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UNIQUE COMMON FIXED POINTS FOR PAIRS OF MULTI-VALUED MAPPINGS IN PARTIAL METRIC SPACES

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Abstract. In this paper, we obtain a unique common fixed point theorems for pairs of multi-valued non-self mappings on a partial Hausdorff metric space without using any continuity or commutativity of the mappings. In doing so, we generalize a theorem by Rao and Rao.

Keywords: partial Hausdorff metric; multi-valued mapping; common fixed points; partial metric space; non-self mapping.

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1. Introduction

In 1969, Nadler [9] introduced the study of fixed points using the Hausdorff metric for multi-valued mappings. Aydi et al. [3] came up with the concept of the partial Hausdorff metric and used it to prove Nadler's theorem on partial metric spaces. Rao and Rao [10] proved a fixed point theorem for a multi-valued self mapping from a partial Hausdorff space into the family of closed and bounded subsets of its partial metric space.

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Using the procedure described by Assad and Kirk [2], we extend the theorem by Rao and Rao [10] to apply to a pair of non-self multi-valued mappings.

2. Preliminaries

We now introduce preliminaries which will be of use in this paper.

Definition 2.1 [8] *A partial metric on a non-empty set X is a mapping $p : X \times X \to [0,+\infty)$, such that for all $x,y,z \in X$.*

P0: $0 \leq p(x,x) \leq p(x,y)$,

P1: $x = y$ *if and only if* $p(x,x) = p(x,y) = p(y,y)$,

P2: $p(x,y) = p(y,x)$ *and*

P3: $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

The pair (X, p) is said to be a partial metric space.

From Definition 2.1, we deduce the following:

$$p(x,y) = 0 \Rightarrow x = y. \tag{2.1}$$

Proof. If $p(x,y) = 0$, then $p(x,x) = 0$ because $0 \leq p(x,x) \leq p(x,y)$ from P0. Similarly, $p(x,y) = 0$ implies $p(y,y) = 0$ because $0 \leq p(y,y) \leq p(x,y)$. Hence $p(x,y) = 0$ implies $p(x,x) = p(x,y) = p(y,y) = 0$. From P1 this means that $x = y$.

From P3, we infer that

$$p(x,y) \leq p(x,z) + p(z,y). \tag{2.2}$$

Example 2.1 *Let $X = \mathbb{R}^+$ and let $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $p(x,y) = \max\{x,y\}$. Then (X, p) is a partial metric space.*

Each partial metric p on X generates a T_0 topology τ_p on X with a base being the family of open balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Definition 2.2 [8] *Let (X, p) be a partial metric space and $\{x_n\}$ be a sequence in X. Then*

(i) $\{x_n\}$ converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to+\infty} p(x,x_n)$.

(ii) $\{x_n\}$ is called a Cauchy sequence if only if there exists (and is finite) $\lim_{n,m\to+\infty} p(x_n,x_m)$.

(iii) A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that

$$p(x,x) = \lim_{n,m\to+\infty} p(x_n,x_m).$$

Lemma 2.1 [8] *If p is a partial metric on X, then the mapping $p^s : X \times X \to [0,+\infty)$ given by*

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{2.3}$$

defines a metric on X.

In this paper, we denote p^s as the metric derived from the partial metric p.

Lemma 2.2 [8]

(a)$\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).

(b) (X,p) is complete if and only if (X,p^s) is complete. Furthermore $\lim_{n\to+\infty} p(x_n,x) = 0$ *if and only if*

$$p(x,x) = \lim_{n\to+\infty} p(x_n,x) = \lim_{n,m\to+\infty} p(x_n,x_m) = 0.$$

It is easy to see that every closed subset of a complete partial metric space is complete [6].

We define a metrically convex metric space.

Definition 2.3 [2] *A complete metric space (X,d) is said to be (metrically) convex if X has the property that for each $x,y \in X$ with $x \neq y$ there exists $z \in X, x \neq z \neq y$, such that $d(x,z) + d(z,y) = d(x,y)$.*

If (X,d) is a metrically convex metric space, and $x,y \in X$, we term

$$seg[x,y] := \{z \in X : d(x,y) = d(x,z) + d(z,y)\}. \tag{2.4}$$

We get the following lemma from Assad and Kirk [2].

Lemma 2.3 [2] *Let C be a closed subset of the complete and convex metric space X. If $x \in C$ and $y \notin C$, then there exists a point $z \in \partial C$ (the boundary of C) such that*

$$d(x,z) + d(z,y) = d(x,y).$$

Using (2.4), we can rephrase Lemma 2.3 as follows:

Lemma 2.4 *Let C be a closed subset of the complete and convex metric space X. If $x \in C$ and $y \notin C$, then there exists a point $z \in \partial C$ (the boundary of C) such that $z \in \operatorname{seg}[x,y]$.*

Now, we introduce the metrically convex partial metric space.

Definition 2.4 *A partial metric space (X,p) is said to be metrically convex if the corresponding metric space (X,p^s) is metrically convex in the sense of Lemma 2.1, where $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ for all $x,y \in X$.*

As an example, the partial metric space $(\mathbb{R}^+, p)$ where $p(x,y) = \max\{x,y\}$ for all $x,y \in \mathbb{R}^+$ is metrically convex because (X,p^s) where $p^s(x,y) = |x-y|$ is the metric derived from the partial metric p, is metrically convex.

Lemma 2.5 *Let (X,p) be a metrically convex partial metric space. Let $x,y \in X$. If $z \in \operatorname{seg}[x,y]$ then:*

(i) $p(x,y) = p(x,z) - p(z,z) + p(z,y)$,

(ii) $p(x,y) \geq p(x,z)$.

Proof. *Applying (2.3) to Definition 2.3, if $z \in seg[x,y]$, then we have*

$$
\begin{aligned}
p^s(x,y) &= p^s(x,z) + p^s(z,y) \\
&\Rightarrow 2p(x,y) - p(x,x) - p(y,y) = 2p(x,z) - p(x,x) - p(z,z) \\
&\qquad + 2p(z,y) - p(z,z) - p(y,y) \\
&\Rightarrow p(x,y) = p(x,z) - p(z,z) + p(z,y).
\end{aligned}
$$

As $\big(-p(z,z) + p(z,y)\big) \geq 0$, from P2 of Definition 2.1, we have $p(x,y) \geq p(x,z)$.

