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SOME RELATED FIXED POINTS THEOREMS OF WEAK CONTRACTION WITH TWO PARTIALLY ORDERED METRIC SPACES

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Abstract. In this article, we prove some fixed point results for generalized weakly contractive mappings defined on two partial metrics space. We provide an example and some applications in order to support the usability of our results. These results generalize some well-known results in the literature.

Keywords: partial metric space; weakly contractive condition; nondecreasing map; fixed point; partially ordered set.

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1. Introduction and preliminaries

Since the appearance of Banach's contraction principle, a variety of generalizations, extensions and applications of this principle have been obtained; see Rhoades[24] for a complete

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survey of this subject, Alber and Guerre-Delabriere [14]defined weakly contractive mappings on a Hilbert spaces and established a fixed point theorem for such mappings, Dutta and Choudhury [13]generalized the weak contractive condition and proved a fixed point theorem for a self-mapping, which in turn extends Theorem 1 in[24] and the corresponding result in [14], Zhang and Song [20] introduced generalized weak contractive mappings and obtained a common fixed point result. Doric [15] extended the result of Zhang and Song using a pair of functions ψ and φ. On the other hand, Banach's contraction principle is broadly applicable in proving the existence of solutions to operator equations, including the ordinary differential equations, partial differential equations and integral equations. This principle has been generalized in many directions. For instance, Matthews [23] introduced the concept of a partial metric as a part of the study of denotational semantics of dataflow networks. He gave a modified version of Banach's contraction principle, more suitable in this context. Many authors followed his idea and gave their contributions in that sense, see for example [4, 5, 6, 11, 12, 19, 21, 22, 25, 27, 28]. Subsequently, several authors studied the problem of existence and uniqueness of a fixed point for mappings satisfying different contractive conditions (e.g., [2, 3, 7, 8, 12, 26, 29]). Existence of fixed points in ordered metric spaces has been initiated in 2004 by Ran and Reurings [9], and further studied by Nieto and López [10]. Finally, several interesting and valuable results have appeared in this direction[16, 17]. The aim of this article is to study the necessary conditions for existence of common fixed points of six maps satisfying generalized weak contractive conditions in the framework of complete two partial metric spaces endowed with a partial order. Our results extend and strengthen various known results[1, 18, 20].

Following are some definitions and known results needed in the sequel.

Definition 1.1.[23] Let X be a nonempty set. A mapping $p : X \times X \to \mathbb{R}^+$ is said to be a partial metric on X if for any $x, y, z \in X$, the following conditions hold true:

(P_1): $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$;

(P_2): $p(x,x) \le p(x,y)$;

(P_3): $p(x,y) = p(y,x)$;

(P_4): $p(x,z) \le p(x,y) + p(y,z) - p(y,y)$.

A nonempty set X equipped with a partial metric p is called partial metric space. We shall denote it by a pair $(X;p)$.

If $p(x,y)=0$, then (P_1) and (P_2) imply that $x=y$. But converse does not hold always.

A trivial example of a partial metric space is the pair $(\mathbb{R}^+,p)$, where $p:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is defined as $p(x,y)=max\{x,y\}$.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon):x\in X,\varepsilon>0\}$, where $B_p(x,\varepsilon)=\{y\in X:p(x,y)<p(x,x)+\varepsilon\}$, for all $x\in X$ and $\varepsilon>0$. Observe (see [23], p. 187) that a sequence x_n in X converges to a point $x\in X$, with respect to τ_p, if and only if $p(x,x)=\lim_{n\to\infty}p(x,x_n)$. If p is a partial metric on X, then the mapping $p^S:X\times X\to\mathbb{R}^+$ (set of all non-negative real numbers) given by

$$p^S(x;y)=2p(x;y)-p(x;x)-p(y;y)$$

is a metric on X.

Example 1.1.[23] If $X=\{[a,b]:a,b\in\mathbb{R},a\le b\}$ then $p([a,b],[c,d])=max\{b,d\}-min\{a,c\}$ defines a partial metric p on X.

Definition 1.2.[23] Let X be a partial metric space.

- **(a):** A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n,m\to\infty}p(x_n,x_m)$ exists and is finite.
- **(b):** X is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x\in X$ such that $\lim_{n\to\infty}p(x,x_n)=p(x,x)$. In this case, we say that the partial metric p is complete.

Aydi [5] obtained the following result in partial metric spaces.

Theorem 1.1. Let $(X,\le_X)$ be a partially ordered set and let p be a partial metric on X such that (X,p) is complete. Let $f:X\to X$ be a nondecreasing map with respect to $\le_X$. Suppose that the following conditions hold: for $y\le x$, we have:

- **(i):** $p(fx,fy)\le p(x,y)-\varphi(p(x,y))$;

  where $\varphi:[0,+\infty[\to[0,+\infty[$ is a continuous and non-decreasing function such that it is positive in $]0,+\infty[$, $\varphi(0)=0$ and $\lim_{t\to+\infty}\varphi(t)=\infty$
- **(ii):** there exist $x_0\in X$ such that $x_0\le_X fx_0$;

(iii): f is continuous in (X,p), or;

(iiii): if a non-decreasing sequence $\{x_n\}$ converges to $x \in X$, then $x_n \leq_X x$ for all n.

