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UNIFYING A MULTITUDE OF COMMON FIXED POINT THEOREMS WITH COMPLEX COEFFICIENTS EMPLOYING AN IMPLICIT RELATION

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Abstract. The concept of complex valued metric spaces was introduced and studied by Azam et al. (cf. [1]). Afterward, by mathematicians many fixed point theorems for mappings that satisfying in a rational inequality with real coefficients were founded in complex valued metric spaces. In this paper, we first, introduce implicit function with complex coefficient. Second, we establish common fixed point theorems involving two pairs of weakly compatible mapping satisfying certain rational expressions with complex coefficients are proved in complex valued metric space. Some related results are also derived besides furnishing illustrative examples to highlight the realized improvements. The presented theorems generalize, extend and improve many existing results in the literature.

Keywords: common fixed point; weakly compatible mapping; contractive type mapping; complex coefficient; complex valued metric space.

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1. Introduction and Preliminaries

The axiomatic development of a metric space was essentially carried out by French mathematician M.Frechet in the year 1906. The utility of metric spaces in the natural growth of

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Functional Analysis is enormous. Inspired from the impact of this natural idea to mathematics in general and to Functional Analysis in particular, several researchers attempted various generalizations of this notion in the recent past such as: rectangular metric spaces, semi metric spaces, quasi metric spaces, quasi semi metric spaces, pseudo metric spaces, probabilistic metric spaces, 2-metric spaces, D-metric spaces, G-metric spaces, K-metric spaces, Cone metric spaces etc and by now there exists considerable literature on all these generalizations of metric spaces.

Most recently, Azam et al. (cf. [1]) and latter Fayyaz et al. (cf. [3]) studied complex valued metric spaces wherein some fixed point theorems for mappings satisfying a rational inequality were established. Naturally, this new idea can be utilized to define complex valued normed spaces and complex valued inner product spaces which ,in turn, offer a lot of scope for further investigation. Though complex valued metric spaces form a special class of cone metric space, yet this idea is intended to define rational expressions which are not meaningful in cone metric spaces and thus many results of analysis cannot be generalized to cone metric spaces. Indeed the definition of a cone metric space banks on the underlying Banach space which is not a division Ring. However, in complex valued metric spaces, we can study improvements of a host of results of analysis involving divisions.

In this paper we prove common fixed point theorems involving two pairs of weakly compatible mappings satisfying certain rational expressions with complex coefficients in complex valued metric space.

In what follows, we recall some notations and definitions that will be utilized in our subsequent discussion.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2).$$

Consequently, one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2), Im(z_1) < Im(z_2)$,

(ii) $Re(z_1) < Re(z_2), Im(z_1) = Im(z_2)$,

(iii) $Re(z_1) < Re(z_2), Im(z_1) < Im(z_2)$,

(iv) $Re(z_1) = Re(z_2), Im(z_1) = Im(z_2)$.

In particular, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we write $z_1 \prec z_2$ if only (iii) is satisfied. Notice that $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$, and $z_1 \precsim z_2, z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

We denote $\mathbb{C}_+$ and $\mathbb{R}^+$ by

$$\mathbb{C}_+ = \{z \in \mathbb{C} : 0 \precsim z\} \text{ and } \mathbb{R}^+ = \{r \in \mathbb{R} : r \geq 0\}$$

Definition 1.1. (cf. [3]) Let X be a nonempty set whereas $\mathbb{C}$ be the set of complex numbers. Suppose that the mapping $d : X \times X \to \mathbb{C}$, satisfies the following conditions:

(d_1). $0 \precsim d(x,y)$, for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(d_2). $d(x,y) = d(y,x)$ for all $x, y \in X$;

(d_3). $d(x,y) \precsim d(x,z) + d(z,y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X, and (X,d) is called a complex valued metric space.

Example 1.1. (cf. [3]) Let $X = \mathbb{C}$ be a set of complex numbers. Define $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by

$$d(z_1, z_2) = |x_1 - x_2| + i|y_1 - y_2|$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then (X,d) is a complex valued metric space.

Definition 1.2. Let (X,d) be a complex valued metric space and $B \subseteq X$.

(i) $b \in B$ is called an interior point of a set B whenever there is $0 \prec r \in \mathbb{C}$ such that

$$N(b,r) \subseteq B$$

where $N(b,r) = \{y \in X : d(b,y) \prec r\}$.

(ii) A point $x \in X$ is called a limit point of B whenever for every $0 \prec r \in \mathbb{C}$,

$$N(x,r) \cap (B \backslash X) \neq \emptyset.$$

(iii) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A. A subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B. The family

$$F = \{N(x,r) : x \in X, 0 \prec r\}$$

is a sub-basis for a topology on X. We denote this complex topology by τ_c. Indeed, the topology τ_c is Hausdorff.

Definition 1.3. Let (X,d) be a complex valued metric space and $\{x_n\}_{n\geq 1}$ be a sequence in X and $x \in X$. We say that

(i) the sequence $\{x_n\}_{n\geq 1}$ converges to x if for every $c \in \mathbb{C}$, with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n,x) \prec c$. We denote this by $\lim_n x_n = x$, or $x_n \to x$, as $n \to \infty$,

(ii) the sequence $\{x_n\}_{n\geq 1}$ is Cauchy sequence if for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n,x_{n+m}) \prec c$,

(iii) the complex valued metric space (X,d) is a complete complex valued metric space if every Cauchy sequence is convergent.

Definition 1.4. (cf. [5]) Two families of self-mappings $\{T_i\}_{i=1}^m$ and $\{S_i\}_{i=1}^n$ are said to be pairwise commuting if:
(i)$T_iT_j = T_jT_i$, $i,j \in \{1,2,...m\}$,
(ii)$S_iS_j = S_jS_i$, $i,j \in \{1,2,...n\}$.
(iii)$T_iS_j = S_jT_i$, $i \in \{1,2,...m\}$, $j \in \{1,2,...n\}$.