This completes the proof.

Lemma 2.6 *Let C be a non-empty subset of a metrically convex partial metric space (X,p) which is closed in (X,p^s). If $x \in C$ and $y \in X \backslash C$, then there exists a point $z \in \partial C$ (the boundary of C with respect to (X,p^s)) such that*

$$p(x,y) + p(z,z) = p(x,z) + p(z,y).$$

Proof. From Definition 2.4, if the partial metric space (X,p) is metrically convex, then (X,p^s) is metrically convex. From Lemma 2.3, this means that if $x \in C$ and $y \in X \backslash C$ then there exists z

in ∂C, (the boundary of C), such that ${}^s(x,y) = p^s(x,z) + p^s(z,y)$. Using (2.3), this means

$$\begin{aligned} p^s(x,y) &= p^s(x,z) + p^s(z,y) \\ \Rightarrow 2p(x,y) - p(x,x) - p(y,y) &= 2p(x,z) - p(x,x) - p(z,z) \\ &\quad + 2p(z,y) - p(z,z) - p(y,y) \\ \Rightarrow 2p(x,y) &= 2p(x,z) + 2p(z,y) - 2p(z,z) \\ \Rightarrow p(x,y) + p(z,z) &= p(x,z) + p(z,y) \\ \Rightarrow p(x,z) + p(z,y) &= p(x,y) + p(z,z). \end{aligned}$$

This completes the proof.

3. The Partial Hausdorff Metric

Now, we describe the partial Hausdorff metric.

Let CB^p be a family of all non-empty, closed and bounded subsets of a partial metric space (X,p), induced by the partial metric p. The set A is said to be a bounded subset in (X,p) if there exists $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_p(x_0, M)$.

Definition 3.1 [3] *For all $A,B \in CB^p(X)$ and $x \in X$, we define*

(i) $p(x,A) = \inf\{p(x,a), a \in A\}$,

(ii) $\delta_p(A,B) = \sup\{p(a,B) : a \in A\}$,

(iii) $\delta_p(B,A) = \sup\{p(b,A) : b \in B\}$,

(iv) $H_p(A,B) = \max\{\delta_p(A,B), \delta_p(B,A)\}$.

The mapping $H_p : CB^p \times CB^p \to [0,+\infty)$ is called the partial Hausdorff metric.

Remark 3.1 [3] *Let (X,p) be a partial metric space and A any non-empty set in (X,p), then $a \in \bar{A}$ if and only if $p(a,A) = p(a,a)$, where $\bar{A}$ denotes the closure of A with respect to the partial metric p.*

We now state some properties of mappings δ_p and H_p.

Lemma 3.1 [3] *Let (X,p) be a partial metric space. For any*
$A,B \in CB^p(X)$ we have

(i) $\delta_p(A,A) = \sup\{p(a,a) : a \in A\}$;

(ii) $\delta_p(A,A) \leq \delta(A,B)$;

(iii) $\delta_p(A,B) = 0$ *implies that* $A \subseteq B$;

(h1) $H_p(A,A) \leq H_p(A,B)$;

(h2) $H_p(A,B) = H_p(B,A)$;

(h3) $H_p(A,B) = 0$ *implies* $A = B$.

We will utilize the following lemma in our proofs.

Lemma 3.2 [3] *Let* (X,p) *be a partial metric space,* $A,B \in CB^p(X)$ *and* $K > 1$. *For any* $a \in A$, *there exists* $b = b(a) \in B$ *such that*

$$p(a,b) \leq KH_p(A,B).$$

The following definitions will be used in the course of our proofs.

Let $T : C \to X$ be a multi-valued mapping, where $C \subseteq X$. We say that T is a *self mapping* if $C = X$, otherwise T is called a *non-self mapping*. If there is an element $x \in C$ such that $x \in Tx$, we say that x is a *fixed point* of T in X.

Suppose we have two multi-valued mappings $S,T : C \to X$, with $C \subseteq X$. If there is an element $x \in C$ such that $x \in (Sx \cap Tx)$ then we call x a *common fixed point* of S and T in X.

We now prove the following lemma, which is modified from Theorem 1 of Assad and Kirk [2], as it is necessary for our work.

Lemma 3.3 *Consider a sequence* $\{w_n\}_{n\in\mathbb{N}} \in \mathbb{R}_+$ *such that, for all* $n \geq 2$ *we have*

$$w_n \leq k\max\{w_{n-2}, w_{n-1}\}, k \in (0,1), \tag{3.1}$$

then

$$w_n \leq k^{n/2}k^{-1/2}\max\{w_0, w_1\}. \tag{3.2}$$

Proof. We prove the lemma by the induction. First we show that Lemma 3.3 holds for $n = 2$.

We note that $k \in (0,1)$ implies $k < k^{1/2}$. Hence if $n = 2$, then (3.1) leads to

$$w_2 \leq k\max\{w_0, w_1\} \leq k^{1/2}\max\{w_0, w_1\} = k^{2/2}k^{-1/2}\max\{w_0, w_1\}. \tag{3.3}$$

We then show that the lemma holds for $n = 3$. If $n = 3$, then (3.1) leads to $w_3 \leq k\max\{w_1, w_2\}$. If $w_1 \geq w_2$, then we get

$$\begin{aligned} w_3 &\leq k\max\{w_1, w_2\} \\ \Rightarrow w_3 &\leq kw_1 \\ &\leq k\max\{w_0, w_1\} \\ &= k^{3/2} \cdot k^{-1/2}\max\{w_0, w_1\}. \end{aligned}$$

If however $w_1 < w_2$, we get

$$\begin{aligned} w_3 &\leq k\max\{w_1, w_2\} \\ \Rightarrow w_3 &\leq kw_2 \\ \Rightarrow w_3 &\leq k \times k^{2/2}k^{-1/2}\max\{w_0, w_1\}, \text{ from (3.3)} \\ &\leq k^{3/2}\max\{w_0, w_1\} \\ &\leq k^{3/2} \cdot k^{-1/2}\max\{w_0, w_1\}, \text{ because } k^{-1/2} \geq 1. \end{aligned}$$

We now show that, if Lemma 3.3 holds for $1 \leq n \leq j$ where $j \geq 2$, then it must be hold for $j+1$. Hence we have from (3.1)

$$w_{j+1} \leq k\max\{w_{j-1}, w_j\}. \tag{3.4}$$

We consider two cases.