Then f has a fixed point $u \in X$. Moreover, $p(u,u) = 0$.

Recently, Abbas and Nazir [1] extended Theorem (1.1) and obtained the following theorem.

Theorem 1.2.[1] Let $(X,\preceq)$ be a partially ordered set such that there exist a complete partial metric p on X and f a nondecreasing selfmap on X. Suppose that for every two elements $x,y \in X$ with $y \preceq x$, we have

$$\psi(p(fx,fy)) \leq \psi(M(x,y)) - \varphi(M(x,y)),$$

where

$$M(x,y) = \max\ \{p(x,y), p(fx,x), p(fy,y), \frac{p(x,fy)+p(y,fx)}{2}\},$$

$\psi,\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, ψ is continuous and nondecreasing, φ is a lower semi continuous, and $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$ and one of the following two conditions is satisfied:

(a): f is continuous self map on (X,p^S);

(b): for any nondecreasing sequence x_n in $(X,\preceq)$ with $\lim_{n\to+\infty} p^S(z,x_n) = 0$ it follows $x_n \preceq z$ for all $n \in \mathbb{N}$,

then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

A nonempty subset W of a partially ordered set X is said to be well ordered if every two elements of W are comparable.

2. Fixed point results

Throughout the rest of this paper, we denote by $(X,\leq,d,\delta)$ be a complete two partially ordered metric spaces, i.e., $\preceq$ is a partial order on the set X and d, δ are complete two partially ordered metric spaces on X. A mapping $F : X \to X$ is said to be nondecreasing if x,y in X, $x \preceq y \Rightarrow Fx \preceq Fy$.

Theorem 2.1. Let $(X, \leq, d, \delta)$ be a complete two partially ordered metric space; let $F : X \to X$ be a continuous and nondecreasing mapping such that for all comparable x, y in X:

$$\begin{cases} \psi(d(Fx,Fy)) \leq \psi(M_\delta(x,y)) - \phi(M_\delta(x,y)), & (1) \\ \psi(\delta(Fx,Fy)) \leq \psi(M_d(x,y)) - \phi(M_d(x,y)). & (2) \end{cases}$$

where:

$$\begin{aligned} M_d(x,y) &= \max\{d(x,y), d(x,Fx), d(y,Fy), d(x,x), d(y,y), \frac{[d(x,Fy)+d(Fx,y)]}{2}\}, \\ M_\delta(x,y) &= \max\{\delta(x,y), \delta(x,Fx), \delta(y,Fy), \delta(x,x), \delta(y,y), \frac{[\delta(x,Fy)+\delta(Fx,y)]}{2}\}, \end{aligned}$$

(a): $\psi : [0,+\infty[\to [0,+\infty[$ is a continuous and monotone nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$.

(b): $\phi : [0,+\infty[\to [0,+\infty[$ is a lower semi-continuous function with $\phi(t) = 0$ if and only if $t = 0$.

If there exists $x_0 \in X$ with $x_0 \preceq Fx_0$, then F has a fixed point .

Proof. Since F is a nondecreasing function, we obtain by induction that:

$$x_0 \preceq Fx_0 \preceq F^2x_0 \preceq ..... \preceq F^nx_0 \preceq F^{n+1}x_0 \preceq ....$$

Put $x_0 \in X$, $x_{n+1} = Fx_n$, Then, for each integer $n = 0, 1, 2, ...$, as the elements x_{n+1} and x_n are comparable, from (1) we get:

$$\psi(d(x_{n+1},x_n)) = \psi(d(Fx_n,Fx_{n-1})) \leq \psi(M_\delta(x_n,x_{n-1})) - \varphi(M_\delta(x_n,x_{n-1})). \tag{3}$$

Which implies $\psi(d(x_{n+1},x_n)) \leq \psi(M_\delta(x_n,x_{n-1}))$. Using the monotone property of the $\psi-function$, we get:

$$d(x_{n+1},x_n) \leq M_\delta(x_n,x_{n-1}). \tag{4}$$

Now, from the triangle inequality, for δ we have:

$$\begin{aligned} M_\delta(x_n,x_{n-1}) &= \max\{\delta(x_n,x_{n-1}),\delta(x_n,Fx_n),\delta(x_{n-1},Fx_{n-1}),\delta(x_n,x_n), \\ &\quad \delta(x_{n-1},x_{n-1}),\frac{[\delta(x_n,Fx_{n-1})+\delta(Fx_n,x_{n-1})]}{2}\} \\ &= \max\{\delta(x_n,x_{n-1}),\delta(x_n,x_{n+1}),\delta(x_{n-1},x_n),\delta(x_n,x_n), \\ &\quad \delta(x_{n-1},x_{n-1}),\frac{[\delta(x_n,x_n)+\delta(x_{n+1},x_{n-1})]}{2}\} \\ &\leq \max\{\delta(x_n,x_{n-1}),\delta(x_{n+1},x_n),\frac{[\delta(x_n,x_{n+1})+\delta(x_n,x_{n-1})]}{2}\} \\ &= \max\{\delta(x_n,x_{n-1}),\delta(x_{n+1},x_n)\}. \qquad (5) \end{aligned}$$

If: $\delta(x_{n+1},x_n) > \delta(x_n,x_{n-1})$, then $M_\delta(x_n,x_{n-1}) = \delta(x_{n+1},x_n) > 0$.

