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FIXED POINT THEOREMS FOR SET VALUED CARISTI TYPE MAPPINGS IN LOCALLY CONVEX SPACE

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Abstract. The aim of this paper is to establish a generalization of Isac's fixed point theorem. We investigate the existence of common fixed and critical points for two set valued maps in locally convex space. We give an example to illustrate our result. As an application, we obtain a common fixed point theorem in probabilistic normed spaces.

Keywords: Isac's fixed point; set-valued map; locally convex space; probabilistic normed spaces.

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1. Introduction

In mathematics, fixed point theory provide an important tools and general criterion guaranteeing existence of solutions of many problems. The existence of a solution is equivalent to the existence of a fixed point for a suitable map.

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In wide range of fixed point theorems, Caristi's fixed point result is the most elegant generalization of Banach contraction principle [3]. It is also known as variation of the ε-variational principle of Ekeland [9]. Moreover, the conclusion of Caristi's theorem is equivalent to metric completeness, as proved by Weston in [21] and Kirk in [14]. Because of its importance for fixed point theory, it has been extended and generalized in many directions (see [8, 5, 13, 17]).

In [8], Downing and Kirk by using an improved version of Caristi's fixed point theorem they were able to give a generalization result in the setting of nonlinear surjective operator. After that, Isac in [13] proved a fixed point theorem of Caristi type mappings in a complete Hausdorff locally convex space and gave an application of Pareto efficiency, for more details see [12]. Recently, Lazaiz et al in [15] by using Brézis-Browder principle [4], they generalize Isac's theorem without assuming the closeness of the intermediary map (see Lemma 2.5. below), and gave an application of their result to prove the existence of solution of Generalized Nonlinear Complementarity Problem.

Since there are many spaces which are not normable, in this paper, we generalize the results of Isac (see [13]) and Lazaiz et al (see [15]). We obtain a several common fixed point theorems for two set-valued maps in a Hausdorff locally convex space and we give an example to illustrate our result. As a consequence, we get some common fixed point results in probabilistic normed spaces.

2. Preliminaries

Before providing the main result, we need to introduce some basic facts about locally convex topological vector spaces. For more on the subject, one can consult [18].

Definition 2.1. *Let X be a vector space. A functional $\rho : X \to [0,+\infty)$ is called a seminorm if it satisfies,*

(1) $\rho(\lambda x) = |\lambda| \rho(x)$ *for all* $x \in X$ *and* $\lambda \in (-\infty,+\infty)$*;*

(2) $\rho(x+y) \leq \rho(x) + \rho(y)$ *for all* $x, y \in X$.

Definition 2.2. *A family* $\{\rho_\alpha\}_{\alpha\in\Lambda}$ *of continuous seminorms on a locally convex topological vector space X is called a basis of continuous seminorms on X if for any continuous seminorm ρ on X there is a seminorm* $\delta \in \{\rho_\alpha\}_{\alpha\in\Lambda}$ *and a constant* $C > 0$ *such that, for all* $x \in X$

$$\rho(x) \leq C\delta(x).$$

The next result characterize the topology defined by continuous seminorms.

Remark 2.3. *Every Hausdorff, locally convex topological vector space has always a basis of continuous seminorms. Indeed, starting from any family* $\{\rho_\alpha\}_{\alpha\in\Lambda}$ *of seminorms generating the topology on the space and putting* $\Im = \{I \subset \Lambda,\ I \text{finite}\}$*, the family* $\{\rho_I\}_{I\in\Im}$ *of seminorms defined by*

$$\rho_I(x) = \sup_{i\in I} \rho_i(x) \quad \forall x \in X, \ \ \forall I \in \Im$$

is a basis of continuous seminorms.

The next result is a reformulation of Isac's fixed point theorem in the context of partially ordered sets.