Definition 1.5. Let $S : \mathbb{C} \to \mathbb{C}$ be a given mapping. We say that S is a non-decreasing mapping with respect $\precsim$ if for every $x,y \in \mathbb{C}, x \precsim y$ implies $Sx \precsim Sy$.

Definition 1.6. Let $S : \mathbb{C} \to \mathbb{C}$ be a given mapping. We say that S is a non-increasing mapping with respect $\precsim$ if for every $x,y \in \mathbb{C}, x \precsim y$ implies $Sy \precsim Sx$.

Definition 1.7.(cf. [6]) Let S and I be self-maps of a set X (i.e., $S,I : X \to X$). If $w = Sx = Ix$ for some $x \in X$, then x is called a coincidence point of S and I, and w is called a point of coincidence of S and I.

Definition 1.8. (cf. [4, 2]) Let S and T be two self-maps defined on set X. Then S and T are said to be weakly compatible if they commute at every coincidence point.

In [1], Azam et al. established the following two lemmas:

Lemma 1.1. (cf. [1]) Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.

Lemma 1.2. (cf. [1]) Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n,x_{n+m})| \to 0$ as $n,m \to \infty$.

2. Implicit Relation with Complex Coefficient

We consider set Φ of functions $\phi(u_1,u_2,u_3,u_4,u_5,u_6) : \mathbb{C}_+^6 \to \mathbb{C}$ satisfying the following properties:

Φ_1: ϕ is continuous;

Φ_2: ϕ is non-increasing with respect to the 5th and 6th variables;

Φ_3: there is $h_1 \in \mathbb{C}$ and $h_2 \in \mathbb{C}$ such that $h = |h_1h_2| < 1$ and if $u,v \in \mathbb{C}_+$ satisfy $\phi(u,v,v,u,u+v,0) \precsim 0$ then $|u| \leq |h_1v|$ and if $u,v \in \mathbb{C}_+$ satisfy $\phi(u,v,u,v,0,u+v) \precsim 0$ then $|u| \leq |h_2v|$;

Φ_4: if $u \in \mathbb{C}_+$ is such that $\phi(u,u,0,0,u,u) \precsim 0$ or $\phi(u,0,u,0,0,u) \precsim 0$ or $\phi(u,0,0,u,u,0) \precsim 0$, then $u = 0$.

Example 2.1. Define $\phi(u_1,u_2,u_3,u_4,u_5,u_6) : \mathbb{C}_+^6 \to \mathbb{C}$ as

$$\phi(u_1,u_2,u_3,u_4,u_5,u_6) = \lambda u_1 - (Au_2 + Bu_3 + Cu_4 + Du_5 + Eu_6)$$

where $D = E = \frac{1}{10}$, $\lambda = 1+i$ and $A = B = C = \frac{1}{5} + \frac{1}{6}i$.

Example 2.2. Define $\phi(u_1,u_2,u_3,u_4,u_5,u_6) : \mathbb{C}_+^6 \to \mathbb{C}$ as

$$\phi(u_1,u_2,u_3,u_4,u_5,u_6) = \lambda u_1 - Au_2 - B(u_3+u_4) - C(u_5+u_6),$$

where $C = \frac{1}{10}$, $\lambda = 2+i$ and $A = B = \frac{1}{5} + \frac{1}{3}i$.

Example 2.3. Define $\phi(u_1,u_2,u_3,u_4,u_5,u_6) : \mathbb{C}_+^6 \to \mathbb{C}$ as

$$\phi(u_1,u_2,u_3,u_4,u_5,u_6) = \lambda u_1 - A\frac{u_2u_3}{1+u_3+u_4} - B(u_5+u_6-u_2)$$

where $B \in \mathbb{R}^+, A, \lambda \in \mathbb{C}_+$ and $A + 2B \prec \lambda$.

Φ_1 and Φ_2 :Obviously;

Φ_3 :Denote

$$h_1 = h_2 = \frac{A}{\lambda - B}.$$

Since $0 \prec A + 2B \prec \lambda$, then

$$0 \prec A \prec \lambda - B \Rightarrow |A| < |\lambda - B| \Rightarrow |\frac{A}{\lambda - B}| < 1,$$

therefore, we have $h = |h_1 h_2| < 1$.

If $\phi(u, v, v, u, u+v, 0) \precsim 0$, we have

$$\lambda u - A\frac{vv}{1+v+u} - B(u+v-v) \precsim 0,$$
$$(\lambda - B)u \precsim A\frac{vv}{1+v+u},$$
$$|\lambda - B||u| \leq |A|\frac{|v||v|}{|1+v+u|} \leq |A|\frac{|v||v|}{|v|},$$

which implies that $|u| \leq |h_1 v|$.

Now, if $\phi(u, v, u, v, 0, u+v) \precsim 0$, we have

$$\lambda u - A\frac{vu}{1+v+u} - B(u+v-v) \precsim 0,$$
$$(\lambda - B)u \precsim A\frac{vu}{1+v+u},$$
$$|\lambda - B||u| \leq |A|\frac{|v||u|}{|1+v+u|} \leq |A|\frac{|v||u|}{|u|},$$

which implies that $|u| \leq |h_2 v|$.

Φ_4 :Suppose that $\phi(u, u, 0, 0, u, u) \precsim 0$. We get

$$\lambda u \precsim Bu \Rightarrow |\lambda||u| \leq |B||u|,$$

on the other hand, since $|B| < |\lambda|$ and $|B||u| \leq |\lambda||u|$ then $|u| = 0$ and $u = 0$. The same result holds if $\phi(u, 0, u, 0, 0, u) \precsim 0$ or $\phi(u, 0, 0, u, u, 0) \precsim 0$. Therefore, $\phi \in \Phi$.

Example 2.4. Define $\phi(u_1, u_2, u_3, u_4, u_5, u_6) : \mathbb{C}_+^6 \to \mathbb{C}$ as

$$\phi(u_1, u_2, u_3, u_4, u_5, u_6) = \lambda u_1 - A\frac{u_2 u_3}{1 + u_2 + u_3 + u_4} - B(u_5 + u_6 - u_2)$$

where $B \in \mathbb{R}^+, A, \lambda \in \mathbb{C}_+$ and $A + 2B \precsim \lambda$.

