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COMMON FIXED POINTS FOR WEAK CONTRACTION OCCASIONALLY WEAKLY BIASED MAPPINGS

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Abstract. We discuss some common fixed point theorems for weakly contractive occasionally weakly biased mappings on metric spaces with illustrative examples.

Keywords: compatible maps; weakly compatible mappings; occasionally weakly compatible mappings; occasionally weakly biased; coincidence points and common fixed point.

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1. Introduction and Preliminaries

Let (X,d) be a metric space. A mapping $f : X \to X$, is called contraction if for each $x, y \in X$, there exists a constant $k \in [0,1)$ such that

$$d(fx, fy) \le kd(x,y) \tag{1.1}$$

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Alber and Guerre- Delabriere[2] defined the concept of weakly contractive mapping on Hilbert spaces and proved the existence of fixed points. Rhoades [11] showed that most results of Alber and Guerre-Delabriere[2] are still true for any Banach space. Note that in Alber and Gurre–Delabriere[2], φ is assumed with an additional condition $\lim_{t\to\infty}\varphi(t)=\infty$. However, Rhoades [11] obtained the result without using this additional condition. Following Rhoades [11], a mapping $f:(X,d)\to(X,d)$ is called a weakly contractive, if for each $x,y\in X$

$$d(fx,fy)\le d(x,y)-\varphi(d(x,y)) \tag{1.2}$$

where $\varphi:[0,\infty)\to[0,\infty)$ is continuous, non-decreasing and positive on $(0,\infty)$ with $\varphi(0)=0$.

Let $f,g:(X,d)\to(X,d)$ be two mappings, then the mapping f is called g-weakly contractive[15] if for each $x,y\in X$

$$d(fx,fy)\le d(gx,gy)-\varphi(d(gx,gy)), \tag{1.3}$$

where $\varphi:[0,\infty)\to[0,\infty)$ is a lower semi-continuous function from right such that φ is positive on $(0,\infty)$ and $\varphi(0)=0$. If $g=I$, an identical operator, then f is reduced to weak contraction.

Further, if $g=I$ and $\varphi(t)=(1-k)t$ where $k\in(0,1)$, then g-weakly contractive is reduced to inequality(1.1). If $\psi(t)=t-\varphi(t)$ and $g=I$, then $\psi(t)$ is upper semi-continuous from right and inequality (1.3) reduces into contractive types of Boyd and Wong [4]. Thus

$$d(fx,fy)\le\psi(d(x,y)) \tag{1.4}$$

Further more, if $k(t)=1-\frac{\varphi(t)}{t}$ for $t>0$ and $k(0)=0$ together with $g=I$, then inequality(1.3) is closely related to Reich type[10]. In fact, the classes of weak contractive are closely related to Boyd and Wong [4], and Riech[10] types (see also [16],[15]).

We denote $C(f,g)=\{x\in X: fx=gx\}$ and $F(f,g)=\{x\in X: fx=gx=x\}$.

In the sequel we need the following definitions.

Definition 1.1[13]**.** Mappings f and g are called weakly commuting if $d(fgx,gfx)\le d(fx,gx)$, for all $x\in X$.

Definition 1.2[1]**(also see Sastry and Murthy**[12]**).** Mappings f and g are called said to satisfy property (E.A) if there exists a sequences $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_x = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Definition 1.3[9]**.** Mappings f and g are called weakly compatible if $fgx = gfx$ for all $x \in C(f,g)$.

Definition 1.4[8]**.** Mappings f and g are called weakly g-biased if $d(gfx,gx) \leq d(fgx,fx)$ for all $x \in C(f,g)$.

If the role of f and g are interchanged in above definition, then the mappings are called weakly f-biased. Note that weakly compatible mappings implies weakly biased mappings (i.e. both f- and g-biased) but the converse is not true in general[14].

Definition 1.5[3]**.** Mappings f and g are called occasionally weakly compatible(owc) if $fgx = gfx$ for some $x \in C(f,g)$.

From above definitions, one may agree that weakly compatible mappings pair implies *owc* but the converse may not be true in general (also see [3]).

Definition 1.6[5]**.** Mappings f and g are called occasionally weakly g-biased if $d(gfx,gx) \leq d(fgx,fx)$ for some $x \in C(f,g)$.