By (3) it furthermore implies that:

$$\begin{aligned} \psi(d(x_{n+1},x_n)) &\leq \psi(\delta(x_{n+1},x_n)) - \phi(\delta(x_{n+1},x_n)) \\ &\leq \psi(\delta(x_{n+1},x_n)) \\ &= \psi(\delta(Fx_n,Fx_{n-1})). \end{aligned}$$

From (2) implies:

$$\psi(d(x_{n+1},x_n)) \leq \psi(M_d(x_n,x_{n-1})) - \phi(M_d(x_n,x_{n-1})). \qquad (6)$$

Similarly the (5) , we can show that:

$$M_d(x_n,x_{n-1}) = \max\{d(x_n,x_{n-1}),d(x_{n+1},x_n)\}.$$

If: $d(x_{n+1},x_n) > d(x_n,x_{n-1})$ then $M_d(x_n,x_{n-1}) = d(x_{n+1},x_n) > 0$. By (6) it furthermore implies that:

$$\psi(d(x_{n+1},x_n)) \leq \psi(d(x_{n+1},x_n)) - \phi(d(x_{n+1},x_n)).$$

Which is a contradiction. So, we have:

$$d(x_{n+1},x_n) \leq M_d(x_n,x_{n-1}) \leq d(x_n,x_{n-1}). \qquad (7)$$

Put in (2) , $x = x_n, y = x_{n-1}$ we get:

$$\psi(\delta(Fx_n,Fx_{n-1})) \leq \psi(M_d(x_n,x_{n-1})) - \phi(M_d(x_n,x_{n-1})).$$

Using (7) implies:

$$\psi(\delta(x_{n+1},x_n)) \leq \psi(d(x_n,x_{n-1})) - \phi(d(x_n,x_{n-1})).$$

Which implies:

$$\psi(\delta(x_{n+1},x_n)) \leq \psi(d(x_n,x_{n-1})).$$

Using the monotone property of the $\psi - function$, we get:

$$\delta(x_{n+1},x_n) \leq d(x_n,x_{n-1}). \tag{8}$$

Put in (1) , $x = x_{n-1}, y = x_{n-2}$ we get:

$$\psi(d(Fx_{n-1},Fx_{n-2})) \leq \psi(M_\delta(x_{n-1},x_{n-2})) - \phi(M_\delta(x_{n-1},x_{n-2})).$$

Which implies

$$\psi(d(x_n,x_{n-1})) \leq \psi(M_\delta(x_{n-1},x_{n-2})).$$

Using the monotone property of the $\psi - function$, and (5) , we have:

$$d(x_n,x_{n-1})) \leq M_\delta(x_{n-1},x_{n-2}) = \delta(x_n,x_{n-1}). \tag{9}$$

(8) and (9) implies

$$\delta(x_{n+1},x_n) \leq \delta(x_n,x_{n-1}).$$

Which is a contradiction. So, we have:

$$\delta(x_{n+1},x_n) \leq M_\delta(x_n,x_{n-1}) \leq \delta(x_n,x_{n-1}). \tag{10}$$

Therefore, (7) and (10) , the sequence $\{d(x_{n+1},x_n)\}$ (resp$\{\delta(x_{n+1},x_n)\}$) is monotone non increasing and bounded, thus, there exists $r \geq 0$ such that:

$$\lim_{n\to+\infty} d(x_{n+1},x_n) = \lim_{n\to+\infty} M_d(x_n,x_{n-1}) = r. \tag{11}$$

Using (3) , (10) , (2) , and (7) , we obtain:

$$\psi(d(x_{n+1},x_n)) \leq \psi(d(x_{n-1},x_{n-2})) - \phi(d(x_{n-1},x_{n-2})). \tag{12}$$

We suppose that $r > 0$, then, letting $n \to \infty$ in the inequality (12) , we get:

$$\psi(r) \leq \psi(r) - \phi(r),$$

Which is a contradiction unless $r = 0$, hence,

$$\lim_{n\to+\infty} d(x_{n+1}, x_n) = 0. \tag{13}$$

Next we prove that $\{x_n\}$ is a d-Cauchy (resp δ-Cauchy) sequence suppose that $\{x_n\}$ is not a d-Cauchy (resp δ -Cauchy) sequence, then, there exists $\varepsilon > 0$ (resp $\varepsilon' > 0$) for which we can find subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ (resp $n_{k'} > m_{k'} > k'$) such that:

$$d(x_{n_k}, x_{m_k}) \geq \varepsilon \ (resp\ \delta(x_{n_{k'}}, x_{m_{k'}}) \geq \varepsilon'). \tag{14}$$

