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SOME COMMON FIXED POINT THEOREMS USING (CLRG) PROPERTY IN CONE METRIC SPACES

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Abstract: In this paper, first we prove some common fixed point theorems for different types of contractive conditions. Secondly, we prove a common fixed point theorem for expansive mappings using (CLRg) property along with weakly compatible maps. In fact, our results generalize the results of Jha [3], Olaleru [8], Kadelburg et. al. [5], Khojasteh et. al. [7] and Kadelburg et. al. [6].

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1. Introduction

In 2007, Huang and Zhang [2] generalized the concept of metric space to cone metric space by replacing the real numbers with ordered Banach space as follows:

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2. Preliminaries

Definition 2.1. Let E be a real Banach space. A subset P of E is called a cone if

(i) P is closed, non-empty and $P \neq \{0\}$;

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$;

(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if

$y-x \in P$. A cone P is called normal if there is a number $m > 0$ such that for all $x, y \in E$,

$0 \leq x \leq y$ implies $\|x\| \leq m\|y\|$

The least positive number satisfying the above inequality is called the normal constant of P, while $x \ll y$ stands for $y-x \in \text{int } P$ (interior of P).

Definition 2.2. Let X be a non-empty set. Suppose that the mapping $d : X \times X \to E$ satisfies

(i) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space. The concept of a cone metric space is more general than that of a metric space, since each metric space is a cone metric space with $E = \mathbb{R}$ and $P = [0, \infty)$ (see [2, Example 1]).

In 1998, Jungck and Rhoades [4] introduced the notion of weakly compatible maps as follows:

Let X be a non-empty set. Two mappings $f, g : X \to X$ are said to be weakly compatible if $fx = gx$ imply $fgx = gfx$ for $x \in X$.

Recently, Sintunavarat et. al. [9] introduced the notion of (CLRg) property in Fuzzy metric spaces as follows:

Suppose that (X, d) is a metric space and $f, g: X \to X$. Two mappings f and g are said to satisfy the common limit in the range of g (CLRg) property if $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$ for some $x \in X$.

Now, we state the Lemma which is useful for proving our main results.

Lemma 2.1. *Let f and g be weakly compatible self-mappings of a set X. If f and g have a unique point of coincidence, that is, t = fx = gx, then t is the common fixed point of f and g.*

3. Fixed point results for contractive condition

In 2008, Jha [3] proved the following fixed point theorem.

Theorem 3.1. *Let (X, d) be a cone metric space and P be a normal cone with normal constant K. Suppose that the mappings f, g : $X \to X$ satisfy the contractive condition*

$d(fx, fy) \leq r\,[d(fx, gy) + d(fy, gx) + d(fx, gx) + d(fy, gy)]$, where $r \in [0, 1/4)$ is a constant.

If the range of g contains the range of f and g(X) is a complete subspace of X, then f and g have a unique coincidence point in X.

Moreover, if f and g are weakly compatible maps, then f and g have a unique common fixed point.

Now, we generalize this result using (CLRg) property as follows:

Theorem 3.2. *Let (X, d) be a cone metric space. Suppose that the mappings f, g be weakly compatible self- mappings of X satisfying the contractive condition*

(3.1) $d(fx, fy) \leq r\,[d(fx, gy) + d(fy, gx) + d(fx, gx) + d(fy, gy)]$, where $r \in [0, \frac{1}{4})$ is a constant.

If f and g satisfy (CLRg) property, then f and g have a unique common fixed point.

Proof. Since f and g satisfy the (CLRg) property, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$ for some $x \in X$.

From (3.1), we have

$d(fx_n, fx) \leq r\,[d(fx_n, gx) + d(fx, gx_n) + d(fx_n, gx_n) + d(fx, gx)]$ for all $n \in \mathbb{N}$.

By making $n \to \infty$, we have gx = fx.

Let w = fx = gx. Since f and g are weakly compatible mappings, therefore fgx = gfx implies that fw = fgx = gfx = gw.

Now, we claim that fw = w.