If the role of mappings are interchanged, then the mappings pair is called occasionally weakly f-biased. Further, it may be noted that the notions of *owc* and weakly g-biased mappings are occasionally weakly g-biased but the converse does not hold in general(see [5]).

Example 1.7. Let $X = [0,1] \subset \mathbb{R}$ with usual metric d. Define $f,g : X \to X$ by $fx = \frac{1}{3}+x$, $gx = \frac{1}{2}$, for $x < \frac{1}{2}$, $f\frac{1}{2} = \frac{2}{3} = g\frac{1}{2}$, $fx = 1$, $gx = 1-x$, for $x > \frac{1}{2}$. Here, $C(f,g) = \{\frac{1}{6}, \frac{1}{2}\}$. Also, we have $f\frac{1}{6} = \frac{1}{2} = g\frac{1}{6}$, $f\frac{1}{2} = \frac{2}{3} = g\frac{1}{2}$ and $fg\frac{1}{6} = \frac{2}{3} = gf\frac{1}{6}$, but $fg\frac{1}{2} = 1 \neq gf\frac{1}{2} = \frac{1}{3}$. The mappings pair (f,g) is occasionally weakly compatible but not weakly compatible. However, the mappings are weakly biased and hence occasionally weakly biased.

Example 1.8. Let $X = [0,1] \subset \mathbb{R}$ with usual metric d. Define $f,g : X \to X$ by $fx = 1$, $gx = \frac{1}{2}$, for $x < \frac{1}{2}$, $f\frac{1}{2} = 0 = g\frac{1}{2}$, $fx = x$, $gx = 1-x$, for $x > \frac{1}{2}$. Here, $C(f,g) = \{\frac{1}{2}\}$. Also, we have $f\frac{1}{2} = 0 = g\frac{1}{2}$ and $|gf\frac{1}{2} - g\frac{1}{2}| = |\frac{1}{2} - 0| = \frac{1}{2} \leq |fg\frac{1}{2} - f\frac{1}{2}| = |1-0| = 1$. The mappings pair

(f,g) is weakly biased and hence occasionally weakly g-biased but neither weakly compatible nor owc.

Example 1.9. Let $X=[0,1]\subset\mathbb{R}$ with usual metric d. Define $f,g:X\to X$ by $fx=2x$, $gx=1-2x$, for $x\leq\frac{1}{4}$, $fx=1, gx=\frac{1}{4}$, for $\frac{1}{4}<x\leq\frac{1}{2}$, $fx=\frac{7}{8}$, $gx=\frac{1+8x}{8}$, for $\frac{1}{2}<x\leq\frac{3}{4}$, $fx=\frac{1}{6}, gx=\frac{3}{4}$, for $\frac{3}{4}<x\leq 1$. Here, $C(f,g)=\{\frac{1}{4},\frac{3}{4}\}$. Also $f\frac{1}{4}=\frac{1}{2}=g\frac{1}{4}$ and $f\frac{3}{4}=\frac{7}{8}=g\frac{3}{4}$ implies that

$$|gf\frac{1}{4}-g\frac{1}{4}|=\frac{1}{4}\leq|fg\frac{1}{4}-f\frac{1}{4}|=\frac{1}{2}$$

and

$$|gf\frac{3}{4}-g\frac{3}{4}|=\frac{1}{8}\not\leq|fg\frac{3}{4}-f\frac{3}{4}|=\frac{17}{24}$$

Therefore, the pair (f,g) is occasionally weakly g-biased, but it is neither weakly g-biased nor weakly compatible(resp. *owc*)

In this paper, we prove some common fixed point theorems for weak contraction occasionally weakly biased mappings pair on metric spaces.

2. Main Results

Song[15] proved the following theorem.

Theorem 1.1 (Song[15]). Let (X,d) be a metric space and $f,g:X\to X$ two self mappings with $\overline{fX}\subset gX$. Assume that either $\overline{fX}$ or gX is complete, and f is g-weakly contractive mapping, then $C(f,g)\neq\phi$. If in addition, (f,g) is weakly compatible, then $F(f,g)$ is singleton.