Further, corresponding to m_k, we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ satisfying (14) , then

$$d(x_{n_k-1}, x_{m_k}) < \varepsilon. \tag{15}$$

Using (14) , (15) and the triangle inequality , we have:

$$\varepsilon \leq d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k}) < d(x_{n_k}, x_{n_k-1}) + \varepsilon.$$

Letting $k \to +\infty$ and using (13) , we obtain:

$$\lim_{k\to+\infty} d(x_{n_k}, x_{m_k}) = \varepsilon \ (\ resp\ \lim_{k\to+\infty} \delta(x_{n_k}, x_{m_k}) = \varepsilon'). \tag{16}$$

Again, the triangle inequality gives us:

$$\begin{aligned} d(x_{n_k-1}, x_{m_k}) &\leq d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{m_k}) \\ d(x_{n_k}, x_{m_k}) &\leq d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k}). \end{aligned}$$

Then we have:

$$| \ d(x_{n_k-1}, x_{m_k}) - d(x_{n_k}, x_{m_k}) \ | \leq d(x_{n_k}, x_{n_k-1}).$$

Letting $k \to +\infty$ in the above inequality and using (13) and (16) , we get:

$$\lim_{k\to+\infty} d(x_{n_k-1}, x_{m_k}) = \varepsilon. \tag{17}$$

Similarly, we can show that:

$$\begin{aligned}\lim_{k\to+\infty} d(x_{n_k},x_{m_k-1}) &= \lim_{k\to+\infty} d(x_{n_k-1},x_{m_k-1})\\ &= \lim_{k\to+\infty} d(x_{n_k},x_{m_k+1})\\ &= \lim_{k\to+\infty} d(x_{n_k-1},x_{m_k}) = \varepsilon. \end{aligned}\tag{18}$$

As:

$$M_d(x_{n_k-1},x_{m_k-1}) = \max\{d(x_{n_k-1},x_{m_k-1}), d(x_{n_k-1},x_{n_k}), d(x_{m_k-1},x_{m_k}), d(x_{n_k-1},x_{n_k-1}), \\ d(x_{m_k-1},x_{m_k-1}), \frac{[d(x_{n_k-1},x_{m_k}) + d(x_{n_k},x_{m_k-1}]}{2}\}.$$

Using (13) and (16) , (18) , we have:

$$\lim_{k\to+\infty} M_d(x_{n_k-1},x_{m_k-1}) = \max\{\varepsilon,0,0,0,0,\varepsilon\} = \varepsilon.\tag{19}$$

For example is chosen $k = \max\{k,k'\}$, similarly (19) , we have:

$$\lim_{k\to+\infty} M_\delta(x_{n_k-1},x_{m_k-1}) = \varepsilon'.\tag{20}$$

As $n_k > m_k$ and x_{n_k-1} , x_{m_k-1} are comparable. Setting $x = x_{n_k-1}$ and $y = x_{m_k-1}$ in (1) , (2) , we obtain:

$$\begin{aligned}\psi(d(Fx_{n_k-1},Fx_{m_k-1})) = \psi(d(x_{n_k},x_{m_k})) &\le \psi(M_\delta(x_{n_k-1},x_{m_k-1}))) - \phi(M_\delta(x_{n_k-1},x_{m_k-1}))\\ \psi(\delta(Fx_{n_k-1},Fx_{m_k-1})) = \psi(\delta(x_{n_k},x_{m_k})) &\le \psi(M_d(x_{n_k-1},x_{m_k-1}))) - \phi(M_d(x_{n_k-1},x_{m_k-1})).\end{aligned}$$

Letting $k\to+\infty$ in the above inequality and using (16) , (19) , (20) , we get:

$$\begin{aligned}\psi(\varepsilon) &\le \psi(\varepsilon') - \phi(\varepsilon')\\ \psi(\varepsilon') &\le \psi(\varepsilon) - \phi(\varepsilon).\end{aligned}$$

Implies:

$$\begin{aligned}\psi(\varepsilon) \le \psi(\varepsilon') &\le \psi(\varepsilon) - \phi(\varepsilon)\\ \psi(\varepsilon') \le \psi(\varepsilon) &\le \psi(\varepsilon') - \phi(\varepsilon').\end{aligned}$$

Which is a contradiction as $\varepsilon > 0$ and $\varepsilon' > 0$. Hence $\{x_n\}$ is a d-Cauchy (resp δ-Cauchy)sequence. By the completeness of X, there exists $z \in X$ such that $\lim_{n\to+\infty} x_n = z$, that is,

$$\lim_{n\to+\infty} d(x_n, z) = d(z,z) = \lim_{m,n\to+\infty} d(x_m, x_n) = 0. \tag{21}$$

Moreover, the continuity of F implies that:

$$\lim_{n\to+\infty} d(x_{n+1}, z) = \lim_{n\to+\infty} d(Fx_n, z) = d(Fz, z) = 0.$$

And this proves that z is a fixed point.

Notice that the continuity of F in Theorem (2.1) is not necessary and can be dropped.

Theorem 2.2. Under the same hypotheses of Theorem (2.1) and without assuming the continuity of F, assume that whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x \in X$ implies $x_n \leq x$ for all $n \in \mathbb{N}$, then F has a fixed point in X.

