as the local weakly commutative and they have not a common fixed point.

The following results provides a criterion of compatibility mapping between f and f' as in [4].

Proposition 3.5. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuously differentiable on $\mathbb{R}$. If* $\lim_n f(x_n) = \lim_n f'(x_n) = x_0 \in \mathbb{R}$ *such that x_0 is a common fixed point, then f and f' are compatible.*

Proof. Suppose that $\lim_n f(x_n) = \lim_n f'(x_n) = x_0$ for some $x_0 \in \mathbb{R}$. By hypothesis $f(x_0) = f'(x_0) = x_0$. Since f is continuously differentiable, we obtain

$$\begin{aligned}
\lim_n |f(f'(x_n)) - f'(f(x_n))| &= |\lim_n f(f'(x_n)) - \lim_n f'(f(x_n))| \\
&= \left| f(x_0) - \lim_n \left(\lim_{h\to 0} \frac{1}{h} \left[f(f(x_n)+h) - f(f(x_n)) \right] \right) \right| \\
&= \left| f(x_0) - \lim_{h\to 0} \frac{1}{h} \left(\lim_n \left[f(f(x_n)+h) - f(f(x_n)) \right] \right) \right| \\
&= \left| f(x_0) - \lim_{h\to 0} \frac{1}{h} \left(\lim_n f(f(x_n)+h) - \lim_n f(f(x_n)) \right) \right| \\
&= \left| f(x_0) - \lim_{h\to 0} \frac{1}{h} \left(f(x_0+h) - f(x_0) \right) \right| \\
&= |f(x_0) - f'(x_0)| \\
&= 0.
\end{aligned}$$

Thus, f and f' are compatible.

Example 3.3. Suppose $f(x) = x^3$ for each $x \in \mathbb{R}$ that its derivative is $f'(x) = 3x^2$. They are compatible. Seem that $\lim_{x_n \to 0} f(x_n) = \lim_{x_n \to 0} f'(x_n) = 0$, where $x_0 = 0$ is a common fixed point of f and f'. It follow that $\lim_{x_n\to 0} |f(f'(x_n)) - f'(f(x_n))| = \lim_{x_n \to 0} |27x_n^6 - 3x_n^6| = \lim_{x_n\to 0} 24|x_n^6| = 0$.

The compatibility mapping of f and f' produces important properties, such as weakly compatible.

Proposition 3.6. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuously differentiable on $\mathbb{R}$ and f, f' be a compatible. If $f(x_0) = f'(x_0)$ for some $x_0 \in \mathbb{R}$, then $f(f'(x_0)) = f'(f(x_0))$.*

Proof. For each $n \in \mathbb{N}$,the sequence of real numbers $x_n = x_0$. Of course the sequence $f(x_n) = f'(x_n)$ convergence to $f(x_0) = f'(x_0)$ by continuously differentiable. Since f and f' are compatible, we obtain $f(f'(x_0)) = f'(f(x_0))$.

Proposition 3.7. *Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be a differentiable on $\mathbb{R}$ and f,f' be a compatible. Then*

(*a*) $\lim_n f'(f(x_n)) = f(c)$, *where $c\in\mathbb{R}$ is a point continuous of f.*

(*b*) *If f' is a continuous at c then $f(c)=f'(c)$ and $f(f'(c))=f'(f(c))$.*

Proof. To prove (a): Suppose that $\lim_n f'(x_n)=c\in\mathbb{R}$. Since f is differentiable at $c\in\mathbb{R}$ (hence continuous at c), we have $\lim_n f(f'(x_n))=f(c)$. Then that

$$\lim_n |f'(f(x_n))-f(c)| \le \lim_n |f'(f(x_n))-f(f'(x_n))| + \lim_n |f(f'(x_n))-f(c)| = 0$$

by the compatibility f and f'. This implies $\lim_n f'(f(x_n))=f(c)$, where $c\in\mathbb{R}$ is a point continuous of f.

