Available online at http://scik.org
Adv. Fixed Point Theory, 7 (2017), No. 3, 441-450
ISSN: 1927-6303

COMMON FIXED POINT UNDER CONTRACTIVE CONDITION OF ĆIRIĆ'S TYPE ON DISLOCATED METRIC SPACES

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Abstract: In this paper, we generalize common fixed point theorems under contractive condition of Ćirić's type on a dislocated metric space. We give basic facts about dislocated metric spaces, and we prove common fixed point theorems under contractive condition of Ćirić's type on a dislocated metric space.

Keywords: dislocated metric space; coincidence point; point of coincidence; common fixed point.

2010 AMS Subject Classification: 47H10, 54H25.

1. Introduction

The Banach contraction principle is the basic result in fixed point theory. It has many applications in various branches of mathematics. The root of fixed point theory lies in work of Brouwer and Banach. In 1912 Brouwer [1] proved a result that a unit closed ball in $\mathbb{R}^n$ has a fixed point Since then, many authors generalized the Banach fixed point theorem in various ways [2-7]. Recently, Samet et al. [8] introduced the notion of α-ψ contractive mappings and proved

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Received March 21, 2017

the related fixed point theorems. The generalization of well known Banach Contraction Principle of metric space to the dislocated metric space proved by P. Hitzler and A. K. Seda, played a key role in the development of logic programming semantics [9]. This concept of dislocated metric space was further generalized into dislocated quasi, right dislocated, left dislocated metric spaces by M.A. Ahmed et.al ([10], [11]). The article is organized as follows. In Section 2, we repeat some definitions and well known results which will be needed in the sequel. In Section 3, we prove common fixed point theorems on a dislocated metric type space.

2. Definitions and notation

Definition 2.1. Let Ψ be a family of functions $\psi:[0,\infty)\to[0,\infty)$ satisfying the following conditions:

(i) ψ is nondecreasing;

(ii) There exist $k_0\in\mathbb{N}$ and $a\in(0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$$\psi^{k+1}(t)\le a\psi^k(t)+v_k \quad \text{for } k\ge k_0 \text{ and any } t\in\mathbb{R}^+.$$

Lemma 2.2. $If\ \psi\in\Psi$, then the following hold:

(i) $\left(\psi^n(t)\right)_{n\in\mathbb{N}}$ converges to 0 as $n\to\infty$ for all $t\in\mathbb{R}^+$

(ii) $\psi(t)<t$ for any $t\in(0,\infty)$

(iii) ψ is continuous at $0;$

(iv) the series $\sum_{n=1}^{\infty}\psi^k(t)$ converges for any $t\in\mathbb{R}^+$

Recently, Samet et al. [8] introduced the following concepts.

Definition 2.3. let $T: X \to X$ and $\alpha : X \times X \to [0, \infty)$. we say that T is α-admissible if for all $x, y \in X$, we have

$\alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1$

Definition 2.4. let (X, d) be a metric space and let $T : X \to X$ be a given mapping. We say that T is an $\alpha - \psi$-contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$\alpha(x, y) d(Tx, Ty) \leq \psi(d(x, y))$

For all $x, y \in X$

Clearly, any contractive mapping, that is, a mapping satisfying Banach contraction, is an $\alpha - \psi$-contractive mapping with $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$, for all $t > 0$ and some $k \in [0,1)$.

Various examples of such mappings are presented in [8]. The main results in [8] are the following fixed point theorems:

Theorem 2.5. Let (X, d) be a complete metric space and $T : X \to X$ be an α-ψ-contractive mapping. Suppose that

(i) T is α-admissible;

(ii) There exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) T is continuous.

Then there exist $u \in X$ such that $Tu = u$.

Theorem 2.6. Let (X, d) be a complete metric space and $T : X \to X$ be an α-ψ-contractive mapping. Suppose that

(i) T is α-admissible;

(ii) There exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.

Then there exist $u \in X$ such that $Tu = u$.

Theorem 2.7. Adding the condition (iv) to the hypothesis of the Theorem 2.5 and Theorem 2.6 we obtain the uniqueness of a fixed point of T.

(iv)For all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.

Hitzler and Seda [9] introduced the concept of dislocated metric space as follows:

Definition 2.8. Let X be a non empty set and let $d: X \times X \to [0, \infty)$ be a function satisfying the following conditions:

(i) $d(x, y) = d(y, x)$

(ii) $d(x, y) = d(y, x) = 0 \text{ implies } x = y$

(iii)$d(x, y) \leq d(x, z) + d(z, y) \text{ for all } x, y, z \in X$

Then d is called dislocated metric (or simply d-metric) on X and the pair (X, d) is called dislocated metric space.