Proof. Following similar arguments to those given in Theorem (2.1), we construct a nondecreasing sequence$\{x_n\}$ in X such that $x_n \to z$ for some $z \in X$. Using the assumption of X, we have $x_n \preceq z$ for every $n \in \mathbb{N}$. Now, we show that $Fz = z$. By (1) , (2) , we have:

$$\begin{cases} \psi(d(Fz, x_{n+1})) = \psi(d(Fz, Fx_n)) \leq \psi(M_\delta(z, x_n)) - \phi(M_\delta(z, x_n)), \\ \psi(\delta(Fz, x_{n+1})) = \psi(\delta(Fz, Fx_n)) \leq \psi(M_d(z, x_n)) - \phi(M_d(z, x_n)). \end{cases}$$

Which implies:

$$\begin{cases} \psi(d(Fz, x_{n+1})) \leq \psi(M_\delta(z, x_n)), \\ \psi(\delta(Fz, x_{n+1})) \leq \psi(M_d(z, x_n)) - \phi(M_d(z, x_n)). \end{cases} \tag{22}$$

Where:

$$\begin{aligned} d(Fz, z) &\leq M_d(z, x_n) = \max\{d(z, x_n), d(Fz, z), d(x_n, x_{n+1}), d(z, z), \\ &\quad d(x_n, x_n), \frac{[d(z, x_{n+1}) + d(Fz, x_n)]}{2}\} \\ &\leq \max\{d(z, x_n), d(Fz, z), d(x_n, x_{n+1}), d(z, z), \\ &\quad d(x_n, x_n), \frac{[d(z, x_{n+1}) + d(Fz, z) + d(z, x_n)]}{2}\}. \end{aligned}$$

Taking limit as $n \to +\infty$, by (21) , we obtain :

$$\lim_{n\to+\infty} M_d(z,x_n) = d(Fz,z).$$

Similarly, we can show that:

$$\lim_{n\to+\infty} M_\delta(z,x_n) = \delta(Fz,z).$$

Therefore, letting $n \to \infty$ in (22) , we get:

$$\begin{cases} \psi(d(Fz,z)) \leq \psi(\delta(Fz,z)), \\ \psi(\delta(Fz,z)) \leq \psi(d(Fz,z)) - \phi(d(Fz,z)). \end{cases} \tag{23}$$

Which implies:

$$\psi(d(Fz,z)) \leq \psi(d(Fz,z)) - \phi(d(Fz,z)).$$

Which is a contradiction unless $d(Fz,z) = 0$, thus $Fz = z$.

Next theorem gives a sufficient condition for the uniqueness of the fixed point.

Theorem 2.3. Let all the conditions of Theorem (2.1) (resp.Theorem (2.2)) be fulfilled and let the following condition be satisfied: For arbitrary two point $x,y \in X$. There exists $z \in X$ which is comparable with both x and y. Then the fixed point of F is unique.

Proof. Suppose that there exist $z,x \in X$ which are fixed points. We distinguish two cases.

Case 1: If x is comparable to z, then $F^n x = x$ is comparable to $F^n z = z$ for $n = 1,2,3,\ldots$ and:

$$\begin{aligned} \psi(d(x,z)) &= \psi(d(F^n x, F^n z)) \\ &\leq \psi(M_\delta(F^{n-1}x, F^{n-1}z)) - \phi(M_\delta(F^{n-1}x, F^{n-1}z)) \\ &\leq \psi(M_\delta(x,z)) - \phi(M_\delta(x,z)). \end{aligned} \tag{24}$$

Where:

$$\begin{aligned} M_\delta(x,z) &= \max\{\delta(x,z), \delta(x,Fx), \delta(z,Fz), \delta(x,x), \delta(z,z), \frac{[\delta(x,Fz) + \delta(Fx,z)]}{2}\} \\ &= \max\{\delta(x,z), \delta(x,x), \delta(z,z), \frac{[\delta(x,z) + \delta(x,z)]}{2}\} \\ &= \delta(x,z). \end{aligned} \tag{25}$$

Using (24) and (25) , we have:

$$\psi(d(x,z)) \leq \psi(\delta(x,z)) - \phi(\delta(x,z)). \tag{26}$$

Similarly, we can show that:

$$\psi(\delta(x,z)) \leq \psi(d(x,z)) - \phi(d(x,z)). \tag{27}$$

Using (26) and (27) , we have:

$$\psi(d(x,z)) \leq \psi(d(x,z)) - \phi(d(x,z)).$$

Which is a contradiction unless $d(x,z) = 0$, this implies that $x = z$

Case 2: If x is not comparable to z, then there exists $y \in X$ comparable to x and z, the monotonicity of F implies that $F^n y$ is comparable to $F^n x = x$ and $F^n z = z$, for $n = 0,1,2,...$