From (3.1), we have

$d(fw,w) = d(fw, fx) \leq r\,[d(fw, gx) + d(fx, gw) + d(fw, gw) + d(fx, gx)]$

$= r\,[d(fw, gx) + d(fx, gw)]$

$= r\,[d(gw, fx) + d(fx, gw)] = 0$, i.e., $fw = w = gw$.

Hence w is a common fixed point of f and g.

For the uniqueness of a common fixed point, we suppose that z is another common fixed point in X such that fz = gz.

From (3.1), we have

$d(gz, gw) = d(fz, fw) \leq r\,[d(fz, gw) + d(fw, gz) + d(fw, gw) + d(fz, gz)]$ implies gz = gw.

By Lemma 2.1, we have f and g have a unique common fixed point.

Example 3.1. Let $E = I^2$ for $I = [0, 1]$, $P = \{(x, y) \in E, x, y \geq 0\} \subset I^2$, $d : I \times I \rightarrow E$ such that

$d(x, y) = (|x-y|, \alpha|x-y|)$, where $\alpha > 0$ is a constant.

Define $fx = \dfrac{\alpha x}{(1+\alpha x)}$ and $gx = \alpha x$ for all $x \in I$. Consider the sequence $\{x_n\} = \{\frac{1}{n}\}$, $n \in \mathbb{N}$, since $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0 = g0$, therefore f and g satisfy the (CLRg) property.

Also x = 0 is the unique common fixed point.

In 2009, Olaleru [8] proved the following theorem.

Let (X, d) be a cone metric space and let f, g : X → X be mappings such that

$d(fx, fy) \leq a_1\, d(fx, gx) + a_2\, d(fy, gy) + a_3\, d(fy, gx) + a_4\, d(fx, gy) + a_5\, d(gy, gx)$ *for all* $x, y \in X$,

where $a_1, a_2, a_3, a_4, a_5 \in [0,1)$ *and* $\sum_{i=1}^{5} a_i < 1$.

Suppose f and g are weakly compatible maps and $f(X) \subset g(X)$ *such that f(X) or g(X) is a complete subspace of X, then the mappings f and g have a unique common fixed point.*

Now, we generalize this result using (CLRg) property along with weakly compatible maps as follows:

Theorem 3.3. *Let (X, d) be a cone metric space and let f, g : X → X be mappings such* *that*

(3.2) $d(fx, fy) \leq a_1 d(fx, gx) + a_2 d(fy, gy) + a_3 d(fy, gx) + a_4 d(fx, gy) + a_5 d(gy, gx)$

for all $x, y \in X$ *where* $a_1, a_2, a_3, a_4, a_5 \in [0,1)$ *and* $\sum_{i=1}^{5} a_i < 1$.

Suppose f and g are weakly compatible maps and satisfy (CLRg) property. Then the mappings f and g have a unique common fixed point.

Proof. Since f and g satisfy the (CLRg) property, there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$ for some $x \in X$.

From (3.2), we have

$d(fx_n, fx) \leq a_1 d(fx_n, gx_n) + a_2 d(fx, gx) + a_3 d(fx, gx_n) + a_4 d(fx_n, gx) + a_5 d(gx, gx_n)$ for all $n \in \mathbb{N}$.

Taking limit as $n\to\infty$, we have

$d(gx, fx) \leq a_1 d(gx, gx) + a_2 d(fx, gx) + a_3 d(fx, gx) + a_4 d(gx, gx) + a_5 d(gx, gx)$

$= (a_2 + a_3) d(fx, gx)$

i.e., $[1- (a_2+ a_3)] d(fx, gx) \leq 0$, i.e., $fx = gx$.

Now let $z = fx = gx$. Since f and g are weakly compatible mappings, therefore $fgx = gfx$ which implies that $fz = fgx = gfx = gz$.

We claim that $gz = z$.

From (3.2), we have

$d(gz, z) = d(fz, fx) \leq a_1 d(fz, gz) + a_2 d(fx, gx) + a_3 d(fx, gz) + a_4 d(fz, gx) + a_5 d(gx, gz) = (a_3 + a_4 + a_5) d(fz, fx) = (a_3 + a_4 + a_5) d(gz, z)$, i.e., $gz = z = fz$.