Example 2.5. Define $\phi(u_1,u_2,u_3,u_4,u_5,u_6):\mathbb{C}_+^6\to\mathbb{C}$ as

$$\phi(u_1,u_2,u_3,u_4,u_5,u_6)=\lambda u_1-A\frac{u_3u_4}{1+u_3+u_4}-B(u_5+u_6-u_2)$$

where $B\in\mathbb{R}^+, A,\lambda\in\mathbb{C}_+$ and $A+2B\precsim\lambda$.

Example 2.6. Define $\phi(u_1,u_2,u_3,u_4,u_5,u_6):\mathbb{C}_+^6\to\mathbb{C}$ as

$$\phi(u_1,u_2,u_3,u_4,u_5,u_6)=\lambda u_1-A\frac{u_3u_4}{1+u_2+u_3+u_4}-B(u_5+u_6-u_2)$$

where $B\in\mathbb{R}^+, A,\lambda\in\mathbb{C}_+$ and $A+2B\precsim\lambda$.

Equating B to zero in Examples 2.3, 2.4, 2.5 and 2.6, in the particular case, we deduce the following example.

Example 2.7. Define $\phi(u_1,u_2,u_3,u_4,u_5,u_6):\mathbb{C}_+^6\to\mathbb{C}$ as

$$\phi(u_1,u_2,u_3,u_4,u_5,u_6)=\lambda u_1-A\frac{u_2u_3}{1+u_3+u_4}$$

where $A,\lambda\in\mathbb{C}_+$ and $A\precsim\lambda$.

Example 2.8. Define $\phi(u_1,u_2,u_3,u_4,u_5,u_6):\mathbb{C}_+^6\to\mathbb{C}$ as

$$\phi(u_1,u_2,u_3,u_4,u_5,u_6)=\lambda u_1-A\frac{u_2u_3}{1+u_2+u_3+u_4}$$

where $A,\lambda\in\mathbb{C}_+$ and $A\precsim\lambda$.

Example 2.9. Define $\phi(u_1,u_2,u_3,u_4,u_5,u_6):\mathbb{C}_+^6\to\mathbb{C}$ as

$$\phi(u_1,u_2,u_3,u_4,u_5,u_6)=\lambda u_1-A\frac{u_3u_4}{1+u_3+u_4}$$

where $A,\lambda\in\mathbb{C}_+$ and $A\precsim\lambda$.

Example 2.10. Define $\phi(u_1,u_2,u_3,u_4,u_5,u_6):\mathbb{C}_+^6\to\mathbb{C}$ as

$$\phi(u_1,u_2,u_3,u_4,u_5,u_6)=\lambda u_1-A\frac{u_3u_4}{1+u_2+u_3+u_4}$$

where $A,\lambda\in\mathbb{C}_+$ and $A\precsim\lambda$.

3. Main Results

We prove our main result as follows:

Theorem 3.1. If S,T,I and J are self-mappings defined on a complex valued metric space (X,d) satisfying $TX \subseteq IX, SX \subseteq JX$ and

$$\phi(d(Sx,Ty),d(Ix,Jy),d(Ix,Sx),d(Jy,Ty),d(Ix,Ty),d(Sx,Jy)) \precsim 0 \tag{1}$$

for all $x,y \in X$ where $\phi \in \Phi$. If one of SX,TX,IX or JX is a complete subspace of X, then:

(a) $\{S,I\}$ and $\{T,J\}$ have a unique point of coincidence in X,

(b) if $\{S,I\}$ and $\{T,J\}$ are weakly compatible, then S,T,I and J have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since $SX \subseteq JX$, we find a point x_1 in X such that $Sx_0 = Jx_1$. Also, since $TX \subseteq IX$, we choose a point x_2 with $Tx_1 = Ix_2$. Thus in general for the point x_{2n-2} one find a point x_{2n-1} such that $Sx_{2n-2} = Jx_{2n-1}$ and then a point x_{2n} with $Tx_{2n-1} = Ix_{2n}$ for $n = 1,2,\cdots$. Repeating such arguments one can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that,

$$y_{2n-1} = Sx_{2n-2} = Jx_{2n-1},\ y_{2n} = Tx_{2n-1} = Ix_{2n},\ n = 1,2,\cdots.$$

Using inequality (1), we have

$$\phi(d(Sx_{2n},Tx_{2n+1}),d(Ix_{2n},Jx_{2n+1}),d(Ix_{2n},Sx_{2n}),d(Jx_{2n+1},Tx_{2n+1}),$$
$$d(Ix_{2n},Tx_{2n+1}),d(Sx_{2n},Jx_{2n+1})) \precsim 0,$$

and

$$\phi\big(d(y_{2n+1},y_{2n+2}),d(y_{2n},y_{2n+1}),d(y_{2n},y_{2n+1}),d(y_{2n+1},y_{2n+2}),$$
$$d(y_{2n},y_{2n+2}),d(y_{2n+1},y_{2n+1})\big) \precsim 0$$

or

$$\phi\big(d(y_{2n+1},y_{2n+2}),d(y_{2n},y_{2n+1}),d(y_{2n},y_{2n+1}),d(y_{2n+1},y_{2n+2}),$$
$$d(y_{2n},y_{2n+2}),0\big) \precsim 0.$$

In view of (Φ_2), we have,

$$\phi\big(d(y_{2n+1},y_{2n+2}),d(y_{2n},y_{2n+1}),d(y_{2n},y_{2n+1}),d(y_{2n+1},y_{2n+2}),\\ d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n+2}),0\big)\precsim 0.$$