Moreover,

$$\psi(d(z,F^n y)) = \psi(d(F^n z,F^n y)) \leq \psi(M_\delta(F^{n-1}z,F^{n-1}y)) - \phi(M_\delta(F^{n-1}z,F^{n-1}y)),$$

$$\psi(\delta(z,F^n y)) = \psi(\delta(F^n z,F^n y)) \leq \psi(M_d(F^{n-1}z,F^{n-1}y)) - \phi(M_d(F^{n-1}z,F^{n-1}y)). \tag{28}$$

Where:

$$\begin{aligned} M_d(F^{n-1}z,F^{n-1}y) &= \max\{d(F^{n-1}z,F^{n-1}y), d(F^{n-1}z,F^n z), d(F^{n-1}y,F^n y), d(F^{n-1}z,F^{n-1}z), \\ &\quad d(F^{n-1}y,F^{n-1}y), \frac{[d(F^{n-1}z,F^n y) + d(F^n z,F^{n-1}y)]}{2}\} \\ &= \max\{d(z,F^{n-1}y), d(z,z), d(F^{n-1}y,F^n y), d(F^{n-1}y,F^{n-1}y), \\ &\quad \frac{[d(z,F^n y) + d(z,F^{n-1}y)]}{2}\} \\ &\leq \max\{d(z,F^{n-1}y), d(z,F^n y)\}. \end{aligned} \tag{29}$$

Similarly, we can show that:

$$M_\delta(F^{n-1}z,F^{n-1}y) \leq \max\{\delta(z,F^{n-1}y), \delta(z,F^n y)\}. \tag{30}$$

For n sufficiently large, because $d(F^{n-1}y,F^{n-1}y) \to 0$ and $d(F^{n-1}y,F^n y) \to 0$ (resp.$\delta(F^{n-1}y,F^{n-1}y) \to 0$ and $\delta(F^{n-1}y,F^n y) \to 0$) when $n \to \infty$.

Similarly as in the proof of Theorem (2.1.), it can be shown that:

$$d(z,F^n y) \leq M_d(z,F^{n-1}y) \leq d(z,F^{n-1}y) \ (resp\ \delta(z,F^n y) \leq M_\delta(z,F^{n-1}y) \leq \delta(z,F^{n-1}y)).$$

It follows that the sequence $\{d(z,F^n y)\}$ (resp $\{\delta(z,F^n y)\}$ is nonnegative decreasing and, consequently, there exists $\alpha \geq 0$ (resp $\alpha' \geq 0$) such that:

$$\lim_{n\to\infty} d(z,F^n y) = \lim_{n\to\infty} M_d(z,F^{n-1}y) = \alpha \; (resp \lim_{n\to\infty} \delta(z,F^n y) = \lim_{n\to\infty} M_\delta(z,F^{n-1}y) = \alpha').$$

We suppose that $\alpha > 0$ (resp $\alpha' > 0$), then letting $n \to \infty$ in (28), we have:

$$\psi(\alpha) \leq \psi(\alpha') - \phi(\alpha'),$$

$$\psi(\alpha') \leq \psi(\alpha) - \phi(\alpha).$$

Which implies:

$$\psi(\alpha) \leq \psi(\alpha'),$$

$$\psi(\alpha') \leq \psi(\alpha) - \phi(\alpha).$$

Then:

$$\psi(\alpha) \leq \psi(\alpha) - \phi(\alpha).$$

Which is a contradiction. Hence $\alpha = 0$. Similarly, it can be proved that:

$$\lim_{n\to\infty} d(x,F^n y) = 0.$$

Now, passing to the limit in $d(x,z) \leq d(x,F^n y) + d(F^n y,z)$, it follows that $d(x,z) = 0$, so $x = z$, and the uniqueness of the fixed point is proved.

Now we present an example to support the useability of our results.

Example 2.1. Let $X = \{0,1,2\}$ and a partial order be defined as $x \preceq y$ whenever $y \preceq x$, and define $d : X \times X \to \mathbb{R}^+$ as follows:

$d(0,0) = 1,$ $\quad d(1,1) = 3,$ $\quad d(2,2) = 0,$

$d(1,0) = d(0,1) = 12,$ $\quad d(2,0) = d(0,2) = 7,$ $\quad d(2,1) = d(1,2) = 5.$

Then $(X,\preceq,d)$ is a complete partially ordered metric space.

And define $\delta : X \times X \to \mathbb{R}^+$ as follows:

$\delta(0,0) = 1,$ $\quad \delta(1,1) = 0,$ $\quad \delta(2,2) = 0,$

$\delta(1,0) = \delta(0,1) = 12,$ $\quad \delta(2,0) = \delta(0,2) = 10,$ $\quad \delta(2,1) = d(1,2) = 6.$

Then $(X,\preceq,\delta)$ is a complete partially ordered metric space.

Let $F : X \to X$ be defined by:

$F0 = 1,$ $\quad F1 = 2,$ $\quad F2 = 2.$

Define ψ , $\phi : [0,+\infty[\to [0,+\infty[$ by $\psi(t) = 2t$ and $\phi(t) = \frac{t}{2}$. We next verify that the function

F satisfies the two inequalities (1) and (2) . For that, given $x, y \in X$ with $x \preceq y$, so $y \preceq x$. Then we have the following cases.