Hence z is a common fixed point of f and g.

For the uniqueness of a common fixed point, we suppose that $w \neq z$ is another common fixed of f and g.

From (3.2), we have

$d(w, z) = d(gw, gz) = d(fw, fz)$

$\leq a_1 d(fw, gw) + a_2 d(fz, gz) + a_3 d(fz, gw) + a_4 d(fw, gz) + a_5 d(gz, gw)$

$= (a_3 + a_4 + a_5) d(gw, gz)$, implies $w = z$.

Hence f and g have a unique common fixed point.

4. Fixed point results for strict contractive condition

Definition 4.1. Let (X, d) be a cone metric space and (f, g) be a pair of self- mappings on X.

For x, y ∈ X, consider the following sets:

$$M_0^{f,g}(x,y) = \{d(gx,gy), d(gx,fx), d(gy,fy), d(gx,fy), d(gy,fx)\}$$

$$M_1^{f,g}(x,y) = \{d(gx,gy), d(gx,fx), d(gy,fy), \frac{d(gx,fy)+d(gy,fx)}{2}\}$$

$$M_2^{f,g}(x,y) = \{d(gx,gy), \frac{d(gx,fx)+d(gy,fy)}{2}, \frac{d(gx,fy)+d(gy,fx)}{2}\}$$

and define the following conditions:

(4.1) for arbitrary x, y ∈ X there exists $u_0(x, y) \in M_0^{f,g}(x,y)$ such that

$d(fx, fy) < u_0(x, y)$;

(4.2) for arbitrary x, y ∈ X there exists $u_1(x, y) \in M_1^{f,g}(x,y)$ such that

$d(fx, fy) < u_1(x, y)$;

(4.3) for arbitrary x, y ∈ X there exists $u_2(x, y) \in M_2^{f,g}(x,y)$ such that

$d(fx, fy) < u_2(x, y)$.

These conditions are called strict contractive conditions.

Definition 4.2. Let (X, d) be a cone metric space. Let f, g be self-maps on X. Then f is called a g-quasi-contraction if for some constant $\alpha \in (0, 1)$ and for every x, y ∈X, there exists $u(x, y) \in M_0^{f,g}(x,y)$ such that $d(fx, fy) \le \alpha\, u(x, y)$.

In 2009, Kadelburg et. al. [5] proved the following fixed point theorems.

Theorem 4.1. *Let f and g be two weakly compatible self-mappings of a cone metric space (X,d) satisfying (4.3) and the following*

(4.4) f(X) ⊂ g(X);

(4.5) (f, g) satisfies property (E.A.).

If g(X) or f(X) is a complete subspace of X, then f and g have a unique common fixed point.

Theorem 4.2. *Let f and g be two weakly compatible self-mappings of a cone metric space (X, d) such that conditions (4.4) and (4.5) of Theorem 4.3 are satisfied and also f is a g-quasi-contraction. If g(X) or f(X) is a complete subspace of X, then f and g have a unique common fixed point.*

Now, we generalize these results using (CLRg) property as follows:

Theorem 4.3. *Let f and g be two weakly compatible self-mappings of a cone metric space (X, d) satisfying (4.3) and the following:*

(4.6) f, g satisfy (CLRg) property.

Then f and g have a unique common fixed point.

Proof. Since f and g satisfy the (CLRg) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx \text{ for some } x \in X.$$

From (4.3), we have

$d(fx_n, fx) < u(x_n, x)$

where $u(x_n,x) \in M_2^{f,g}(x_n,x)$

$$= \left\{ d(gx_n, gx), \frac{d(gx_n, fx_n) + d(gx, fx)}{2}, \frac{d(gx_n, fx) + d(gx, fx_n)}{2} \right\}$$

We will show that fx = gx.

Suppose that $fx \neq gx$.

From (4.3), we have three cases.

Case1. $d(fx_n, fx) < d(gx_n, gx)$

Taking limit as $n\to\infty$, we have

$d(gx, fx) < d(gx, gx) = 0$, a contradiction.