By (Φ_3), there exist $h_1\in\mathbb{C}_+$ such that

$$|d(y_{2n+1},y_{2n+2})|\leq|h_1||d(y_{2n},y_{2n+1})|. \tag{2}$$

Again, using inequality (1),

$$\phi(d(Sx_{2n},Tx_{2n-1}),d(Ix_{2n},Jx_{2n-1}),d(Ix_{2n},Sx_{2n}),d(Jx_{2n-1},Tx_{2n-1}),\\ d(Ix_{2n},Tx_{2n-1}),d(Sx_{2n},Jx_{2n-1}))\precsim 0,$$

and

$$\phi\big(d(y_{2n+1},y_{2n}),d(y_{2n},y_{2n-1}),d(y_{2n},y_{2n+1}),d(y_{2n-1},y_{2n}),d(y_{2n},y_{2n}),\\ d(y_{2n+1},y_{2n-1})\big)\precsim 0,$$

or

$$\phi\big(d(y_{2n+1},y_{2n}),d(y_{2n},y_{2n-1}),d(y_{2n},y_{2n+1}),d(y_{2n-1},y_{2n}),0,\\ d(y_{2n+1},y_{2n-1})\big)\precsim 0.$$

In view of (Φ_2), we have,

$$\phi\big(d(y_{2n+1},y_{2n}),d(y_{2n},y_{2n-1}),d(y_{2n},y_{2n+1}),d(y_{2n-1},y_{2n}),0,\\ d(y_{2n+1},y_{2n})+d(y_{2n},y_{2n-1})\big)\precsim 0.$$

By Φ_3, there exist $h_2\in\mathbb{C}_+$ such that,

$$|d(y_{2n+1},y_{2n})|\leq|h_2||d(y_{2n},y_{2n-1})|. \tag{3}$$

Combining (2) and (3), we have,

$$|d(y_{2n+1},y_{2n+2})|\leq h|d(y_{2n},y_{2n-1})|.$$

Continuing this process, we get,

$$|d(y_{2n+1},y_{2n+2})| \leq h^n |d(y_1,y_2)|. \tag{4}$$

By using inequality (1), we have,

$$\phi(d(Sx_{2n+2},Tx_{2n+1}),d(Ix_{2n+2},Jx_{2n+1}),d(Ix_{2n+2},Sx_{2n+2}),d(Jx_{2n+1},Tx_{2n+1}),$$
$$d(Ix_{2n+2},Tx_{2n+1}),d(Sx_{2n+2},Jx_{2n+1})) \precsim 0,$$

and

$$\phi\big(d(y_{2n+3},y_{2n+2}),d(y_{2n+2},y_{2n+1}),d(y_{2n+2},y_{2n+3}),d(y_{2n+1},y_{2n+2}),$$
$$d(y_{2n+2},y_{2n+2}),d(y_{2n+1},y_{2n+3})\big) \precsim 0,$$

$$\phi\big(d(y_{2n+3},y_{2n+2}),d(y_{2n+2},y_{2n+1}),d(y_{2n+2},y_{2n+3}),d(y_{2n+1},y_{2n+2}),0,$$
$$d(y_{2n+1},y_{2n+3})\big) \precsim 0,$$

in view of (Φ_2), we have,

$$\phi\big(d(y_{2n+3},y_{2n+2}),d(y_{2n+2},y_{2n+1}),d(y_{2n+2},y_{2n+3}),d(y_{2n+1},y_{2n+2}),0,$$
$$d(y_{2n+1},y_{2n+2})+d(y_{2n+2},y_{2n+3})\big) \precsim 0.$$

From Φ_3, there exist $h_2 \in \mathbb{C}_+$, such that,

$$|d(y_{2n+2},y_{2n+3})| \leq |h_2||d(y_{2n+2},y_{2n+1})|.$$

Using (4), we obtain,

$$|d(y_{2n+2},y_{2n+3})| \leq h^n |h_2||d(y_1,y_2)|. \tag{5}$$

From (4) and (5), we get,

$$|d(y_n,y_{n+1})| \leq \frac{\max\{1,|h_2|\}}{h}(\sqrt{h})^n |d(y_1,y_2)|, \text{for } n = 2,3,\cdots.$$

Therefore, for any $m > n$, we have,

$$\begin{aligned}
|d(y_n,y_m)| &\leq |d(y_n,y_{n+1})|+|d(y_{n+1},y_{n+2})|+|d(y_{n+2},y_{n+3})|+\cdots+|d(y_{m-1},y_m)| \\
&\leq \frac{\max\{1,|h_2|\}}{h}[\sqrt{h}^n+\sqrt{h}^{n+1}+\sqrt{h}^{n+2}+\cdots+\sqrt{h}^{m-1}]|d(y_1,y_2)| \\
&\leq [\frac{\sqrt{h}^n}{h(1-\sqrt{h})}]\max\{1,|h_2|\}|d(y_1,y_2)|
\end{aligned}$$

since $0 < h < 1$, so that

$$|d(y_n,y_m)| \leq [\frac{\sqrt{h}^n}{h(1-\sqrt{h})}]\max\{1,|h_2|\}|d(y_1,y_2)| \to 0 \quad \text{as} \quad n\to\infty.$$

In view of Lemma 1.2, the sequence $\{y_n\}$ is Cauchy sequence in (X,d). Now suppose that IX is complete subspace of X. Then the subsequence $y_{2n} = Tx_{2n-1} = Ix_{2n}$ converges to some u in IX, that is,

$$y_{2n} = Ix_{2n} = Tx_{2n-1} \to u \quad \text{as} \quad n\to\infty. \tag{6}$$

As $\{y_n\}$ is a Cauchy sequence which contains a convergent subsequence $\{y_{2n}\}$, therefore the sequence $\{y_n\}$ also converges implying thereby the convergence of the subsequence $\{y_{2n-1}\}$ being a subsequence of convergent sequence $\{y_n\}$. Consequently, we can find $v \in X$ such that

$$Iv = u. \tag{7}$$

We claim that $Sv = u$. Using inequality (1) and (7),we have,

$$\phi(d(Sv,Tx_{2n-1}),d(Iv,Jx_{2n-1}),d(Iv,Sv),d(Jx_{2n-1},Tx_{2n-1}),$$
$$d(Iv,Tx_{2n-1}),d(Sv,Jx_{2n-1})) \precsim 0,$$

and

$$\phi\big(d(Sv,y_{2n}),d(u,y_{2n-1}),d(u,Sv),d(y_{2n-1},y_{2n}),d(u,y_{2n}),d(Sv,y_{2n-1})\big) \precsim 0.$$