Let $\varphi:[0,\infty)\to[0,\infty)$ be a lower semi-continuous function with $\varphi(t)=0$ if and only it $t=0$. Let f and g be two self mappings on a metric space (X,d). We denote

$$M(x,y)=max\left\{d(gx,gy),d(fx,gy),d(fy,gx),\frac{1}{2}[d(fx,gx)+d(fy,gy)]\right\} \tag{2.1}$$

and

$$N(x,y)=max\left\{d(gx,fy),d(fx,fy),d(gx,gy),\frac{1}{2}[d(fx,gx)+d(fy,gy)]\right\} \tag{2.2}$$

Theorem 2.2. Let f and g be two self mappings of a metric space (X,d) satisfying the following inequality

$$d(fx,fy) \leq M(x,y) - \varphi\Big(M(x,y)\Big), \forall x,y \in X \tag{2.3}$$

If (f,g) satisfies property-(E.A) and gX is closed in X, then $C(f,g) \neq \phi$. Further, if (f,g) is occasinally weakly g-biased, then $F(f,g)$ is singleton.

Proof. Since f and g satisfy property (E.A), there exists a sequence in X such that $fx_n, gx_n \to t$ for some $t \in X$. As gX is closed and $t \in X$, there exists $u \in X$ such that $t = gu$. We claim that $fu = gu$. By (2.1) and (2.3), we obtain

$$d(fx_n, fu) \leq M(x_n,u) - \varphi\Big(M(x_n,u)\Big)$$

and

$$M(x_n,u) = max\left\{d(gx_n,gu), d(fx_n,gu), d(fu,gx_n), \frac{1}{2}\left[d(fx_n,gx_n) + d(fu,gu)\right]\right\}$$

On letting $n \to \infty$, we obtain

$$\begin{aligned} d(gu,fu) &\leq max\left\{0,0,d(fu,gu),\frac{1}{2}d(fu,gu)\right\} \\ &\qquad - \varphi\left(max\left\{0,0,d(fu,gu),\frac{1}{2}d(fu,gu)\right\}\right) \\ &= d(fu,gu) - \varphi(d(fu,gu)) \end{aligned}$$

which gives $fu = gu$. Therefore, $C(f,g) \neq \phi$. Since (f,g) is occasionally weakly g-biased mappings, then $fu = gu$ for some $u \in C(f,g)$ and

$$d(gfu,gu) \leq d(fgu.fu). \tag{2.4}$$

Also, $fu = gu$ yields $ffu = fgu$ and $gfu = ggu$. Now we show that $ffu = fu$, otherwise by (2.1), (2.3) and (2.4), we obtain

$$\begin{aligned}
d(ffu, fu) &\le M(fu,u) - \varphi\Big(M(fu,u)\Big) \\
&= max\left\{d(gfu,gu), d(ffu,gu), \frac{1}{2}[d(ffu,gfu)+d(fu,gu)]\right\} \\
&\quad - \varphi\left(max\left\{d(gfu,gu), d(ffu,gu), \frac{1}{2}[d(ffu,gfu)+d(fu,gu)]\right\}\right) \\
&\le d(ffu,fu) - \varphi(d(ffu,fu))
\end{aligned}$$

which gives $\varphi(d(ffu, fu)) = 0 \Rightarrow ffu = fu$. By occasionally weakly g-biased of f and g, we obtain

$$d(gfu,gu) \le d(fgu,fu) = d(ffu,fu) = 0,$$

which in turn gives $gfu = fu$. Therefore, $fu = z$ is a common fixed point of f and g. For the uniqueness, let $z \ne z' \in X$ such that $fz = gz = z$ and $fz' = gz' = z'$, then by (2.1) and (2.3), we obtain

$$\begin{aligned}
d(z,z') &= d(fz, fz') \\
&\le M(z,z') - \varphi\Big(M(z,z')\Big) \\
&= d(z,z') - \varphi(d(z,z'))
\end{aligned}$$

which yields $\varphi(d(d,z')) = 0$ and $z = z'$. This completes the proof.

The following example illustrate the validity of above theorem.