Case (i): Suppose $w_{j-1} \leq w_j$. Then (3.4) becomes

$$\begin{aligned} w_{j+1} &\leq kw_j \\ &\leq k \cdot k^{j/2}k^{-1/2}\max\{w_0, w_1\} \text{ from (3.2)} \\ &= k^{(j+2)/2}k^{-1/2}\max\{w_0, w_1\}. \end{aligned} \tag{3.5}$$

Case (ii): Suppose $w_{j-1} > w_j$. Then (3.4) becomes

$$\begin{aligned} w_{j+2} &\leq kw_{j-1} \\ &\leq k \cdot k^{(j-1)/2}k^{-1/2}\max\{w_0, w_1\} \text{ from (3.2)} \\ &= k^{(j+1)/2}k^{-1/2}\max\{w_0, w_1\}. \end{aligned} \tag{3.6}$$

We note that for $j \geq 2$ and $k \in (0,1)$ we have $k^{(j+1)/2} > k^{(j+2)/2}$. Hence (3.5) and (3.6) imply that

$$w_{j+1} \leq k^{(j+1)/2} k^{-1/2} \max\{w_0, w_1\}.$$

This completes the proof.

Aydi et al. proved the following theorem.

Theorem 3.1. [3] *Let (X,p) be a complete partial metric space. If $T: X \to CB^p(X)$ is a multi-valued mapping such that for all $x, y \in X$ we have*

$$H_p(Tx, Ty) \leq kp(x,y), \tag{3.7}$$

where $k \in (0,1)$, then T has a fixed point.

4. Main Results

We start by proving an extension of Theorem 3.1 which will then be used to establish Theorem 4.3.

Theorem 4.1 *Let (X,p) be a complete metrically convex partial metric space and C a non-empty closed subset of X, the closure being with respect to (X,p^s). Let ∂C, the boundary of C with respect to (X,p^s), be non-empty. Let $S,T: C \to CB^p(X)$ be multi-valued mappings such that for all $x,y \in C$ we have*

$$H_p(Tx, Sy) \leq kp(x,y), \tag{4.1}$$

where $k \in (0, \frac{1}{4})$. Furthermore, let $x \in \partial C$ imply $Tx \subset C$ and $Sx \subset C$. Then there exists a common fixed point $x^\star$ of S and T in C and $p(x^\star, x^\star) = 0$.

Proof. We commence with an arbitrary $x_0 \in \partial C$. This implies from the assumption that we can choose $x_1 \in Tx_0 \subset C$. By Lemma 3.2 with $K = \dfrac{1}{\sqrt{k}}$, there exists $y_2 \in Sx_1$ such that

$$p(x_1, y_2) \leq \frac{1}{\sqrt{k}} H_p(Tx_0, Sx_1). \tag{4.2}$$

If $y_2 \in C$, we set $x_2 = y_2$. Thus (4.2) becomes

$$p(x_1, x_2) \leq \frac{1}{\sqrt{k}} H_p(Tx_0, Sx_1). \tag{4.3}$$

By (4.1), we have $H_p(Tx_0,Sx_1) \leq kp(x_0,x_1)$. This means

$$p(x_1,x_2) \leq \sqrt{k}p(x_0,x_1).$$

If $y_2 \notin C$, then by Lemma 2.4, there is $x_2 \in \partial C$ such that $x_2 \in \text{seg}[x_1,y_2]$. Using Lemma 2.5 (ii), we get

$$\begin{aligned} p(x_1,x_2) &\leq p(x_1,y_2) \\ &= p(y_1,y_2), \text{ because } x_1 = y_1 \\ &\leq \frac{1}{\sqrt{k}}H_p(Tx_0,Sx_1) \\ &\leq \sqrt{k}p(x_0,x_1). \end{aligned}$$

Continuing in this way, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way, using $K = \frac{1}{\sqrt{k}} > 1$:

(4.i) $x_0 \in \partial C, y_1 \in Tx_0 \subset C$.

(4.ii) For all $n \geq 1, y_{2n} \in Sx_{2n-1}, y_{2n+1} \in Tx_{2n}$.

(4.iii) Here we apply Lemma 3.2. For all $n \geq 1$, we choose y_{2n+1} such that $p(y_{2n+1},y_{2n}) \leq \frac{1}{\sqrt{k}}H_p(Tx_{2n},Sx_{2n-1})$. Similarly we choose y_{2n+2} such that $p(y_{2n+1},y_{2n+2}) \leq \frac{1}{\sqrt{k}}H_p(Tx_{2n},Sx_{2n+1})$.

(4.iv) For all $n \geq 1$, if $y_n \in C$, then $x_n = y_n$. However if $y_n \notin C$, then applying Lemma 2.4, we choose $x_n \in \partial C$ such that $x_n \in \text{seg}[x_{n-1},y_n]$.

Let us partition the elements in the sequence $\{x_n\}$ into two sets P and Q, where $P = \{x_i \in \{x_n\} : x_i = y_i\}$ and $Q = \{x_i \in \{x_n\} : x_i \neq y_i\}$.