Letting $n\to\infty$ in the above inequality, using (6) and the continuity ϕ, we have,

$$\phi\big(d(Sv,u),0,d(u,Sv),0,0,d(Sv,u)\big) \precsim 0.$$

From Φ_4, this implies that $d(Sv,u) = 0$, that is,

$$Sv = u. \tag{8}$$

Now, combining (7) and (8), we have,

$$Iv = Sv = u,$$

that is, u is a point of coincidence of I and S.

Since $u = Sv \in SX \subseteq JX$, there exists $w \in X$ such that,

$$u = Jw. \tag{9}$$

We claim that $Tw = u$.

Using inequality (1), we have,

$$\phi\big(d(Sv,Tw),d(Iv,Jw),d(Iv,Sv),d(Jw,Tw),d(Iv,Tw),d(Sv,Jw)\big) \precsim 0,$$

or,

$$\phi\big(d(u,Tw),0,0,d(u,Tw),d(u,Tw),0\big) \precsim 0,$$

which by using Φ_4: we have $d(u,Tw) = 0$, that is,

$$u = Tw. \tag{10}$$

Combining (9) and (10), we have,

$$u = Jw = Tw,$$

that is, u is a point of coincidence of J,T.

Now, suppose that u' is another point of coincidence of I and S, that is,

$$u' = Iv' = Sv',$$

for some $v' \in X$. Using inequality (1), we have,

$$\phi\big(d(Sv',Tw),d(Iv',Jw),d(Iv',Sv'),d(Jw,Tw),d(Iv',Tw),d(Sv',Jw)\big) \precsim 0,$$

or,

$$\phi\big(d(u',u),d(u',u),d(u',u'),d(u,u),d(u',u),d(u',u)\big) \precsim 0$$

$$\Rightarrow \phi\big(d(u',u),d(u',u),0,0,d(u',u),d(u',u)\big) \precsim 0,$$

which implies (by using Φ_4) $d(u',u)=0$, that is, $u'=u$.

Now, suppose that $\overline{u}$ is another point of coincidence of J and T, that is ,

$$\overline{u}=Jw'=Tw',$$

for some $w'\in X$. Using inequality (1), we get

$$\phi\big(d(u,\overline{u}),d(u,\overline{u}),0,0,d(u,\overline{u}),d(u,\overline{u})\big)\precsim 0,$$

which implies (by using Φ_4) $d(u,\overline{u})=0$, that is, $u=\overline{u}$.

Therefore, we proved that u is the unique point of coincidence of $\{I,S\}$ and $\{J,T\}$.

Since $\{I,S\}$ and $\{J,T\}$ are weakly compatible, and $u=Iv=Sv=Jw=Tw$, we can write

$$Su=S(Iv)=I(Sv)=Iu=w_1 \text{ (say)}$$

and,

$$Tu=T(Jw)=J(Tw)=Ju=w_2 \text{ (say)}.$$

By using inequality (1), we get

$$\phi\big(d(w_1,w_2),d(w_1,w_2),0,0,d(w_1,w_2),d(w_1,w_2)\big)\precsim 0,$$

which implies that $w_1=w_2$, that is,

$$Su=Iu=Tu=Ju,$$

which by using inequality (1) we have

$$\phi\big(d(Sv,Tu),d(Iv,Ju),d(Iv,Sv),d(Ju,Tu),d(Iv,Tu),d(Sv,Ju)\big)\precsim 0$$

$$\Rightarrow \phi\big(d(Sv,Tu),d(Sv,Tu),0,0,d(Sv,Tu),d(Sv,Tu)\big)\precsim 0,$$

we deduce (by using Φ_4) that $Sv=Tu$, that is, $u=Tu$. This implies that

$$u=Su=Iu=Tu=Ju.$$

Then, u is the unique common fixed point of S,I,J and T.

The proof for the cases in which SX,JX, or TX is complete are similar, and are omitted.

Corollary 3.1. If S, T, I and J are self-mappings defined on complex valued metric space (X, d) satisfying $TX \subseteq IX, SX \subseteq JX$ and

$$\lambda d(Sx, Ty) \precsim Ad(Ix, Jy) + Bd(Ix, Sx) + Cd(Jy, Ty) + Dd(Ix, Ty) + Ed(Sx, Jy), \tag{11}$$

for all $x, y \in X$, where $D, E \in \mathbb{R}^+$, $\lambda, A, B, C \in \mathbb{C}_+$ and $0 \prec A + B + C + D + E \prec \lambda$. If one of SX, TX, IX or JX is a complete subspace of X, then:

(a) $\{S, I\}$ and $\{T, J\}$ have a unique point of coincidence in X,

(b) if $\{S, I\}$ and $\{T, J\}$ are weakly compatible, then S, T, I and J have a unique common fixed point in X.

Corollary 3.2. If S, T, I and J are self-mappings defined on complex valued metric space (X, d) satisfying $TX \subseteq IX, SX \subseteq JX$ and

$$\lambda d(Sx, Ty) \precsim Ad(Ix, Jy) + B(d(Ix, Sx) + d(Jy, Ty)) + C(d(Ix, Ty) + d(Sx, Jy)), \tag{12}$$

for all $x, y \in X$, where $C \in \mathbb{R}^+$, $\lambda, A, B \in \mathbb{C}_+$ and $0 \prec A + 2B + 2C \prec \lambda$. If one of SX, TX, IX or JX is a complete subspace of X, then:

(a) $\{S, I\}$ and $\{T, J\}$ have a unique point of coincidence in X,

(b) if $\{S, I\}$ and $\{T, J\}$ are weakly compatible, then S, T, I and J have a unique common fixed point in X.