Case 1:If $x = 1$, $y = 0$, then:

$d(F1, F0) = d(2,1) = 5, \qquad \delta(F1, F0) = \delta(2,1) = 6.$

And:

$$\begin{aligned} M_d(1,0) &= \max\{d(1,0), d(1,F1), d(0,F0), d(1,1), d(0,0), \frac{[d(1,F0)+d(F1,0)]}{2}\}, \\ &= \max\{12,5,12,3,1,\frac{3+7}{2}\} = 12. \end{aligned}$$

$$\begin{aligned} M_\delta(1,0) &= \max\{\delta(1,0), \delta(1,F1), \delta(0,F0), \delta(1,1), \delta(0,0), \frac{[\delta(1,F0)+\delta(F1,0)]}{2}\}, \\ &= \max\{12,6,12,0,1,\frac{0+10}{2}\} = 12. \end{aligned}$$

As:

$$\begin{cases} \psi(d(F1,F0) = 10 \leq 24 - \frac{12}{2} = 18 = \psi(M_\delta(1,0)) - \phi(M_\delta(1,0)), \\ \psi(\delta(F1,F0) = 12 \leq 24 - \frac{12}{2} = 18 = \psi(M_d(1,0)) - \phi(M_d(1,0)). \end{cases}$$

The inequality (1) and (2) are satisfied in this case.

Case 2: If $x = 2$, $y = 0$, then:

$d(F2, F0) = d(2,1) = 5, \qquad \delta(F2, F0) = \delta(2,1) = 6.$

And:

$$\begin{aligned} M_d(2,0) &= \max\{d(2,0), d(2,F2), d(0,F0), d(2,2), d(0,0), \frac{[d(2,F0)+d(F2,0)]}{2}\}, \\ &= \max\{7,0,12,0,1,\frac{5+7}{2}\} = 12. \end{aligned}$$

$$\begin{aligned} M_\delta(2,0) &= \max\{\delta(2,0), \delta(2,F2), \delta(0,F0), \delta(2,2), \delta(0,0), \frac{[\delta(2,F0)+\delta(F2,0)]}{2}\}, \\ &= \max\{10,0,12,0,1,\frac{6+10}{2}\} = 12. \end{aligned}$$

As:

$$\begin{cases} \psi(d(F2,F0) = 10 \leq 24 - \frac{12}{2} = 18 = \psi(M_\delta(2,0)) - \phi(M_\delta(2,0)), \\ \psi(\delta(F2,F0) = 12 \leq 24 - \frac{12}{2} = 18 = \psi(M_d(2,0)) - \phi(M_d(2,0)). \end{cases}$$

The inequality (1) and (2) are satisfied in this case.

Case 3: If $x=2$, $y=1$, then as $d(F2,F1)=0,\ \delta(F2,F1)=0$, $M_d(2,1)=5$ and $M_\delta(2,1)=6$, the inequality (1) and (2) are satisfied in this case.

Case 4: If $x=0$, $y=0$, then as $d(F0,F0)=3,\ \delta(F0,F0)=0$, $M_d(0,0)=12$ and $M_\delta(0,0)=12$, the inequality (1) and (2) are satisfied in this case.

Case 5: If $x=1$, $y=1$, then as $d(F1,F1)=0,\ \delta(F1,F1)=0$, $M_d(1,1)=5$ and $M_\delta(1,1)=6$, the inequality (1) and (2) are satisfied in this case.

Case 6: If $x=2$, $y=2$, then as $d(F2,F2)=0,\ \delta(F2,F2)=0$, $M_d(2,2)=0$ and $M_\delta(2,2)=0$, the inequality (1) and (2) are satisfied in this case.

So, F, ψ and ϕ satisfy all the hypotheses of Theorem (2.1). Therefore F has a unique fixed point. Here 2 is the unique fixed of F.

If we take $\psi(t)=t$ in Theorem (2.1) , we have the following corollary.

Corollary 2.1. *Let $(X,\leq,d,\delta)$ be a complete two partially ordered metric space; let $F:X\to X$ be a nondecreasing mapping such that for all comparable x,y in X with:*

$$\begin{cases} d(Fx,Fy)\leq M_\delta(x,y)-\phi(M_\delta(x,y)), \\ \delta(Fx,Fy)\leq M_d(x,y)-\phi(M_d(x,y)). \end{cases}$$

where:

$$\begin{aligned} M_d(x,y) &= \max\{d(x,y),d(x,Fx),d(y,Fy),d(x,x),d(y,y),\frac{[d(x,Fy)+d(Fx,y)]}{2}\}, \\ M_\delta(x,y) &= \max\{\delta(x,y),\delta(x,Fx),\delta(y,Fy),\delta(x,x),\delta(y,y),\frac{[\delta(x,Fy)+\delta(Fx,y)]}{2}\}, \end{aligned}$$

$\phi:[0,+\infty[\to[0,+\infty[$ is a lower semi-continuous function with $\phi(t)=0$ if and only if $t=0$. If there exists $x_0\in X$ with $x_0\preceq Fx_0$ and one of the following two conditions is satisfied:

- **(i):** *F is a continuous in (X,d,δ).*
- **(ii):** *$\{x_n\}$ is a nondecreasing sequence in X such that $x_n\to x\in X$ implies $x_n\leq x$ for all $n\in\mathbb{N}$.*

Then F has a fixed point. Moreover, if the following condition is satisfied: For arbitrary two points x,y in X , there exists $z\in X$ which is comparable with both x and y , then the fixed point of F is unique.