We consider the following cases

Case 4.1 Consider the case where $(x_n,x_{n+1}) \in P \times P, n \geq 1$. Suppose n is even, that is $n = 2m$ for some $m \in \mathbb{N}$. Then, from (4.iv) we have $x_{2m} = y_{2m}$ and $x_{2m+1} = y_{2m+1}$. Applying (4.iii) we have

$$\begin{aligned}p(x_{2m},x_{2m+1}) &= p(y_{2m},y_{2m+1})\\ &= p(y_{2m+1},y_{2m})\\ &\leq \frac{1}{\sqrt{k}}H_p(Tx_{2m},Sx_{2m-1}), \text{ by (4.iii)}\\ &\leq \frac{1}{\sqrt{k}}\times kp(x_{2m},x_{2m-1}) \text{ by (4.1)}\\ &= \sqrt{k}p(x_{2m-1},x_{2m}).\end{aligned}$$

Using a similar argument, when n is odd, that is, when $n = 2m+1$ for some $m \in \mathbb{N}$, we get

$$p(x_{2m+1},x_{2m+2}) \leq \sqrt{k}p(x_{2m},x_{2m+1}).$$

Thus in general, when $(x_n,x_{n+1}) \in P\times P, n \geq 1$, we have

$$p(x_n,x_{n+1}) \leq \sqrt{k}p(x_{n-1},x_n). \tag{4.4}$$

Case 4.2 Let us now consider the situation where $(x_n,x_{n+1}) \in P\times Q$, $n \geq 1$. Suppose n is even, that is $n = 2m$ for some $m \in \mathbb{N}$. Then, from (4.iv) we have $x_{2m} = y_{2m}$.

We also have $x_{2m+1} \in \partial C$ and $x_{2m+1} \in \text{seg}[y_{2m},y_{2m+1}]$. From Lemma 2.5 (ii), we note that $p(x_{2m},x_{2m+1}) = p(y_{2m},x_{2m+1}) \leq p(y_{2m},y_{2m+1})$. Applying (4.iii) we have

$$\begin{aligned}p(x_{2m},x_{2m+1}) &\leq p(y_{2m},y_{2m+1})\\ &\leq \sqrt{k}p(x_{2m-1},x_{2m}),\end{aligned}$$

using the argument in Case 4.1.

Using a similar procedure, we can show that

$$p(x_{2m+1},x_{2m+2}) \leq \sqrt{k}p(x_{2m},x_{2m+1}).$$

In general, when $(x_n,x_{n+1}) \in P\times Q, n \geq 1$, we have

$$p(x_n,x_{n+1}) \leq \sqrt{k}p(x_{n-1},x_n). \tag{4.5}$$

Case 4.3 We consider the situation where $(x_n,x_{n+1}) \in Q\times P$, $n \geq 1$. In this case, we can show by contradiction that $x_{n-1} \in P$.

We assume $x_{n-1} \in Q$. This implies $x_{n-1} \in \partial C$. This in turn implies that $x_n = y_n \in Tx_{n-1} \subset C$, implying $x_n \in P$, which is a contradiction. Hence $x_{n-1} \in P$, implying $x_{n-1} = y_{n-1}$.

Let us consider when n is even, that is $n = 2m$ for some $m \in \mathbb{N}$. Then, from (4.iv), we have $x_{2m+1} = y_{2m+1}$. We also have $x_{2m} \in \partial C$ and $x_{2m} \in \mathrm{seg}[y_{2m-1}, y_{2m}]$. Hence

$$\begin{aligned}
p(x_{2m}, x_{2m+1}) &= p(x_{2m}, y_{2m+1}) \\
&\leq p(x_{2m}, y_{2m}) + p(y_{2m}, y_{2m+1}), \text{ according to (2.2),} \\
&\leq p(y_{2m-1}, y_{2m}) + p(y_{2m}, y_{2m+1}), \text{ using Lemma 2.5 (ii),} \\
&\leq \frac{1}{\sqrt{k}} H_p(Tx_{2m-2}, Sx_{2m-1}) + \frac{1}{\sqrt{k}} H_p(Sx_{2m-1}, Tx_{2m}), \text{ by (4.iii)} \\
&= \frac{1}{\sqrt{k}} H_p(Tx_{2m-2}, Sx_{2m-1}) + \frac{1}{\sqrt{k}} H_p(Tx_{2m}, Sx_{2m-1}) \\
&\leq \frac{1}{\sqrt{k}} \times k\big(p(x_{2m-2}, x_{2m-1}) + p(x_{2m}, x_{2m-1})\big), \text{ by (4.1)} \\
&= \sqrt{k}\big(p(x_{2m-2}, x_{2m-1}) + p(x_{2m-1}, x_{2m})\big) \\
&\leq 2\sqrt{k} \max\{p(x_{2m-2}, x_{2m-1}), p(x_{2m-1}, x_{2m})\}.
\end{aligned}$$

We get a similar result when n is odd.

In general, when $(x_n, x_{n+1}) \in Q \times P$, and $n \geq 2$, then we have

$$p(x_n, x_{n+1}) \leq 2\sqrt{k} \max\big\{p(x_{n-2}, x_{n-1}), p(x_{n-1}, x_n)\big\}. \tag{4.6}$$

The case where $(x_n, x_{n+1}) \in Q \times Q$ is not possible.

Thus in all cases, according to (4.4), (4.5) and (4.6), we have

$$p(x_n, x_{n+1}) \leq t \max\big\{p(x_{n-2}, x_{n-1}), p(x_{n-1}, x_n)\big\}, \tag{4.7}$$

where $t = 2\sqrt{k} < 1$, implying $k < \dfrac{1}{4}$.

According to Lemma 3.3, (4.7) implies

$$p(x_n, x_{n+1}) \leq t^{n/2}\delta, \tag{4.8}$$

where $\delta = t^{-1/2} \max\{p(x_0, x_1), p(x_1, x_2)\}$.

Consider $n,m \in \mathbb{N}$ with $n > m$. Then, we have inductively from (2.2)

$$\begin{aligned} p(x_m,x_n) &\leq \sum_{i=m}^{n-1} p(x_i,x_{i+1}) \\ &\leq \sum_{i=m}^{n-1} t^{i/2} t^{-1/2}\delta \\ &\leq t^{-1/2}\delta \sum_{i=m}^{+\infty} t^{i/2} \\ &= \delta \frac{t^{m/2}}{1-t^{1/2}} t^{-1/2}. \end{aligned}$$

As $m,n \to +\infty$ we get

$$\lim_{m,n\to+\infty} p(x_m,x_n) = 0 < +\infty.$$

From Definition 2.2 (ii), this shows that the sequence $\{x_n\} \in C$ is a Cauchy sequence. Because C is closed in (X,p^s), it is complete in (X,p^s) and hence is complete in (X,p).