Proof. Define $\phi(u_1, u_2, u_3, u_4, u_5, u_6) : \mathbb{C}_+^6 \to \mathbb{C}_+$ as

$$\phi(u_1, u_2, u_3, u_4, u_5, u_6) = \lambda u_1 - Au_2 - B(u_3 + u_4) - C(u_5 + u_6),$$

where $C \in \mathbb{R}^+$, $\lambda, A, B \in \mathbb{C}_+$ and $0 \prec A + 2B + 2C \prec \lambda$.

We show $\phi \in \Phi$.

Φ_1 and Φ_2 :Obviously;

Φ_3 :Denote

$$h_1 = h_2 = \frac{A + B + C}{\lambda - (B + C)}.$$

Since $0 \prec A + 2B + 2C \prec \lambda$, then

$$0 \prec A + B + C \prec \lambda - (B + C) \Rightarrow |A + B + C| < |\lambda - (B + C)|$$

and,

$$|\frac{A+B+C}{\lambda-(B+C)}|<1,$$

therefore, we have $h=|h_1h_2|<1$.

If $\phi(u,v,v,u,u+v,0)\precsim 0$, we have

$$\lambda u-Av-Bu-Bv-Cu-Cv\precsim 0,$$

which implies that $|u|\leq|h_1v|$. Now, if $\phi(u,v,u,v,0,u+v)\precsim 0$, we have

$$\lambda u-Av-Bu-Bv-Cu-Cv\precsim 0,$$

which implies that $|u|\leq|h_2v|$.

Φ_4 :Suppose that $\phi(u,u,0,0,u,u)\precsim 0$. We get

$$\lambda u\precsim Au+2Cu\Rightarrow|\lambda||u|\leq|A+2C||u|,$$

on the other hand, since $|A+2C|<|\lambda|$ and $|A+2C||u|\leq|\lambda||u|$ then $|u|=0$ and $u=0$. The same result holds if $\phi(u,0,u,0,0,u)\precsim 0$ or $\phi(u,0,0,u,u,0)\precsim 0$. Therefore, $\phi\in\Phi$.

Moreover,inequality (12) is equivalent to inequality (1). Then to obtain the desired result, we have only to apply Theorem 3.1 for the considered function ϕ.

Remark 3.1. If $S=T$ and I and J are identity mapping, $\lambda=1, A=B=0$ and $C\neq 0$, in the particular case, when (X,d) is a metric space, we obtain Kannan fixed point theorem (cf. [7]).

Remark 3.2. If $S=T$ and I and J are identity mapping, $A=C=0, B\in\mathbb{C}_+$ and $B\neq 0$, in the particular case, when (X,d) is a metric space, we obtain Chatterjia theorem (cf. [8]).

Remark 3.3. If $S=T$ and I and J are identity mapping, $A,B,C\in\mathbb{R}^+$ and $\lambda=1$ in the particular case, when (X,d) is a metric space, we obtain Hardy and Rogers theorem (cf. [9]).

Corollary 3.3. The conclusions of Theorem 3.1 remain true if implicit relation (1) is replaced by any one of the following conditions.

$$\lambda d(Sx,Ty)\precsim A\frac{d(x,Sx)d(y,Ty)}{1+d(x,Sx)+d(y,Ty)}, \tag{13}$$

where $A,\lambda\in\mathbb{C}_+$ and $A\precsim\lambda$.

$$\lambda d(Sx,Ty)\precsim A\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)+d(x,Sx)+d(y,Ty)}, \tag{14}$$

where $A,\lambda \in \mathbb{C}_+$ and $A \precsim \lambda$.

$$\lambda d(Sx,Ty) \precsim A\frac{d(x,y)d(x,Sx)}{1+d(x,y)+d(x,Sx)+d(y,Ty)}, \tag{15}$$

where $A,\lambda \in \mathbb{C}_+$ and $A \precsim \lambda$.

$$\lambda d(Sx,Ty) \precsim A\frac{d(x,y)d(x,Sx)}{1+d(x,Sx)+d(y,Ty)}, \tag{16}$$

where $A,\lambda \in \mathbb{C}_+$ and $A \precsim \lambda$.

4. Applications

As an application of Theorem 3.1, we prove the following theorem for two finite families of mappings.

Theorem 4.1. If $\{T_i\}_1^m$,$\{J_i\}_1^p$ and $\{S_i\}_1^l$,$\{I_i\}_1^n$ are two finite pairwise commuting finite families of self-mapping defined on a complex valued metric space (X,d) such that the mappings S,T,I and J (with $T=T_1T_2...T_m$, $J=J_1J_2...J_p$, $I=I_1I_2...I_n$ and $S=S_1S_2...S_l$) satisfy $TX\subset IX$ and $SX\subset JX$ and the inequality (1).If one of TX,SX,IX or JX are complete subspace of X, then the component maps of the two families $\{T_i\}_1^m$,$\{J_i\}_1^p$ and $\{S_i\}_1^l$,$\{I_i\}_1^n$ have a unique common fixed point.