Example 2.9. Let (X, d) be a metric space. The function $f: X \times X \to \mathbb{R}^+$, defined as

$$d(x, y) = \max(x, y); \qquad \text{for all } x, y \in X \text{ is a dislocatedmetric on } X.$$

Definition 2.10. A sequence $\{x_n\}$ in a dislocated metric space (X, d) is said to be dislocated convergent if for every given $\in > 0$ there exist an $n \in N$ and $x \in X$ such that $d(x_n, x) < \in$ for all $n > N$ and it is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Definition 2.11. A sequence $\{x_n\}$ in a dislocated metric space (X, d) is said to be dislocated Cauchy sequence if for every $\in > 0$ there exist $n_0 \in N$ such that $d(x_n, x_m) < \in$ for all $m, n \in N$.

Definition 2.12. A dislocated metric space (X, d) is called complete if every Cauchy sequence is convergent.

Lemma 2.13. let (X, d) be a dislocated metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $x_n \to x(n \to \infty)$ if and only if $d(x_n, x) \to 0(n \to \infty)$.

Lemma 2.14: let (X, d) be a dislocated metric space and let $\{x_n\}$ be a sequence in X. If the sequence $\{x_n\}$ is convergent then the limit point is unique.

Theorem 2.15 Let (X,d) be a complete dislocated metric space and let $T:X\to X$ be a mapping satisfying the following condition for all $x,y\in X$:

$$d(Tx,Ty)\leq kd(x,y),$$

where $k\in[0,1)$. then T has a unique fixed point.

3. Main results

Theorem 3.1. Let (X,d) be a complete dislocated metric space . Let {*F*,T} be a pair of self mappings on X such that for some constant $\lambda\in(0,1/2)$ for all x,y ∈ X there exists

$$u(x,y)\in\{d(x,y),d(x,Fx),d(y,Ty),d(y,Fx),d(x,Ty)\}, \tag{3.1}$$

such that the following inequality

$$d(Fx,Tx)\leq\lambda u(x,y) \tag{3.2}$$

holds.Then F and T have a unique common fixed point.

Proof. Let us choose $x_0\in X$ arbitrary and define sequence $\{x_n\}$ as follows:

$x_{2n+1}=Fx_{2n}$, $x_{2n+2}=Tx_{2n+1}$, n = 0,1,2……..we shall show that

$$d(x_{k+1},x_k)\leq\alpha d(x_k,x_{k-1}),\ k\geq1, \tag{3.3}$$

where $\alpha\in(0,1)$. In order to prove this, we consider the cases of an odd integer *k* and of an even integer *k*.

For k = 2n + 1, from (3.2) we have $d(x_{2n+2}\ x_{2n+l})=d(Fx_{2n},Tx_{2n+1})\leq\lambda u(x_{2n},x_{2n+1})$,

where, according to (3.1),

$$u(x_{2n},x_{2n+1})\in\{d(x_{2n},x_{2n+l}),d(x_{2n}\ Fx_{2n}),d(x_{2n}+_1,Tx_{2n+1}),d(x_{2n+l},Fx_{2n}),d(x_{2n},Tx_{2n+l})\}$$

$$=\{d(x_{2n},x_{2n+l}),d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2}),d(x_{2n+l},x_{2n+1}),d(x_{2n},x_{2n+2})\}$$

$$= \{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2})\}$$

Thus, we get the following cases:

- $d(x_{2n+2}, x_{2n+1}) \leq \lambda d(x_{2n+1}, x_{2n+2})$ which Implies $d(x_{2n+2}, x_{2n+1}) = 0$
- $d(x_{2n+2}\ x_{2n+1}) \leq \lambda d(x_{2n}, x_{2n+1})$;
- $d(x_{2n+2}, x_{2n+1}) \leq \lambda d(x_{2n}, x_{2n+2})$ that is beacause of (iii) of definition 2.8 , $d(x_{2n+2}, x_{2n+1}) \leq \lambda(d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}))$

which implies $d(x_{2n+2}, x_{2n+1}) \leq \frac{\lambda}{1-\lambda}\, d(x_{2n}, x_{2n+1})$

Hence, (3.3) is satisfied, where $\alpha = \max\{\lambda, \frac{\lambda}{1-\lambda}\} = \frac{\lambda}{1-\lambda} \Longrightarrow \alpha \in (0,1)$

Now, for k = 2n + 2,we have $d(x_{2n+3}, x_{2n+2}) = d(Fx_{2n+2}, Tx_{2n+1}) \leq \lambda u\ (x_{2n+2}, x_{2n+1})$

where

$u(x_{2n+2}, x_{2n+1}) \in \{d(x_{2n+2}, x_{2n+1}), d(x_{2n+2}, Fx_{2n+2}), d(x_{2n+1}, Tx_{2n+1}), d(x_{2n+2}, Tx_{2n+1}), d(x_{2n+1}, Fx_{2n+2})\}$