This means, according to Lemma 2.2, there is $x^\star \in C$ such that

$$\lim_{m,n\to+\infty} p(x_m,x_n) = \lim_{n\to+\infty} p(x^\star,x_n) = p(x^\star,x^\star) = 0.$$

We now show that $x^\star$ is a fixed point of S and T.

Consider a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ for which each $x_{n_j} \in P$. If n_j is odd, that is $n_j = 2m_j+1$, then we have from the assumption, $H_p(Tx^\star, Sx_{2m_j+1}) \leq kp(x^\star,x_{2m_j+1})$. This implies

$$\lim_{j\to+\infty} H_p(Tx^\star, Sx_{2m_j+1}) = p(x^\star,x^\star) = 0. \tag{4.9}$$

Now consider n_j being an even number, that is $n_j = 2m_j$ for some m_j. Because $x_{2m_j} \in Sx_{2m_j-1}$, we have

$$p(Tx^\star, x_{2m_j}) \leq \delta_p(Tx^\star, Sx_{2m_j-1}) \leq H_p(Tx^\star, Sx_{2m_j-1}). \tag{4.10}$$

Taking $j \to +\infty$ in (4.10) and applying (4.9), we get

$$\begin{aligned} \lim_{j\to+\infty} p(Tx^\star, x_{2m_j}) &\leq \lim_{j\to+\infty} H_p(Tx^\star, Sx_{2m_j-1}) = 0 \\ &\Rightarrow p(Tx^\star,x^\star) = 0 = p(x^\star,x^\star) \\ &\Rightarrow x^\star \in Tx^\star. \end{aligned} \tag{4.11}$$

This shows that $x^\star$ is a fixed point of T. Using a similar argument we conclude that $x^\star$ is also a fixed point of S.

Rao and Rao [10] proved the following fixed point theorem (Theorem 2.8) involving the Hausdorff partial metric for a pair of multi-valued self mappings.

Theorem 4.2. [10] *Let (X,p) be a complete partial metric space and let $S,T : X \to CB^p(X)$ be mappings satisfying*

$$H_p(Sx,Ty) \leq \alpha \max\left\{p(x,y), p(x,Sx), p(y,Ty), \frac{1}{2}[p(x,Ty)+p(y,Sx)]\right\}$$

for all $x,y \in X$ and $0<\alpha<1$. Then S and T have a common fixed point in X. Further, if we assume that $p(x,y) \leq p(y,Sx)$ or $p(x,y) \leq p(y,Tx)$ for all $x,y \in X$, then S and T have a unique common fixed point in X.

In this research, we modify the Theorem 4.2 so that it applies to a pair of non-self multi-valued mappings in a metrically convex partial metric space.

We provide a proof for the following assumption.

Theorem 4.3. *Let (X,p) be a complete metrically convex partial metric space and C a non-empty closed subset of X, the closure being with respect to (X,p^s). Let ∂C, the boundary of C with respect to (X,p^s), be non-empty. Let $S,T : C \to CB^p(X)$ be mappings satisfying*

$$H_p(Sx,Ty) \leq \alpha \max\left\{p(x,y), p(x,Sx), p(y,Ty), \frac{1}{2}[p(x,Ty)+p(y,Sx)]\right\}$$

for all $x,y \in X$ and $0<\alpha<\frac{1}{4}$. Let the following conditions apply:
(i) $x \in \partial C$ implies $Tx \subset C$,
(ii) $x \in \partial C$ implies $Sx \subset C$.
Then S and T have a common fixed point in X. Further, if we assume that $p(x,y) \leq p(y,Sx)$ or $p(x,y) \leq p(y,Tx)$ for all $x,y \in X$, then S and T have a unique common fixed point z in C with $p(z,z)=0$.

Proof. We construct sequences $\{x_n\} \in C$ and $\{y_n\} \in X$ in the following way.

We commence by choosing an arbitrary $x_0 \in \partial C$. According to (i), we choose $x_1 \in C$ such that $x_1 \in Tx_0$. We set $y_1 = x_1$. Because $\alpha \in \left(0,\frac{1}{4}\right)$ implies $\frac{1}{\sqrt{\alpha}} > 1$, by Lemma 3.2, there exists

$y_2 \in Sx_1$ such that

$$p(y_1, y_2) \leq \frac{1}{\sqrt{\alpha}} H_p(Tx_0, Sx_1).$$

If $y_2 \in C$, then we set $x_2 = y_2$.

If however $y_2 \notin C$, then, according to Lemma 2.4, there is $x_2 \in \partial C$ such that $x_2 \in \text{seg}[x_1, y_2]$. Using Lemma 3.2, and recalling that $y_2 \in Sx_1$, we choose $y_3 \in Tx_2$ such that

$$p(y_3, y_2) \leq \frac{1}{\sqrt{\alpha}} H_p(Tx_2, Sx_1).$$

From (i) in the assumption, we have $y_3 \in C$.

In general, the sequences $\{x_n\} \in C$ and $\{y_n\}_{n\geq 1} \in X$ are constructed in the same way as we did when proving Theorem 4.1.

We partition the elements of $\{x_n\}$ into sets P and Q such that $P = \big\{x_i \in \{x_n\} : x_i = y_i\big\}$ and $Q = \big\{x_i \in \{x_n\} : x_i \neq y_i\big\}$.

Now for $n \geq 2$, we consider the following cases.

Case 4.4 Consider $x_n \in P \times P$. This means $x_n = y_n$.

If n is even, that is, if $n = 2m$ for some $m \in \mathbb{N}$, we have $x_n = x_{2m} = y_{2m}$. As $x_{2m} = y_{2m} \in Sx_{2m-1}$, from (4.ii), we can choose $y_{2m+1} \in Tx_{2m}$ such that

$$p(x_{2m}, y_{2m+1}) \leq \frac{1}{\sqrt{\alpha}} H_p(Sx_{2m-1}, Tx_{2m}). \tag{4.12}$$

We consider two scenarios.