Case2. $d(fx_n, fx) < \frac{1}{2}[d(gx_n, fx_n) + d(gx, fx)]$

Making limit as $n\to\infty$, we have

$d(gx, fx) < \frac{1}{2}[d(gx, gx) + d(gx, fx)] = \frac{1}{2} d(gx, fx)$, a contradiction.

Case3. $d(fx_n, fx) < \frac{1}{2}[d(gx_n, fx) + d(gx, fx_n)]$

Making limit as $n\to\infty$, we have

d(gx, fx) < $\frac{1}{2}$ [d(gx, fx) + d(gx, gx) = $\frac{1}{2}$ d(gx, fx), a contradiction.

Hence gx = fx in all cases.

Let z = fx = gx. Since f and g are weakly compatible mappings, therefore fgx = gfx which implies that

fz = fgx = gfx = gz.

We claim that fz = z. Let, if possible, fz $\neq$ z.

From (4.3), we have

d(fz, z) = d(fz, fx) < u(z, x)

where u(z, x) $\in \left\{ d(gz, gx), \frac{d(gz, fz) + d(gx, fx)}{2}, \frac{d(gz, fx) + d(gx, fz)}{2} \right\}$

$= \{ d(fz, z), 0, d(fz, z) \}$

So, we have only two possible cases:

Case1. d(fz, z) < d(fz, z), a contradiction.

Case2. d(fz, z) < 0, a contradiction.

Hence fz = z = gz.

Hence z is a common fixed point of f and g.

Uniqueness: We suppose that w is another common fixed point in X such that fw = gw.

We shall prove that z = w. Let, if possible, z $\neq$ w.

From (4.3), we have

d(z, w) = d(gz, gw) = d(fz, fw) < u(z, w), where

$$u(z, w) \in \left\{ d(gz, gw), \frac{d(gz, fz) + d(gw, fw)}{2}, \frac{d(gz, fw) + d(gw, fz)}{2} \right\}$$

So, we have only two possible cases:

Case1. d(gz, gw) < d(gz, gw), a contradiction.

Case2. d(gz, gw) < 0, a contradiction.

Hence gz = gw implies z = w.

So, we can say that f and g have a unique common fixed point.

Example 4.1. Let X = ℝ, E = $C_{\mathbb{R}}^1[0,1]$ and P = { $\phi(t)$: $\phi(t) \geq 0$, t $\in$[0, 1]}. d(x, y) (t)

= $|x-y|\phi$, where $\phi(t) > 0$ is an arbitrary fixed function. Consider the functions f, g : X $\rightarrow$ X defined by fx = $\frac{x}{3}$ and gx = $\frac{x}{2}$ .Consider the sequence $\{x_n\}$ = $\{\frac{1}{n}\}$, $n \in \mathbb{N}$, since $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0 = g0$, therefore f and g satisfy the (CLRg) property. Also x=0 is the unique common fixed point. Also

$$d(fx, fy) = |fx - fy|\phi(t) = \left|\frac{x}{3} - \frac{y}{3}\right|\phi(t) = \frac{1}{3}|x-y|\phi(t)$$

$$\leq \frac{2}{3}\left|\frac{x}{2} - \frac{y}{2}\right|\phi(t) \leq \left|\frac{x}{2} - \frac{y}{2}\right|\phi(t) = d(gx, gy).$$

Hence all the conditions of theorem are fulfilled.

Theorem 4.4. *Let f and g be two weakly compatible self-mappings of a cone metric space (X,d) such that*

(4.7) f is a g-quasi-contraction;

(4.8) f and g satisfy (CLRg) property.

Then f and g have a unique common fixed point.

Proof. Since f and g satisfy the (CLRg) property, there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$ for some $x \in X$.

From (4.7), we have

$d(fx_n, fx) \leq \lambda\, u(x_n, x)$ for some $u(x_n, x) \in \{d(gx_n, gx), d(gx_n, fx_n), d(gx_n, fx), d(gx, fx_n), d(gx, fx)\}$

Now the following cases arise:

Case1. $d(fx_n, fx) \leq \lambda\, d(gx_n, gx)$

Making limit as $n\to\infty$, we have gx = fx.