To prove (b): From (a) that $\lim_n f'(f(x_n))=f(c)$. While, $\lim_n f'(f(x_n))=f'(c)$ by the continuity of f' at c. Thus $f(c)=f'(c)$ by uniqueness of limit. According to the Proposition 3.6, we have $f(f'(c))=f'(f(c))$. This completes the proof.

There are fact interesting that the derivative of a function f satisfies properties (ε,δ)-f-contraction such as the following lemma.

Lemma 3.1. *If $f:\mathbb{R}\longrightarrow\mathbb{R}$ is a differentiable on $\mathbb{R}$, then f' is (ε,δ)-f-contraction .*

Proof. For each ε we choose $\delta(\varepsilon)=\frac{\varepsilon}{4}>0$ such that $\frac{\varepsilon}{4}\le|f(x)-f(y)|<\frac{\varepsilon}{4}+\frac{\varepsilon}{4}$ for each $x\neq y\in\mathbb{R}$. Since f is differentiable (hence continuous) we have

$$\begin{aligned}
|f'(x)-f'(y)| &= \left|\lim_{h\to 0}\frac{1}{h}(f(x+h)-f(x)) - \lim_{h\to 0}\frac{1}{h}(f(y+h)-f(y))\right| \\
&= \lim_{h\to 0}\frac{1}{h}\,|f(x+h)-f(x)-f(y+h)+f(y)| \\
&\le +\lim_{h\to 0}\frac{1}{h}|f(y)-f(x)| + \lim_{h\to 0}\frac{1}{h}|f(x+h)-f(y+h)| \\
&\le |f(x)-f(y)| + (\frac{\varepsilon}{4}+\frac{\varepsilon}{4}) \\
&< 2(\frac{\varepsilon}{4}+\frac{\varepsilon}{4}) = \varepsilon.
\end{aligned}$$

and If $f(x)=f(y)$, then $f'(x)=\lim_{h\to 0}\dfrac{1}{h}(f(x+h)-f(x))=\lim_{h\to 0}\dfrac{1}{h}(f(y+h)-f(y))=f'(y)$. According to Definition 2.1, the function f' is (ε,δ)-f-contraction.

The following proposition is similar to Lemma 2.2 in [6] that applied on the pairs f and f'. This proposition describes the role of compatibility in producing common fixed points.

Proposition 3.8. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable on $\mathbb{R}$ and the pairs of function f and f' is compatible. If there exists $z \in \mathbb{R}$ such that $f(z) = f'(z)$ then the point $x_0 = f(z)$ is the unique common fixed point of f and f'.*

Proof. By Lemma 3.1 for $x \neq y$, $|f'(x) - f'(y)| < |f(x) - f(y)|$. Let $x_0 = f(z)$ and suppose $x_0 \neq f(x_0)$. Since f and $f'$are compatible and $f(z) = f'(z)$ for some $z \in \mathbb{R}$,we have $f(f'(z) = f'(f(z))$ by Proposition 3.6. Then we obtain

$$
\begin{aligned}
|x_0 - f(x_0)| &= |f'(z) - f(f'(z))| = |f'(z) - f'(f(z))| \\
&< |f(z) - f(f(z))| = |x_0 - f(x_0)|.
\end{aligned}
$$

It follow that $x_0 = f(x_0)$. Similarly for $x_0 = f'(z)$ and suppose $x_0 \neq f'(x_0)$, we would obtained $x_0 = f'(x_0)$. Thus x_0 is a common fixed point of f and f'. Furthermore, we show that the point x_0 is unique. If x_1 were another common fixed point of f and f' we would have $|x_0 - x_1| = |f'(x_0) - f'(x_1)| < |f(x_0) - f(x_1)| = |x_0 - x_1|$. Thus $x_0 = x_1$ is unique.

Example 3.4. Let $f(x) = x^2 - x + 1$ for $x \in \mathbb{R}$. Then $f'(x) = 2x - 1$ and there exists the point $z = 1$ or $z = 2$ that are solution of $f(z) = f'(z)$. The point $x_0 = 1$ is the common fixed point f and f' because $f(1) = f'(1) = 1$.