(4.4.1) If $y_{2m+1} \in P$, then $x_{2m+1} = y_{2m+1}$. Hence, (4.12) becomes

$$\begin{aligned}
p(x_{2m}, x_{2m+1}) &= p(y_{2m}, y_{2m+1}) \\
&\leq \frac{1}{\sqrt{\alpha}} H_p(Sx_{2m-1}, Tx_{2m}) \\
&\leq \frac{1}{\sqrt{\alpha}} \times \alpha \max\Big\{ p(x_{2m-1}, x_{2m}), p(x_{2m-1}, Sx_{2m-1}), \\
&\qquad p(x_{2m}, Tx_{2m}), \frac{1}{2}[p(x_{2m-1}, Tx_{2m}) + p(x_{2m}, Sx_{2m-1})] \Big\} \\
&\leq \sqrt{\alpha} \max\Big\{ p(x_{2m-1}, x_{2m}), p(x_{2m-1}, y_{2m}), p(x_{2m}, y_{2m+1}), \\
&\qquad \frac{1}{2}[p(x_{2m-1}, y_{2m+1}) + p(x_{2m}, y_{2m})] \Big\}
\end{aligned}$$

$$= \sqrt{\alpha}\max\Big\{p(x_{2m-1},x_{2m}),p(x_{2m-1},x_{2m}),$$
$$p(x_{2m},x_{2m+1}),\frac{1}{2}[p(x_{2m-1},x_{2m+1})+p(x_{2m},x_{2m})]\Big\}$$
$$\Rightarrow p(x_{2m},x_{2m+1}) \le \sqrt{\alpha}\max\Big\{p(x_{2m-1},x_{2m}),\frac{1}{2}[p(x_{2m-1},x_{2m})+p(x_{2m},x_{2m+1})]\Big\}. \tag{4.13}$$

If $p(x_{2m-1},x_{2m}) < \frac{1}{2}[p(x_{2m-1},x_{2m})+p(x_{2m},x_{2m+1})]$ implying $p(x_{2m-1},x_{2m}) < p(x_{2m},x_{2m+1})$, then we have

$$\begin{aligned} p(x_{2m},x_{2m+1}) &\le \frac{\sqrt{\alpha}}{2}[p(x_{2m-1},x_{2m})+p(x_{2m},x_{2m+1})] \\ &\le \frac{\sqrt{\alpha}}{2-\sqrt{\alpha}}p(x_{2m-1},x_{2m}) \\ &< p(x_{2m-1},x_{2m}),\ \text{as}\ \frac{\sqrt{\alpha}}{2-\sqrt{\alpha}} < 1. \end{aligned}$$

This is a contradiction.

Hence $p(x_{2m-1},x_{2m}) \ge \frac{1}{2}[p(x_{2m-1},x_{2m})+p(x_{2m},x_{2m+1})]$ implying

$$p(x_{2m},x_{2m+1}) =\le \sqrt{\alpha}p(x_{2m-1},x_{2m}).$$

(4.4.2) If $y_{2m+1} \in Q$, then $x_{2m+1} \neq y_{2m+1}$. From the construction of proof, we have $x_{2m+1} \in \operatorname{seg}[x_{2m},y_{2m+1}]$. Using Lemma 2.5 (ii), we get

$$\begin{aligned} p(x_{2m},x_{2m+1}) &\le p(x_{2m},y_{2m+1)} \\ &= p(y_{2m},y_{2m+1}) \\ &\le \sqrt{\alpha}p(x_{2m-1},x_{2m}), \end{aligned}$$

using the argument in (4.4.1).

We get the following similar result when n is odd, that is, when $n = 2m+1$ for some $m \in \mathbb{N}$,

$$p(x_n,x_{n+1}) = p(x_{2m+1},x_{2m+2}) \le \sqrt{\alpha}p(x_{2m},x_{2m+1}).$$

Thus, for $x_n \in P$, we have

$$p(x_n,x_{n+1}) \le \sqrt{\alpha}p(x_{n-1},x_n). \tag{4.14}$$

Case 4.5 Consider the case where $(x_n,x_{n+1}) \in Q \times P$. We claim that for $n \ge 1$, $x_n \in Q$ implies $x_{n-1} \in P$.

Let $x_{n-1} \in Q$, then $x_{n-1} \in \partial C$. This means, according to (ii), $x_n = y_n \in C$. This implies $x_n \in P$, which is a contradiction.

Hence we have

$$x_{n-1}, x_{n+1} \in P \text{ and } x_n \in \text{seg}[x_{n-1}, y_n].$$

Consider when n is even, that is, when $n = 2m$ for some $m \in \mathbb{N}$. According to (4.iii), $y_{2m+1} \in Tx_{2m} \subset C$ was chosen in such a way that

$$p(y_{2m}, y_{2m+1}) \leq \frac{1}{\sqrt{\alpha}} H_p(Sx_{2m-1}, Tx_{2m}). \tag{4.15}$$

We apply (2.2) and get

$$\begin{aligned} p(x_{2m}, x_{2m+1}) &= p(x_{2m}, y_{2m+1}) \\ &\leq p(x_{2m}, y_{2m}) + p(y_{2m}, y_{2m+1}) \end{aligned}$$

$$\Rightarrow p(x_{2m}, x_{2m+1}) \leq 2\max\{p(x_{2m}, y_{2m}), p(y_{2m}, y_{2m+1})\}. \tag{4.16}$$

If $p(x_{2m}, y_{2m}) \geq p(y_{2m}, y_{2m+1})$, (4.16) becomes

$$\begin{aligned} p(x_{2m}, x_{2m+1}) &\leq 2p(x_{2m}, y_{2m}) \\ &\leq 2p(x_{2m-1}, y_{2m}), \text{ as per Lemma 2.5 (ii)} \\ &= 2p(y_{2m-1}, y_{2m}), \text{ as } x_{2m-1} = y_{2m-1} \\ &\leq 2\sqrt{\alpha}\, p(x_{2m-2}, x_{2m-1}), \end{aligned} \tag{4.17}$$

using the argument in (4.4.2).