Case2. $d(fx_n, fx) \leq \lambda\, d(gx_n, fx_n)$

Making limit as $n\to\infty$, we have gx = fx.

Case3. $d(fx_n, fx) \leq \lambda\, d(gx_n, fx)$

Making limit as $n\to\infty$, we have gx = fx.

Case4. $d(fx_n, fx) \leq \lambda\, d(gx, fx_n)$

Making limit as $n\to\infty$, we have gx = fx.

Case5. $d(fx_n, fx) \leq \lambda\, d(gx, fx)$

Making limit as $n \to \infty$, we have gx = fx.

Thus, in all possible cases, gx = fx.

Now, let z = fx = gx. Since f and g are weakly compatible mappings fgx = gfx which implies that fz = fgx = gfx = gz.

We claim that fz = z.

From (4.7), we have

$d(fz, z) = d(fz, fx) \leq \lambda\, u(z, x)$

where $u(z, x) \in \{d(gz, gx), d(gz, fz), d(gx, fx), d(gz, fx), d(gx, fz)\}$

$= \{d(fz, z), 0, 0, d(fz, z), d(z, fz)\}.$

Now, we have only two possible cases.

Case1. $d(fz, z) \leq \lambda\, d(fz, z)$ implies fz = z.

Case2. $d(fz, z) \leq \lambda 0$ implies fz = z.

Hence fz = z = gz.

Hence z is a common fixed point of f and g.

For the uniqueness, we suppose that w is another common fixed point of f and g in X such that fw = gw.

From (4.7), we have

$d(gz, gw) = \lambda\, d(fz, fw) \leq \lambda\, u(z, w)$,

where $u(z, w) \in \{d(gz, gw), d(gz, fz), d(gw, fw), d(gz, fw), d(gw, fz)\}$

$= \{d(fz, fw), 0, 0, d(fz, fw), d(fw, fz)\}.$

Now, we have two cases.

Case1. $d(fz, fw) \leq \lambda\, d(fz, fw)$ implies fz = fw.

Case2. $d(fz, fw) \leq \lambda\, 0$ implies fz = fw.

So, we can say that f and g have a unique common fixed point.

5. Fixed point results for integral type mappings

In 2002, Branciari in [1] introduced a general contractive condition of integral type as follows:

Theorem 5.1. *Let (X, d) be a complete metric space, $\alpha \in (0, 1)$, and $f : X \to X$ be a mapping such that for all $x, y \in X$,*

$$\int_0^{d(fx,fy)} \phi(t)dt \quad \leq \alpha \int_0^{d(x,y)} \phi(t)dt\,,$$

where $\phi : [0, \infty) \to [0, \infty)$ *is non-negative and Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of* $[0,\infty)$ *such that for each* $\varepsilon > 0$, $\int_0^{\varepsilon} \phi(t)dt > 0$, *then f has a unique fixed point* $a \in X$ *, such that for each* $x \in X$,

$\lim_{n\to\infty} f^n\, x = a$.

In 2010, Khojasteh et. al. [7] proved the following fixed point theorem:

Theorem 5.2. *Let(X, d) be a complete cone metric space and P a normal cone. Suppose that*

$\phi : P \to P$ *is a non-vanishing map and a subadditive cone integrable on each [a, b]* $\subset$ *P such that for each* $\in >> 0$; $\int_0^{\in} \phi dp >> 0$. *If* $f : X \to X$ *is a map such that, for all* $x, y \in X$

$$\int_0^{d(fx,fy)} \phi dp \quad \leq \alpha \int_0^{d(x,y)} \phi dp$$

for some $\alpha \in (0,1)$ *then f has a unique fixed point in X.*

We generalize this result using (CLRg) property for a pair of mappings as follows:

Theorem 5.3. *Let(X, d) be a cone metric space with cone P. Suppose that* $\phi : P \to P$ *is a non-vanishing map integrable on each [a, b]* $\subset$ *P such that for each* $\in >> 0$;

$\int_0^{\in} \phi dp >> 0$.