$= \{d(x_{2n+2}\ x_{2n+1}), d(x_{2n+2}, x_{2n+3}), d(x_{2n+1}\ x_{2n+3})\}$

and we get the following cases:

- $d(x_{2n+3}, x_{2n+2}) \leq \lambda d(x_{2n+2}, x_{2n+1})$,
- $d(x_{2n+3}, x_{2n+2}) \leq \lambda\ d(x_{2n+3}, x_{2n+2})$ which gives $d(x_{2n+3}, x_{2n+2}) = 0$.
- $d(x_{2n+3}\ x_{2n+2}) \leq \lambda\ d(x_{2n+3}, x_{2n+1}) \leq \lambda(\ d(x_{2n+3}, x_{2n+2}) + d(x_{2n+2}, x_{2n+1}))$

  Which implies

$$d(x_{2n+3}, x_{2n+2}) \leq \frac{\lambda}{1-\lambda}\ d(x_{2n+2}, x_{2n+1}).$$

So, inequality (3.3) is satisfied in this case, too. Therefore, (3.3) is satisfied for all $k \in N_0$ and by iterating we get

$$d(x_k, x_{k+1}) \leq \alpha^k d(x_0, x_1) \tag{3.4}$$

Since $k \geq 1$, for $m > k$ we have

$d(x_k, x_m) \leq d(x_k, x_{k+1}) + d(x_{k+1}, x_{k+2}) + \ldots.. + d(x_{m-1}, x_m)$

$$\leq (\alpha^{\kappa} + \alpha^{\kappa+1} + \ldots + \alpha^{m-1})\, d(x_o, x_1)$$

$$\leq \frac{\alpha^k}{1-\alpha}\, d(x_0, x_1) \to 0, \text{as } k \to \infty.$$

Hence, $\{x_k\}$ is a Cauchy sequence in X

Since X is complete dislocated metric space, there exists $v \in X$ such that $x_k \to v$, as $k \to \infty$.

Let us show that $Fv = Tv = v$. We have $d(Fx_{2n}, Tv) \leq \lambda u(x_{2n}, v)$,

Where

$u(x_{2n}, v) \in \{d(x_{2n}\, v), d(x_{2n}, Fx_{2n}), d(v, Tv), d(x_{2n}\, Tv), d(v, Fx_{2n})\}$.

Thus, for sufficiently large n, at least one of the following cases hold:

- $d(Fx_{2n}, Tv) \leq \lambda d(x_{2n}, v) \to 0$

- $d(Fx_{2n}, Tv) \leq \lambda d(x_{2n}, Fx_{2n})$, i.e., $d(Fx_{2n}\, Tv) \leq \lambda(d(x_{2n}, v) + d(v, x_{2n+1})) \to 0$

- $d(Fx_{2n}, Tv) \leq \lambda d(v, Tv) \leq \lambda d(v, Fx_{2n}) + d(Fx_{2n}, Tv))$, i.e., $d(Fx_{2n}, Tv) \leq \frac{\lambda}{1-\lambda} d(v, x_{2n+1}) \to 0$

- $d(Fx_{2n}, Tv) \leq \lambda d(x_{2n}\, Tv) \leq \lambda(d(x_{2n}, v) + d(v, Fx_{2n}) + d(Fx_{2n}, Tv))$, i.e.,

  $d(Fx_{2n}, Tv) \leq \frac{\lambda}{1-\lambda} d(x_{2n}, v) + \frac{\lambda}{1-\lambda} d(v, x_{2n+1}) \to 0$

- $d(Fx_{2n}, T\nu) \leq \lambda d(\nu, Fx_{2n}) = \lambda d(\nu, x_{2n+1}) \to 0.$

In all these cases, we obtain that $Fx_{2n} \to T\nu$, as $n \to \infty$, that is $x_n \to T\nu, n \to \infty$.

Since the limit of a convergent sequence in a dislocated metric space is unique, we have that $\nu = T\nu$. Now, we have to prove that $F\nu = T\nu$. Since

$d(F\nu, \nu) = d(F\nu, T\nu) \leq \lambda u(\upsilon, \upsilon)$,

where

$u(\upsilon,\upsilon) \in \{d(\upsilon,\upsilon) d(\upsilon, F\upsilon), d(\upsilon, T\upsilon), d(\upsilon, T\upsilon), d(\upsilon, F\upsilon)\} = \{d(\upsilon,F\upsilon), d(\upsilon,\upsilon)\}$.

Hence, we get the following cases: $d(F\upsilon, \upsilon) \leq \lambda d(\upsilon, \upsilon) = \lambda\{d(v,Fv) + (Fv,v)\}$

and According to (iii) of theorem 2.8, it follows that $F\upsilon = \upsilon$, that is, υ is a common fixed point of F and T. It can be easily verified that v is the unique common fixed point of F and T.

Conflict of Interests

The authors declare that there is no conflict of interests.

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