If $p(x_{2m}, y_{2m}) < p(y_{2m}, y_{2m+1})$, (4.16) becomes

$$p(x_{2m}, x_{2m+1}) \leq 2p(y_{2m}, y_{2m+1}). \tag{4.18}$$

Let us consider the term $p(y_{2m}, y_{2m+1})$. From (4.15) and Theorem 3.1 we have

$$\begin{aligned} p(y_{2m}, y_{2m+1}) &\leq \frac{1}{\sqrt{\alpha}} H_p(Sx_{2m-1}, Tx_{2m}) \\ &\leq \sqrt{\alpha} \max\big\{p(x_{2m-1}, x_{2m}), p(x_{2m-1}, Sx_{2m-1}), p(x_{2m}, Tx_{2m}), \\ &\quad \frac{1}{2}[p(x_{2m-1}, Tx_{2m}) + p(x_{2m}, Sx_{2m-1})]\big\} \end{aligned}$$

$$\begin{aligned} \Rightarrow p(y_{2m}, y_{2m+1}) \leq \sqrt{\alpha} \max\{&p(x_{2m-1}, x_{2m}), p(x_{2m-1}, y_{2m}), p(x_{2m}, y_{2m+1}), \\ &\frac{1}{2}[p(x_{2m-1}, y_{2m+1}) + p(x_{2m}, y_{2m})]\}. \end{aligned} \tag{4.19}$$

As $x_{2m} \in \text{seg}[x_{2m-1}, y_{2m}]$, from Lemma 2.5 (ii), we have

$$p(x_{2m-1}, y_{2m}) \geq p(x_{2m-1}, x_{2m}).$$

From P3 of Definition 2.1, we also have

$$\begin{aligned} &\leq [p(x_{2m-1}, x_{2m}) + p(x_{2m}, y_{2m+1}) \\ &\quad - p(x_{2m}, x_{2m}) + p(x_{2m}, y_{2m})] \\ &= [p(x_{2m-1}, y_{2m}) + p(x_{2n}, y_{2m+1})]. \end{aligned} \tag{4.20}$$

The expression (4.20) is because $x_{2m} \in \text{seg}[x_{2m-1}, y_{2m}]$ and Lemma 2.5 (i).

Hence (4.19) becomes

$$\begin{aligned} p(y_{2m}, y_{2m+1}) \leq \sqrt{\alpha} \max\{&p(x_{2m-1}, y_{2m}), p(x_{2m}, y_{2m+1}), \\ &\frac{1}{2}[p(x_{2m-1}, y_{2m}) + p(x_{2m}, y_{2m+1})]\}. \end{aligned} \tag{4.21}$$

Suppose $p(x_{2m-1}, y_{2m}) < p(x_{2m}, y_{2m+1})$, implying
$p(x_{2m}, y_{2m+1}) > \frac{1}{2}[p(x_{2m-1}, y_{2m}) + p(x_{2m}, y_{2m+1})]$. Then (4.21) becomes

$$p(y_{2m}, y_{2m+1}) \leq \sqrt{\alpha} p(x_{2m}, y_{2m+1}). \tag{4.22}$$

We continue from (4.18) and get

$$\begin{aligned} p(x_{2m}, x_{2m+1}) &\leq 2p(y_{2m}, y_{2m+1}) \\ &\leq 2\sqrt{\alpha} p(x_{2m}, y_{2m+1}) \\ &= 2\sqrt{\alpha} p(x_{2m}, x_{2m+1}), \text{ as } x_{2m+1} = y_{2m+1} \\ &< p(x_{2m}, x_{2m+1}), \text{ because } 2\sqrt{\alpha} < 1. \end{aligned}$$

This is a contradiction.

Hence $p(x_{2m-1}, y_{2m}) \geq p(x_{2m}, y_{2m+1})$, implying

$$p(x_{2m-1}, y_{2m}) \geq \frac{1}{2}[p(x_{2m-1}, y_{2m}) + p(x_{2m}, y_{2m+1})].$$

Then (4.21) becomes

$$p(y_{2m}, y_{2m+1}) \leq \sqrt{\alpha} p(x_{2m-1}, y_{2m}). \tag{4.23}$$

We continue from (4.18) and get

$$\begin{aligned} p(x_{2m},x_{2m+1}) &\leq 2p(y_{2m},y_{2m+1}) \\ &\leq 2\sqrt{\alpha}p(x_{2m-1},y_{2m}) \\ &= 2\sqrt{\alpha}p(y_{2m-1},y_{2m}), \text{ because } x_{2m-1}=y_{2m-1} \\ &\leq 2\sqrt{\alpha}\times\sqrt{\alpha}p(x_{2m-2},x_{2m-1}), \text{ as per (4.4.2)} \\ \Rightarrow p(x_{2m},x_{2m+1}) &\leq \sqrt{\alpha}p(x_{2m-2},x_{2m-1}), \text{ because } 2\sqrt{\alpha}<1. \end{aligned} \tag{4.24}$$

Hence, in observing (4.17) and (4.24), when $x_{2m}\in Q$, we have

$$p(x_{2m},x_{2m+1}) \leq 2\sqrt{\alpha}p(x_{2m-2},x_{2m-1}). \tag{4.25}$$

Using a similar argument, we can show that, when n is odd, that is, when $n=2m+1$ for some $m\in\mathbb{N}$, we have

$$p(x_{2m+1},x_{2m+2}) \leq 2\sqrt{\alpha}p(x_{2m-1},x_{2m}).$$

Hence in general, when $(x_n,x_{n+1})\in P\times Q$ we have

$$p(x_n,x_{n+1}) \leq 2\sqrt{\alpha}p(x_{n-2},x_{n-1}).$$

The case of $(x_n,x_{n+1})\in Q\times Q$ is not possible.

For all cases 4.4 and 4.5 we have

$$p(x_n,x_{n+1}) \leq t\max\{p(x_{n-2},x_{n-1}),p(x_{n-1},x_n)\}, \tag{4.26}$$

where

$$t=2\sqrt{\alpha}<1.$$

According to Lemma 3.3, (4.27) implies

$$p(x_n,x_{n+1}) \leq t^{n/2}t^{-1/2}\max\{p(x_0,x_1),p(x_1,x_2)\}. \tag{4.27}$$

Using the same argument used during the proof of Theorem 4.1, (4.27) shows that there is $z\in C$ such that

$$\lim_{m,n\to+\infty}p(x_m,x_n)=\lim_{n\to+\infty}p(z,x_n)=p(z,z)=0.$$

We now prove that z is a fixed point of both S and T.