Proposition 3.8 above can be also viewed as a generalization of the result by Park and Bae ([6] Theorem 2) with replacing commutative requirements by compatible. In addition, the Proposition 3.8 does not mention explicitly the terms (ε, δ)-f-contraction because for these requirements automatically by Lemma 3.1.

In next the result the above we would present the other result in below. For these needed the notion the iteration as the following. Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable. For a point $x_0 \in \mathbb{R}$ that be given, we consider a sequence $f(x_n)$ recursively given by rule $f(x_n) = f'(x_{n-1})$ for each $n \in \mathbb{N}$. Such a sequence is called an *iteration-f* of x_0 under f'.

Lemma 3.2 and Lemma 3.3 below just replacing function g by f' in [4] and [6].

Lemma 3.2. (Lemma 3.1 in [4]) *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable on $\mathbb{R}$. If there exists an $x_0 \in \mathbb{R}$ and a sequence $f(x_n)$ is an f-iteration of x_0 under f', then $f(x_n)$ is a Cauchy sequence.*

Lemma 3.3. (Lemma 2.1 in [6]) *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable on $\mathbb{R}$. If there exists an $x_0 \in \mathbb{R}$ and a sequence $f(x_n)$ is an f-iteration of x_0 under f', then a sequence $|f(x_n) - f(x_{n+1})|$, $n \in \mathbb{N}$ is monotone decreasing to* 0.

Now we have one main result that important.

Theorem 3.2. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable on $\mathbb{R}$. and the pairs of function f and f' is compatible. If a sequence $f(x_n)$ is an iteration-f of x_0 under f' with a limit point $p \in \mathbb{R}$, then $f(x_n)$ converges to p and $f(p)$ is the unique common fixed point of f and f'.*

Proof. By Proposition 3.8, it is sufficient to show $f(p) = f'(p)$ for some point $p \in X$. Since a sequence $f(x_n)$ is an iteration-f of x_0 under f', if there exist for some $n \in \mathbb{N}$ such that $f(x_{n+1}) = f(x_n)$, then $f(x_{n+1}) = f'(x_n) = f(x_n)$, and here we are finished.

Now suppose that $f(x_{n+1}) \neq f(x_n)$ for each $n \in \mathbb{N}$ and a sequence $f(x_n)$ is Cauchy sequence by Lemma 3.2. Since $p \in \mathbb{R}$ is limit points of a sequence $f(x_n)$, it converges to p. Since the function f is differentiable on $\mathbb{R}$ (hence f is continuous at p), sequence $f(f(x_n))$ converges to $f(p)$. In addition, a sequence $f'(f(x_{n-1})) = f(f'(x_{n-1})) = f(f(x_n))$ converges to $f(p)$ by compatibility and iteration-f of x_0 under f'. Suppose there exists $m \in \mathbb{N}$ such that

$$f(f(x_m)) = f(f(x_{m+1})) = f(f(x_{m+2})) = ... = f(f(x_{m+k})) = ...$$

Of course the sequence $f(f(x_n))$ converges to $f(f(x_m))$ and $f(f(x_m)) = f(f(x_{m+1})) = f(f'(x_m)) = f'(f(x_m))$ by compatibility and iteration-f of x_0 under f'. Hence $f(f(x_m)) = f(p)$ and $f(x_m) = p$ is a coincidence point of f and f' (i.e. $f(p) = f'(p)$). Thus we are done.

If there is no an index $m \in \mathbb{N}$ that satisfies $f(f(x_m)) = f(f(x_{m+1})) = f(f(x_{m+2}))...$, then for $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for each $m \geq N$,

$$|f(f(x_m)) - f(p)| < \frac{\varepsilon}{4}.$$

Suppose that there exists $n \geq N$ such that $f(f(x_n)) \neq f(p)$, we have

$$\begin{aligned} |f(p) - f'(p)| &\leq |f(p) - f(f'(x_n))| + |f(f'(x_n)) - f'(p)| \\ &= |f(p) - f(f(x_{n+1}))| + |f'(f(x_n)) - f'(p)| \\ &< \frac{\varepsilon}{4} + |f(f(x_n)) - f(p)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon. \end{aligned}$$

Therefore $f(p) = f'(p)$, and this completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

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