If f and g are weakly compatible self-mappings on X satisfying (CLRg) property such that for all $x, y \in X$

$$\int_0^{d(fx,fy)} \phi dp \quad \leq \alpha \int_0^{d(gx,gy)} \phi dp \tag{5.1}$$

for some $\alpha \in (0,1)$ *then f and g have a unique common fixed point in X.*

Proof. Since f and g satisfy the (CLRg) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$$

From (5.1), we have

$$\int_0^{d(fx_n, fx)} \phi dp \le \alpha \int_0^{d(gx_n, gx)} \phi dp .$$

Taking limit as n→∞, we get

$$\int_0^{d(gx, fx)} \phi dp \le \alpha \int_0^{d(gx, gx)} \phi dp \text{ implying } \int_0^{d(gx, fx)} \phi dp \le 0 \text{, i.e., } gx = fx.$$

Now, let z = gx = fx. Since f and g are weakly compatible mappings, therefore fgx = gfx implies that fz = fgx = gfx = gz.

We claim that fz = z.

From (5.1), we have $\int_0^{d(fz, z)} \phi dp = \int_0^{d(fz, fx)} \phi dp \le \alpha \int_0^{d(gz, gx)} \phi dp = \alpha \int_0^{d(fz, fx)} \phi dp$,

i.e., (1- α) $\int_0^{d(fz, fx)} \phi dp \le 0$, i.e., fz = fx = z .

So fz = z = gz.

Hence z is a common fixed point of f and g.

For the uniqueness of a common fixed point, we suppose that w is another common fixed point in X such that fw = gw.

From (5.1), we have

$$\int_0^{d(gw, gz)} \phi dp = \int_0^{d(fw, fz)} \phi dp \le \alpha \int_0^{d(gw, gz)} \phi dp \text{ implying } (1-\alpha) \int_0^{d(gw, gz)} \phi dp = 0 \text{, i.e., } gw = gz.$$

So, by Lemma 1.1, we have f and g have a unique common fixed point.

Lemma 5.1. *Let E =* $\mathbb{R}^2$*, P = {x, y* $\in$ *E, x, y* $\ge$ *0}, and X =* $\mathbb{R}$*. Suppose that d : X × X → E is defined by d(x, y) =* $(|x-y|, \alpha|x-y|)$*, where* $\alpha \ge 0$ *is a constant. Suppose that* ϕ *: [(0,0), (a,b)] →P is defined by* $\phi(x, y) = (\phi_1(x), \phi_2(y))$*, where* ϕ_1, ϕ_2 *: [0, +∞) → [0, +∞) are two Riemann-integrable functions. Then*

$$\int_{(0,0)}^{(a,b)} \phi \, dp = \sqrt{a^2 + b^2} \left(\frac{1}{a} \int_0^a \phi_1(t) dt, \frac{1}{b} \int_0^b \phi_2(t) dt \right)$$

Example 5.1. Let X = $\{\frac{1}{n}, n \in \mathbb{N}\}$, E = $\mathbb{R}^2$ and P = {(x, y) ∈ E, x, y ≥ 0}. Suppose that d(x, y) = $(|x-y|, \ \alpha|x-y|)$, for some constant $\alpha > 0$. Here (X, d) is a cone metric space. If f, g : X → X and ϕ : P → P are defined by

$$f(x) = \begin{cases} \frac{1}{n+1} & if \ x = \frac{1}{n} \\ 0 & if \ x = 0 \end{cases}, \ n \in \mathbb{N}$$

$$g(x) = \begin{cases} \frac{1}{n} & if \ x = \frac{1}{n} \\ 0 & if \ x = 0 \end{cases}, \ n \in \mathbb{N}$$

and $\phi(t, s) = \begin{cases} (t^{\frac{1}{t-2}}(1 - l_n(t)), s^{\frac{1}{s-2}}(1 - l_n(s))), & (t, s) \in P\backslash\{(0,0)\} \\ (0,0) & , (t, s) = (0,0) \end{cases}$ respectively.

Consider the sequence $\{x_n\} = \{\frac{1}{n}\}$, then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0 = g0$, therefore f and g satisfy the (CLRg) property. Also x = 0 is the unique common fixed point.