Proof. Appealing to componentwise commutativity of various pairs, one immediately concludes that $SI=IS$ and $TJ=JT$ and hence, obviously both the pairs (S,I) and (T,J) are weak compatible. Note that all the conditions of Theorem 3.1 (for mappings S,T,I and J) are satisfied ensuring the existence of unique common fixed point u in X, i.e. $Su=Tu=Iu=Ju=u$. We are required to show that u is common fixed point of all the components maps of the families. For this consider

$$\begin{aligned} S(S_ku) &= ((S_1S_2...S_l)S_k)u=(S_1S_2...S_{l-1})((S_lS_k)u)\\ &= (S_1...S_{l-2})(S_{l-1}S_k(S_lu))=(S_1...S_{l-2})(S_kS_{l-1}(S_lu))=...\\ &= S_1S_k(S_2S_3S_4...S_lu)=S_kS_1(S_2S_3S_4...S_lu)=S_k(Su)=S_ku \end{aligned}$$

Similarly, one can show that,

$$T_k u = T_k J u = J T_k u, T_k u = T_k T u = T T_k u$$

$$J_k u = T J_k u = J J_k u, S_k u = I S_k u = S S_k u$$

$$I_k u = I I_k u = S I_k u, T_k u = T T_k u = J T_k u,$$

which show that (for every k)$S_k u, T_k u, I_k u$ and $J_k u$ are other fixed points of S, T, I *and* J.
By using the uniqueness of common fixed point for S, T, I *and* J, we can write $S_k u = T_k u = I_k u = J_k u = u$ (for every k) which shows that u is a common fixed point of the family $\{T_i\}_1^m$, $\{S_i\}_1^l$, $\{I_i\}_1^p$ and $\{J_i\}_1^n$ (for every k). This completes the proof of the theorem.

By setting $S_1 = S_2 = ... = S_l = G, T_1 = T_2 = ... = T_m = F, I_1 = I_2 = ... = I_n = Q$ and $J_1 = J_2 = ... = J_p = R$ in Theorem 4.1, we derive the following common fixed point theorem involving iterates of mappings.

Corollary 4.1. If F, R and G, Q are two commuting self-mappings defined on a complex valued metric space (X, d) satisfying $F^m X \subseteq Q^n X, G^l X \subseteq R^p X$ and

$$\phi(d(G^l x, F^m y), d(Q^n x, R^p y), d(Q^n x, G^l x), d(R^p y, F^m y), d(Q^n x, F^m y), d(G^l x, R^p y)) \precsim 0$$

for all $x, y \in X$ where $\phi \in \Phi$. If one of $G^l X, F^m X, Q^n X$ or $R^p X$ is a complete subspace of X, then G, F, Q and R have a unique common fixed point in X.

By setting $m = n = l = p$ and $F = R = G = Q = H$ in Corollary 4.1, we deduce the following corollary.

Corollary 4.2. If $H : X \to X$ is a mapping defined on a complex valued metric space (X, d) satisfying the condition (for some fixed n):

$$\phi(d(H^n x, H^n y), d(H^n x, H^n y), 0, 0, d(H^n x, H^n y), d(H^n x, H^n y)) \precsim 0$$

for all $x, y \in X$ where $\phi \in \Phi$. If (for some fixed n) $H^n X$ is complete subspace of X, then H have unique fixed point in X.

As an application of Corollary 3.1, we prove the following theorem for two finite families of mappings.

Theorem 4.2. If $\{T_i\}_1^m$,$\{J_i\}_1^p$ and $\{S_i\}_1^l$,$\{I_i\}_1^n$ are two finite pairwise commuting finite families of self-mapping defined on a complex valued metric space (X,d) such that the mappings S,T,I and J (with $T = T_1T_2...T_m$, $J = J_1J_2...J_p$, $I = I_1I_2...I_n$ and $S = S_1S_2...S_l$) satisfy $TX \subset IX$ and $SX \subset JX$ and the inequality (12). If one of TX, SX, IX or JX are complete subspace of X, then the component maps of the two families $\{T_i\}_1^m$,$\{J_i\}_1^p$ and $\{S_i\}_1^l$,$\{I_i\}_1^n$ have a unique common fixed point.

Proof. The proof of this theorem is identical to that of Theorem 4.1.

By setting $S_1 = S_2 = ... = S_l = G, T_1 = T_2 = ... = T_m = F, I_1 = I_2 = ... = I_n = Q$ and $J_1 = J_2 = ... = J_p = R$ in Theorem 4.2, we derive the following common fixed point theorem involving iterates of mappings.

Corollary 4.3. If F,R and G,Q are two commuting self-mappings defined on a complex valued metric space (X,d) satisfying $F^mX \subseteq Q^nX, G^lX \subseteq R^pX$ and

$$\lambda d(G^lx,F^my) \precsim Ad(Q^nx,R^py)+B(d(Q^nx,G^lx)+d(R^py,F^my))+C(d(Q^nx,F^my)+d(G^lx,R^py))$$

for all $x,y \in X$, where $C \in \mathbb{R}^+$, $\lambda,A,B \in \mathbb{C}_+$ and $0 \prec A+2B+2C \prec \lambda$. If one of G^lX, F^mX, Q^nX or R^pX is a complete subspace of X, then G,F,Q and R have a unique common fixed point in X.

By setting $m = n = l = p$ and $F = R = G = Q = H$ in Corollary 4.3, we deduce the following corollary.

Corollary 4.4. If $H : X \to X$ is a mapping defined on a complex valued metric space (X,d) satisfying (for some fixed n).

$$(\lambda - A - 2C)d(H^nx,H^ny) \precsim 0$$

for all $x,y \in X$, where $C \in \mathbb{R}^+$, $\lambda,A,B \in \mathbb{C}_+$ and $0 \prec A+2B+2C \prec \lambda$.If H^nX is complete subspace of X, then H have unique fixed point in X.

Remark 4.1. Equating λ and A,B,C to zero suitably in Theorem 4.2 and corollaries 3.1, 4.3 and 4.4 one can derive a multitude of common fixed point theorems which are often new results in the setting of complex valued nature space.

5. Illustrative Examples

Now we furnish examples to demonstrate the validity of the hypotheses and degree of generality of Theorem 3.1 .

Example 5.1. Let $X=[0,1]$ and define $d: X\times X\to\mathbb{C}$ by $d(x,y)=i|x-y|$, for all$x,y\in X$. Define self-mappings $S,T,I$and$J$on$X$ as

$$S(x)=T(x)=\frac{x}{3} \text{ and } I(x)=J(x)=\frac{x}{2}$$

One may note that the pairs (S,I) and (T,J) commute at 0 which is their common coincidence point. Also $S(X)=[0,\frac{1}{3}]\subseteq J(X)=[0,\frac{1}{2}]$ and $T(X)=[0,\frac{1}{3}]\subseteq I(X)=[0,\frac{1}{2}]$. All the needed pairwise commutativity at coincidence point 0 are immediate.