Example 2.3. Let $X = [0,1) \subset \mathbb{R}$ with usual metric $d(x,y) = |x-y|$. Define $f, g : X \to X$ by $fx = \frac{1}{2}$, for $0 \le x \le \frac{1}{2}$ $fx = \frac{1}{4}$, for $\frac{1}{2} < x < 1$ and $gx = \frac{1}{2}(1+x)$, for $0 \le x < \frac{1}{2}$, $g\frac{1}{2} = \frac{1}{2}$, $gx = \frac{3}{4}$, for $\frac{1}{2} < x < 1$. Here, $fX = \{\frac{1}{4}, \frac{1}{2}\}$ is not contained in $gX = [\frac{1}{2}, \frac{3}{4}]$, and gX is closed in X. Mappings f and g satisfy property (E.A), to verify this, let $\{x_n\}$ be a sequence in X, $x_n > 0$, $n = 1,2,3,...$ such that $x_n \to 0$ as $n \to \infty$ then $fx_n, gx_n \to \frac{1}{2} \in X$. One can also verify that (f,g) satisfies inequality (2.3) for every $x, y \in X$ taking with $\varphi(t) = \frac{t}{2}$. Also, $C(f,g) = \{0, \frac{1}{2}\}$ and $f0 = \frac{1}{2} = g0$ which implies f and g are occasionally weakly g-biased mappings. Thus, all the conditions of the theorem are satisfied and $\frac{1}{2}$ is the unique common point.

Corollary 2.4 Let f and g be two self mappings of a metric space (X,d) satisfying the following: for every $x,y \in X$,

$$d(fx,fy) \leq \psi(M(x,y)) \tag{2.5}$$

where $\psi : [0,\infty) \to [0,\infty)$ is a function such that $0 < \psi(t) < t$ for $t > 0$ and $\psi(0) = 0$. If (f,g) satisfies the property (E.A) and gX is closed in X, then $C(f,g) \neq \phi$. Further, if (f,g) is occasionally weakly g-biased, then $F(f,g)$ is singleton.

Proof. Letting $\varphi(t) = t - \psi(t)$, then $0 < \psi(t) = t - \varphi(t) < t$ for $t > 0$ (by definition of ψ) and inequality (2.5) implies that

$$d(fx,fy) \leq M(x,y) - \varphi\Big(M(x,y)\Big)$$

Therefore, the result follows from Theorem 2.2.

Corollary 2.5 Let f and g be two self mappings of a metric space (X,d) such that for every $x,y \in X$

$$d(fx,fy) \leq \alpha\Big(M(x,y)\Big)M(x,y) \tag{2.6}$$

where $\alpha : [0,\infty) \to [0,1)$ is a function. If (f,g) satisfies the property (E.A) and gX is closed in X, then $C(f,g) \neq \phi$. Further, if (f,g) is occasionally weakly g-biased, then $F(f,g)$ is singleton.

Proof. Setting $\varphi(t) = [1-\alpha(t)]t$, then equation (2.6) implies that

$$d(fx,fy) \leq M(x,y) - \varphi(M(x,y))$$

The result follows from Theorem 2.2.

Theorem 2.6. Let f and g be two self mappings of a metric space (X,d) satisfying

$$d(fx,gy) \leq N(x,y) - \varphi\Big(N(x,y)\Big), \forall x,y \in X \tag{2.7}$$

If (f,g) satisfies the property-(E.A) and fX is closed in X, then $C(f,g) \neq \phi$. Further, if (f,g) is occasionally weakly g-biased, then $F(f,g)$ is singleton.

Proof. Since f and g satisfy property-(E.A), there exists a sequence $\{x_n\}$ in X such that $fx_n, gx_n \to t$ for some $t \in X$. As fX is closed and $t \in X$, there exists $u \in X$ such that $t = fu$. We claim that $fu = gu$. By (2.2) and (2.7), we obtain

$$d(fx_n, gu) \leq N(x_n, u) - \varphi\left(N(x_n, u)\right)$$

and

$$N(x_n, u) = max\left\{d(gx_n, fu), d(fx_n, fu), d(gx_n, gu), \frac{1}{2}[d(fx_n, gx_n) + d(fu, gu)]\right\}$$

On letting $n \to \infty$, we obtain

$$\begin{aligned} d(gu, fu) &\leq max\left\{0, 0, d(fu, gu), \frac{1}{2}d(fu, gu)\right\} \\ &\qquad - \varphi\left(max\left\{0, 0, d(fu, gu), \frac{1}{2}d(fu, gu)\right\}\right) \\ &= d(fu, gu) - \varphi\left(d(fu, gu)\right) \end{aligned}$$

which gives $fu = gu$. Therefore, $C(f, g) \neq \phi$.