Also $\int_0^{d(fx, fy)} \phi dp \le \frac{1}{2} \int_0^{d(gx, gy)} \phi dp$

In order to obtain above inequality, set gx = $\frac{1}{n}$, gy = $\frac{1}{m}$ where m > n.

Hence

$$d(fx, fy) = \left(\frac{m-n}{(m+1)(n+1)}, \frac{\alpha(m-n)}{(m+1)(n+1)}\right)$$

$$d(gx, gy) = \left(\frac{m-n}{mn}, \frac{\alpha(m-n)}{mn}\right)$$

Suppose

$\phi_1(t) = \phi_2(t) = t^{\frac{1}{t-2}}(1 - l_n(t))$ for all t > 0 and $\phi_1(0) = \phi_2(0) = 0$, thus $\phi(t, s) = (\phi_1(t), \phi_2(s))$.

By Lemma 5.1, we have

$$\int_0^{d(fx,fy)} \phi dp = \int_{(0,0)}^{(\frac{m-n}{(m+1)(n+1)},\frac{\alpha(m-n)}{(m+1)(n+1)})} (\phi_1,\phi_2) dp$$

=

$$\left(\frac{m-n}{(m+1)(n+1)}\sqrt{1+\alpha^2}\right)\left(\frac{(m+1)(n+1)}{m-n}\int_0^{\frac{m-n}{(m+1)(n+1)}} \phi_1(t)dt, \frac{(m+1)(n+1)}{\alpha(m-n)}\int_0^{\frac{\alpha(m-n)}{(m+1)(n+1)}} \phi_2(t)dt\right),$$

$$=(\sqrt{1+\alpha^2})\left(\int_0^{\frac{m-n}{(m+1)(n+1)}} \phi_1(t)dt, \frac{1}{\alpha}\int_0^{\frac{\alpha(m-n)}{(m+1)(n+1)}} \phi_2(t)dt\right).$$

Since $\int_0^h t^{\frac{1}{t-2}}(1-l_n(t))dt = h^{\frac{1}{h}}$, Thus

$$\int_0^{\frac{m-n}{(m+1)(n+1)}} \phi_1(t)dt = \left[\frac{m-n}{(m+1)(n+1)}\right]^{\frac{(m+1)(n+1)}{m-n}},$$

$$\int_0^{\frac{\alpha(m-n)}{(m+1)(n+1)}} \phi_2(t)dt = \left[\frac{\alpha(m-n)}{(m+1)(n+1)}\right]^{\frac{(m+1)(n+1)}{\alpha(m-n)}},$$

It means that

$$\int_0^{d(fx,fy)} \phi\, dp = (\sqrt{1+\alpha^2})\left(\left[\frac{m-n}{(m+1)(n+1)}\right]^{\frac{(m+1)(n+1)}{m-n}}, \frac{1}{\alpha}\left[\frac{\alpha(m-n)}{(m+1)(n+1)}\right]^{\frac{(m+1)(n+1)}{\alpha(m-n)}}\right). \quad (5.2)$$

On the other hand, Branciari in [1] shows that

$\left[\frac{m-n}{(m+1)(n+1)}\right]^{\frac{(m+1)(n+1)}{m-n}} \le \frac{1}{2}\left[\frac{m-n}{mn}\right]^{\frac{mn}{m-n}}$ for all m, n ∈ ℕ. Therefore

$$\left(\left[\frac{m-n}{(m+1)(n+1)}\right]^{\frac{(m+1)(n+1)}{m-n}}, \frac{1}{\alpha}\left[\frac{\alpha(m-n)}{(m+1)(n+1)}\right]^{\frac{(m+1)(n+1)}{\alpha(m-n)}}\right) \le \frac{1}{2}\left(\left[\frac{m-n}{mn}\right]^{\frac{mn}{m-n}}, \frac{1}{\alpha}\left[\frac{\alpha(m-n)}{mn}\right]^{\frac{mn}{\alpha(m-n)}}\right).$$

(5.3)