If we take $\phi(t)=(1-k)t$ for $k\in[0,1[$ in Corollary (2.1), we have the following corollary:

Corollary 2.2. *Let $(X,\leq,d,\delta)$ be a complete two partially ordered metric space ;let $F:X\to X$ be a nondecreasing mapping such that for all comparable x,y in X with:*

$$\begin{cases} d(Fx,Fy)\leq kM_{\delta}(x,y),\\ \delta(Fx,Fy)\leq kM_{d}(x,y),\end{cases}$$

where:

$$\begin{aligned} M_d(x,y) &= \max\{d(x,y),d(x,Fx),d(y,Fy),d(x,x),d(y,y),\frac{[d(x,Fy)+d(Fx,y)]}{2}\},\\ M_{\delta}(x,y) &= \max\{\delta(x,y),\delta(x,Fx),\delta(y,Fy),\delta(x,x),\delta(y,y),\frac{[\delta(x,Fy)+\delta(Fx,y)]}{2}\},\end{aligned}$$

And $k\in[0,1[$. If there exists $x_0\in X$ with $x_0\preceq Fx_0$ and one of the following two conditions is satisfied:

- **(i):** *F is a continuous in (X,d,δ).*
- **(ii):** *$\{x_n\}$ is a nondecreasing sequence in X such that $x_n\to x\in X$ implies $x_n\leq x$ for all $n\in\mathbb{N}$.*

Then F has a fixed point. Moreover, if the following condition is satisfied: For arbitrary two points x,y in X , there exists $z\in X$ which is comparable with both x and y , then the fixed point of F is unique.

Corollary 2.3. *Let $(X,\leq,d,\delta)$ be a complete two partially ordered metric space; let $F:X\to X$ be a nondecreasing mapping such that for all comparable x,y in X with:*

$$\begin{cases} d(Fx,Fy)\leq \delta(x,y)-\phi(M_{\delta}(x,y)),\\ \delta(Fx,Fy)\leq d(x,y)-\phi(M_{d}(x,y)),\end{cases}$$

where:

$\phi:[0,+\infty[\to[0,+\infty[$ is a lower semi-continuous function with $\phi(t)=0$ if and only if $t=0$.

If there exists $x_0\in X$ with $x_0\preceq Fx_0$ and one of the following two conditions is satisfied:

- **(i):** *F is a continuous in (X,d,δ).*
- **(ii):** *$\{x_n\}$ is a nondecreasing sequence in X such that $x_n\to x\in X$ implies $x_n\leq x$ for all $n\in\mathbb{N}$.*

Then F has a fixed point. Moreover, if the following condition is satisfied:For arbitrary two points x,y in X , there exists $z\in X$ which is comparable with both x and y , then the fixed point of F is unique.

Remark. So our results can be viewed as the generalization and extension of corresponding results in [1, 5, 12] and several other comparable results.

3. Application

De note by Λ the set of functions $\alpha : [0,+\infty[\to [0,+\infty[$ satisfying the following hypotheses:

(h_1): : α is Lebesgue-integrable mapping on each compact subset of $[0,+\infty[$

(h_2): : For every $\varepsilon > 0$, we have:

$$\int_0^{\varepsilon} \alpha(s)ds > 0.$$

We have the following results.

Corollary 3.1. Let $(X,\leq,d,\delta)$ be a complete two partially ordered metric space; let $F : X \to X$ be a nondecreasing mapping such that for all comparable x,y in X with:

$$\begin{cases} \int_0^{d(Fx,Fy)} \alpha_1(s)ds \leq \int_0^{M_\delta(x,y)} \alpha_1(s)ds - \int_0^{M_\delta(x,y)} \alpha_2(s)ds, \\ \int_0^{\delta(Fx,Fy)} \alpha_1(s)ds \leq \int_0^{M_d(x,y)} \alpha_1(s)ds - \int_0^{M_d(x,y)} \alpha_2(s)ds, \end{cases}$$

where: $\alpha_1,\alpha_2 \in \Lambda$. If there exists $x_0 \in X$ with $x_0 \preceq Fx_0$, Then F has a fixed point.

Proof. Follows from Theorem (2.1) by taking $\psi(t) = \int_0^t \alpha_1(s)ds$ and $\phi(t) = \int_0^t \alpha_2(s)ds$.

Corollary 3.2. *Under the same hypotheses of Corollary (3.1) and without assuming the continuity of F, assume that whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x \in X$ implies $x_n \leq x$ for all $n \in \mathbb{N}$, then F has a fixed point in X.*

Proof. Follows from Theorem (2.2) by taking $\psi(t) = \int_0^t \alpha_1(s)ds$ and $\phi(t) = \int_0^t \alpha_2(s)ds$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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