Since (f, g) is occasionally weakly g-biased mappings, then $fu = gu$ for some $u \in C(f, g)$ and

$$d(gfu, gu) \leq d(fgu.fu). \tag{2.8}$$

Also, $fu = gu$ yields $ffu = fgu$ and $gfu = ggu$. Now we show that $ffu = fu$, otherwise by (2.7), (2.2)and (2.8), we obtain

$$\begin{aligned} d(ffu, fu) &= d(ffu, gu) \\ &\leq N(fu, u) - \varphi\Big(N(fu, u)\Big) \\ &= max\left\{d(gfu, fu), d(ffu, fu), \frac{1}{2}d(ffu, gfu)\right\} \\ &\quad - \varphi\left(max\left\{d(gfu, fu), d(ffu, fu), \frac{1}{2}d(ffu, gfu)\right\}\right) \\ &\leq max\left\{d(ffu, fu), d(ffu, fu), d(ffu, fu)\right\} \\ &\qquad - \varphi\left(max\left\{d(ffu, fu), d(ffu, fu), d(ffu, fu)\right\}\right) \\ &= d(ffu, fu) - \varphi(d(ffu, fu)) \end{aligned}$$

which gives $\varphi(d(ffu, fu)) = 0 \Rightarrow ffu = fu$.

By occasionally weakly g-biased of f and g, we obtain

$$d(gfu, gu) \leq d(fgu, fu) = d(ffu, fu) = 0,$$

which in turn gives $gfu = fu$. Therefore, $fu = z$ is a common fixed point of f and g. For the uniqueness, let $z \neq z' \in X$ such that $fz = gz = z$ and $fz' = gz' = z'$, then by (2.2) and (2.7), we obtain

$$\begin{aligned} d(z, z') &= d(fz, gz') \\ &\leq N(z, z') - \varphi(N(z, z')) \\ &= d(z, z') - \varphi(d(z, z')) \end{aligned}$$

which yields $\varphi(d(d, z')) = 0$ and $z = z'$. This completes the proof.

The validity of above theorem is illustrated by the following example.

Example 2.7. Let $X = [0, 1) \subset \mathbb{R}$ with usual metric d. Define $f, g : X \to X$ by $fx = \frac{1}{2}$, for $0 \leq x \leq \frac{1}{2}$, $fx = 0$, for $x > \frac{1}{2}$ and $gx = \frac{1}{2}(1 + x)$, for $0 \leq x < \frac{1}{2}$, $g\frac{1}{2} = \frac{1}{2}$, $gx = \frac{3}{5}$, for $x > \frac{1}{2}$. Here, $fX = \{0, \frac{1}{2}\}$ is not contained in $gX = [\frac{1}{2}, \frac{3}{4})$, and fX is closed in X. Mappings f and g satisfy property (E.A), to verify this, let $\{x_n\}$ be a sequence in X, $x_n > 0$, $n = 1, 2, 3, ...$ such that $x_n \to 0$ as $n \to \infty$ then $fx_n, gx_n \to \frac{1}{2} \in X$. One can also verify that f and g satisfy inequality(2.7) for every $x, y \in X$ taking with $\varphi(t) = \frac{t}{2}$. Also, $C(f, g) = \{0, \frac{1}{2}\}$ and $f0 = \frac{1}{2} = g0$ implies f and g are occasionally weakly g-biased mappings. Thus, all the conditions of the theorem are satisfied and $f0 = \frac{1}{2}$ is the unique common point.

Corollary 2.8. Let f and g be two self mappings of a metric space (X, d) satisfying

$$d(fx, gy) \leq N(x, y) - \varphi\Big(N(x, y)\Big), \forall x, y \in X$$

If (f, g) satisfies the property-(E.A) and fX is closed in X, then $C(f, g) \neq \phi$. Further, if (f, g) is occasionally weakly compatible, then $F(f, g)$ is singleton.

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