Thus inequalities (5.2) and (5.3) imply that

$$\int_0^{d(fx,fy)} \phi\, dp \le \frac{1}{2}\sqrt{1+\alpha^2}\left(\left[\frac{m-n}{mn}\right]^{\frac{mn}{m-n}}, \frac{1}{\alpha}\left[\frac{\alpha(m-n)}{mn}\right]^{\frac{mn}{\alpha(m-n)}}\right)$$

$$= \frac{1}{2}\int_0^{d(gx,gy)} \phi\, dp.$$

6. Fixed point results for expansive mappings

In 2011, Kadelburg et. al. [6] proved the following fixed point theorem:

Theorem 6.1. *Let (X, d) be cone metric space and let f, g : X → X be two maps such* *that f(X) ⊃ g(X) and one of the subsets f(X) and g(X) is complete. Suppose that*

(6.1) $d(fx, fy) \geq \alpha\, d(gx, gy)$ for some $\alpha > 1$ and all $x, y \in X$.

Then f and g have a unique point of coincidence.

If, moreover, the pair (f, g) is weakly compatible, then f and g have a unique common fixed point.

Now, we generalize this result using (CLRg) property along with weakly compatible maps as follows:

Theorem 6.2. *Let (X, d) be a cone metric space and f, g be two self-maps on X satisfying (CLRg) property and the following :*

(6.2) $d(gx, gy) \geq \alpha\, [\max\{d(fx, fy), d(gx, fx), d(gy, fy), d(gy, fx), d(gx, fy)\}]$ for some $\alpha>1$ and all $x, y \in X$.

Then f and g have a unique point of coincidence.

If, moreover, the pair (f, g) is weakly compatible then f and g have a unique common fixed point.

Proof. Since f and g satisfy (CLRg) property, there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$ for some $x \in X$.

From (6.2), we have

$d(gx_n, gx) \geq \alpha\, [\max\{d(fx_n, fx), d(gx_n, fx_n), d(gx, fx), d(gx, fx_n), d(gx_n, fx)\}]$ for all $n \in \mathbb{N}$.

Taking limit as $n\to\infty$, we have

$d(gx, gx) \geq \alpha\, [\max\{d(gx, fx), d(gx, gx), d(gx, fx), d(gx, gx), d(gx, fx)\} = \alpha\, d(gx, fx)$,

i.e., gx = fx.

Now, let z = fx = gx. Since f and g are weakly compatible mappings, therefore, fgx = gfx implies that fz = fgx = gfx = gz.

We claim that gz = z.

From (6.2), we have

$d(gz, z) = d(gz, gx) \geq \alpha\, [\max\{d(fz, fx), d(gz, fz), d(gx, fx), d(gx, fz), d(gz, fx)\}]$

$= \alpha\, [\max\{d(fz, fx), 0, 0, d(fx, fz), d(fz, fx)\}]$

= α d(fz, fx) = α d(gz, z), i.e., gz = z = fz.

Hence z is a common fixed point of f and g.

For the uniqueness of a common fixed point, we suppose that w is another common fixed point in X such that fw = gw.

From (6.2), we have

d(gw, gz) ≥ α [max{d(fw, fz), d(gw, fw), d(gz, fz), d(gz, fw), d(gw, fz)}] = α d(gw, gz),

i.e., gw = gz.

So, we can say that f and g have a unique common fixed point.

Example 6.1. Let E = $C_{\mathbb{R}}^1[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and P = {x∈E: x(t) ≥ 0 on [0,1]}. Then P is a non-normal cone on E. Let X = [0, 1] and let d : X × X → E be defined by d(x, y)(t) = $|x-y|\phi(t)$ where $\phi(t)>0$ is an arbitrary fixed function. Consider the functions f, g : X → X defined by fx = $\frac{x}{3}$ and gx = $\frac{x}{2}$ and take arbitrary α∈(1, $\frac{3}{2}$]. Consider the sequence $\{x_n\}$ = $\{\frac{1}{n}\}$, n ∈ $\mathbb{N}$, since $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0 = g0$, therefore, f and g satisfy the (CLRg) property. Also x = 0 is the unique common fixed point.

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