Consider the subsequence $\{x_{n_j}\}$ of $\{x_n\}$ each of whose terms is in P. This means $x_{n_j} = y_{n_j}$ for $j = 1,2,\ldots$. Consider the case where n_j is odd, that is $n_j = 2m_j + 1$ for some $m_j \in \mathbb{N}$.

As $x_{2m_j+1} \in Tx_{2m_j}$, we have

$$p(z,Tx_{2m_j}) \leq p(z,x_{2m_j+1}).$$

This implies $\lim_{j\to+\infty} p(z,Tx_{2m_j}) = 0$.

Using a similar argument, we can show that $\lim_{j\to+\infty} p(z,Sx_{2m_j+1}) = 0$.

Now consider

$$\begin{aligned} p(z,Sz) &\leq p(z,Tx_{2m_j}) + p(Tx_{2m_j},Sz) \\ &\leq p(z,Tx_{2m_j}) + \alpha \max\{p(x_{2m_j},z), p(x_{2m_j},Tx_{2m_j}), \\ &\qquad p(z,Sz), \frac{1}{2}[p(x_{2m_j},Sz) + p(z,Tx_{2m_j})]\}. \end{aligned}$$

Taking $j \to +\infty$, we have

$$\begin{aligned} p(z,Sz) &\leq 0 + \alpha \max\{0,0,p(z,Sz), \frac{1}{2}[p(z,Sz)]\} \\ &= \alpha p(z,Sz) \\ &\leq p(z,Sz), \text{ because } \alpha < 1 \\ \Rightarrow p(z,Sz) &= 0. \end{aligned}$$

This implies $z \in Sz$ meaning z is a fixed point in S. Using a similar argument, we have z is a fixed point in T.

We show the uniqueness of the fixed point. Let z and y be fixed points of both S and T. As $z \in Sz$ we have

$$p(y,Sz) = \inf_{a\in Sz} p(y,a) \leq p(y,z) = p(z,y). \tag{4.28}$$

Suppose, as per assumption, we have $p(z,y) \leq p(y,Sz)$. Then, (4.28) leads us to conclude that

$$p(z,y) = p(y,Sz). \tag{4.29}$$

Because $y \in Ty$, we have

$$\begin{aligned} p(z,y) = p(y,Sz) &\leq H_p(Ty,Sz) \\ &\leq \alpha \max\{p(z,y), p(y,Ty), p(z,Sz), \frac{1}{2}[p(y,Sz) + p(z,Ty)]\} \end{aligned}$$

$$\Rightarrow p(z,y) = \frac{\alpha}{2}[p(y,Sz) + p(z,Ty)] \tag{4.30}$$

$$\begin{aligned}\Rightarrow p(z,y) &\leq \frac{\alpha}{2-\alpha} p(z,Ty) \\ &\leq p(z,Ty), \text{ as } \frac{\alpha}{2-\alpha} < 1.\end{aligned} \tag{4.31}$$

Let us consider (4.30). We also consider (4.29) which states that $p(z,y) = p(y,Sz)$. We then have

$$\begin{aligned}p(z,y) &\leq \frac{\alpha}{2}[p(y,Sz) + p(z,Ty)] \\ &= \frac{\alpha}{2}[p(z,y) + p(z,Ty)] \\ &\leq \frac{\alpha}{2}[p(z,y) + p(z,y)], \text{ because } y \in Ty \\ &= \alpha p(z,y) \\ &\Rightarrow p(z,y) = 0, \text{ as } \alpha < 1 \\ &\Rightarrow z = y, \text{ by (2.1).}\end{aligned}$$

We will reach the same conclusion if we assume $p(z,y) \leq p(z,Ty)$. This shows that the common fixed point z is unique. The proof has been completed.

Remark 4.1 *Theorem 4.3 is valid when we have $S = T$.*

Remark 4.2 *If we set $S = T$, and assume $C = X$, only **(4.4.1)** applies, and we get Theorem 4.2 by Rao and Rao* [10].

When we set $T = f$ where f is a single valued mapping we get the following corollary:

Corollary 4.1 *Let (X,p) be a complete metrically convex partial metric space and C a non-empty closed subset of X, the closure being with respect to (X,p^s). Let ∂C, the boundary of C with respect to (X,p^s), be non-empty. Let $S,f : C \to CB^p(X)$ be mappings satisfying*

$$p(Sx,fy) \leq \alpha \max\left\{p(x,y), p(x,Sx), p(y,fy), \frac{1}{2}[p(x,fy) + p(y,Sx)]\right\}$$

for all $x,y \in X$ and $0 < \alpha < \frac{1}{4}$. Let the following conditions apply:

(i) $x \in \partial C$ implies $fx \in C$,

(ii) $x \in \partial C$ implies $Sx \subset C$.

Then S and f have a common fixed point in X. Further, if we assume that $p(x,y) \leq p(y,Sx)$ or

$p(x,y) \leq p(y,fx)$ for all $x,y \in X$, then S and f have a unique common fixed point z in C with $p(z,z) = 0$.

If we set $T = f, S = g$, where both f and g are single valued mappings we get the following corollary:

Corollary 4.2 *Let (X,p) be a complete metrically convex partial metric space and C a non-empty closed subset of X, the closure being with respect to (X,p^s). Let ∂C, the boundary of C with respect to (X,p^s), be non-empty. Let $g,f : C \to X$ be mappings satisfying*
$p(gx,fy) \leq \alpha \max\left\{p(x,y), p(x,gx), p(y,fy), \frac{1}{2}[p(x,fy)+p(y,gx)]\right\}$ for all $x,y \in X$ and $0 < \alpha < \dfrac{1}{4}$. Let the following condition apply: $x \in \partial C$ implies $fx \in C$ and $gx \subset C$,

Then g and f have a common fixed point in X. Further, if we assume that $p(x,y) \leq p(y,gx)$ or $p(x,y) \leq p(y,fx)$ for all $x,y \in X$, then g and f have a unique common fixed point z in C with $p(z,z) = 0$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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