Define $\phi(u_1,u_2,u_3,u_4,u_5,u_6):\mathbb{C}_+^6\to\mathbb{C}_+$ as

$$\phi(u_1,u_2,u_3,u_4,u_5,u_6)=\lambda u_1-Au_2-B(u_3+u_4)-C(u_5+u_6),$$

where $C\in\mathbb{R}^+$, $\lambda,A,B\in\mathbb{C}_+$ and $0\prec A+2B+2C\prec\lambda$. By a routine calculation one can verify that contraction conditions (1) is satisfied for $\lambda=3+3i, A=2+2i$ and $B=C=\frac{1}{4}$. If $x,y\in X$, then

$$\begin{aligned}\lambda d(Sx,Ty)=(3+3i)(i|\frac{x}{3}-\frac{y}{3}|)&\precsim(2+2i)(i|\frac{x}{2}-\frac{y}{2}|)\\ +\frac{1}{4}(i|\frac{x}{2}-\frac{x}{3}|+i|\frac{y}{2}-\frac{y}{3}|&+i|\frac{x}{2}-\frac{y}{3}|+i|\frac{x}{3}-\frac{y}{2}|)\\ &=Ad(Ix,Jy)+B(d(Ix,Sx)+d(Jy,Ty))\\ &+C(d(Ix,Ty)+d(Sx,Jy))\end{aligned}$$

Thus all the conditions of Theorem 3.1, are satisfied and 0 is the unique common fixed point of S,T,I and J.

Example 5.2. Let $X=\{0,1,\frac{1}{2},\frac{1}{2^2},\frac{1}{2^3},\ldots\}$ and define $d: X\times X\to\mathbb{C}_+$ by $d(x,y)=i|x-y|$, for all$x,y\in X$.

Define self-mappings $S,T,I,J: X\to X$ as

$S(0)=T(0)=\frac{1}{2^2}, S(\frac{1}{2^n})=T(\frac{1}{2^n})=\frac{1}{2^{n+2}}$ and $I(0)=J(0)=\frac{1}{2}, S(\frac{1}{2^n})=T(\frac{1}{2^n})=\frac{1}{2^{n+1}}$, for $n=$

$0, 1, 2, \ldots$ respectively. Clearly,

$$S(X) = T(X) = \{\frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, ..\} \subseteq I(X) = J(X) = \{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, ..\}$$

Considering the same implicit function as in Example 5.1, one can verify that the contraction condition (1) is satisfied for $\lambda = \frac{1}{4} + i, A = \frac{1}{8}(1+i), B = \frac{1}{32}$ and $C = \frac{1}{34}$.

(i) : If $x = y = 0$, we get

$$\begin{aligned}\lambda d(Sx, Ty) = (\frac{1}{4} + i)(\frac{1}{2^2} - \frac{1}{2^2}|) \precsim \frac{1}{8}(1+i)(0) + \frac{1}{32}(i|\frac{1}{2} - \frac{1}{2^2}| + i|\frac{1}{2} - \frac{1}{2^2}|) \\ + \frac{1}{34}(i|\frac{1}{2} - \frac{1}{2^2}| + i|\frac{1}{2^2} - \frac{1}{2}) \\ = Ad(Ix, Jy) + B(d(Ix, Sx) + d(Jy, Ty)) \\ + C(d(Ix, Ty) + d(Sx, Jy))\end{aligned}$$

(ii) : If $x \neq 0, y \neq 0$,

$$\begin{aligned}\lambda d(Sx, Ty) = (\frac{1}{4} + i)(i|\frac{1}{4} - \frac{1}{2^{n+2}}|) \precsim \frac{1}{8}(1+i)(i|\frac{1}{2^{n+1}} - \frac{1}{2^{n'+1}}|) \\ + \frac{1}{32}(i|\frac{1}{2^{n+1}} - \frac{1}{2^{n+2}}| + i|\frac{1}{2^{n'+1}} - \frac{1}{2^{n'+2}}|) \\ + \frac{1}{34}(i|\frac{1}{2^{n+1}} - \frac{1}{2^{n'+2}}| + i|\frac{1}{2^{n+2}} - \frac{1}{2^{n'+1}}) \\ = Ad(Ix, Jy) + B(d(Ix, Sx) + d(Jy, Ty)) \\ + C(d(Ix, Ty) + d(Sx, Jy))\end{aligned}$$

(iii) : If $x = 0$ and $y \neq 0$,

$$\begin{aligned}\lambda d(Sx, Ty) = (\frac{1}{4} + i)(i|\frac{1}{2^{n+2}} - \frac{1}{2^{n'+2}}|) \precsim \frac{1}{8}(1+i)(i|\frac{1}{2} - \frac{1}{2^{n+1}}|) \\ + \frac{1}{32}(i|\frac{1}{2} - \frac{1}{4}| + i|\frac{1}{2^{n+1}} - \frac{1}{2^{n+2}}|) \\ + \frac{1}{34}(i|\frac{1}{2} - \frac{1}{2^{n+2}}| + i|\frac{1}{4} - \frac{1}{2^{n+1}}) \\ = Ad(Ix, Jy) + B(d(Ix, Sx) + d(Jy, Ty)) \\ + C(d(Ix, Ty) + d(Sx, Jy))\end{aligned}$$

Thus all the conditions of Theorem 3.1 are satisfied except the completeness of the subspaces $S(X), I(X), T(X)$ and $J(X)$. Note that S, I and T, J have no point of coincidence. Here it is fascinating to note that in the set up of Theorem 3.1 even the completeness of the space cannot ensure the existence of coincidence point as the space X is complete in the present example. Also note that mappings S and I are not continuous at 0.

Conflict of Interests

The authors declare that there is no conflict of interests.

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