Theorem 2.4. [13] *Let* $(X, \{\rho\}_{\alpha\in\Lambda})$, $(Y, \{\delta_i\}_{i\in I})$ *be two complete Hausdorff locally convex spaces, and* $f : X \to Y$ *a closed map. Define a binary relation on* X *by for all* $x, y \in X$

$$x \preceq y \Leftrightarrow \begin{cases} \max\{c_\alpha \rho_\alpha(x-y), c_i \delta_i(f(x)-f(y))\} \leq \varphi_{\alpha i}(f(x)) - \varphi_{\alpha i}(f(y)) \\ \forall (\alpha, i) \in \Lambda \times I \end{cases} \tag{2.1}$$

where $c_\alpha, c_i > 0$ *and* $\varphi_{\alpha i} : X \to [0, \infty)$ *is lower semi-continuous function.*
Then $(X, \preceq)$ *is partially ordered set and has a maximal element.*

Recently, Lazaiz et al (see [15]) generalize the above result in the context of pseudo-metric space without the closeness assumption of the map f.
In the sequel we assume that $(X, \{\rho\}_{\alpha\in\Lambda})$ and $(Y, \{\delta_i\}_{i\in I})$ are two complete Hausdorff locally convex spaces and $f : X \to 2^Y$ an arbitrary set valued map.

Assume that for every $(\alpha, i) \in \Lambda \times I$, there exist $c_\alpha, c_i > 0$ and lower semi-continuous functions $\varphi_{\alpha i} : Y \to [0, \infty)$.

Denote by $M_0^{\alpha i}$ a subset of $G_f = \{(x, y) \in X \times Y; y \in fx\}$ defined by

$$M_0^{\alpha i} = \left\{(x, y) \in G_f; \max\{c_\alpha \rho_\alpha(x - x_0), c_i \delta_i(y - y_0)\} \leq \varphi_{\alpha i}(y_0) - \varphi_{\alpha i}(y)\right\}$$

for some $(x_0, y_0) \in G_f$. Then we have the following result.

Lemma 2.5. *Under the above notations, define the binary relation* $\preceq$ *on* G_f *as follows*

$$(x_2, y_2) \preceq (x_1, y_1) \Leftrightarrow \max\{c_\alpha \rho_\alpha(x_1 - x_2), c_i \delta_i(y_1 - y_2)\} \leq \varphi_{\alpha i}(y_1) - \varphi_{\alpha i}(y_2) \tag{2.2}$$

for some $(\alpha,i)\in\Lambda\times I$.

Then $\left(M_0^{\alpha i},\preceq\right)$ *is partially ordered set and has a minimal element* $(\overline{x},\overline{y})$*, that is, if* $(x,y)\preceq(\overline{x},\overline{y})$ *we get* $\overline{x}=x$ *and* $\overline{y}=y$*.*

Throughout this paper we may assume that (see [2])

(1) $h:[0,\infty)\to[0,\infty)$ is right locally bounded from above that is there exists $\lambda>0$ such that $\sup_{t_0\geq 0}h([t_0,t_0+\lambda])<\infty$;

(2) $K:[0,\infty)\times[0,\infty)\to[0,\infty)$ a locally bounded function.

3. Common fixed point results

We start by the following existence result for two set valued maps.

Theorem 3.1. *Let* $\left(X,\{\rho\}_{\alpha\in\Lambda}\right)$ *and* $\left(Y,\{\delta_i\}_{i\in I}\right)$ *be two complete Hausdorff locally convex spaces. Let* $T,S:X\to 2^X$ *and* $f:X\to 2^Y$ *be nonempty set-valued maps.*

Assume that for every $(\alpha,i)\in\Lambda\times I$ *there exist* $c_\alpha,c_i>0$ *and lower semi-continuous functions* $\varphi_{\alpha i}:Y\to[0,\infty)$ *satisfies for each* $(x,y)\in G_f$ *and* $(u,v)\in Tx\times Sx$ *there exist* $(w,z)\in fu\times fv$ *such that:*

$$\begin{cases}\max\left\{c_\alpha\rho_\alpha(x-u),c_i\delta_i(y-w)\right\}\leq K\left(h\left(\varphi_{\alpha i}(y)\right),h\left(\varphi_{\alpha i}(z)\right)\right)\left[\varphi_{\alpha i}(y)-\varphi_{\alpha i}(z)\right]\\ \max\left\{c_\alpha\rho_\alpha(x-v),c_i\delta_i(y-z)\right\}\leq K\left(h\left(\varphi_{\alpha i}(y)\right),h\left(\varphi_{\alpha i}(w)\right)\right)\left[\varphi_{\alpha i}(y)-\varphi_{\alpha i}(w)\right]\end{cases}\tag{3.1}$$

Then T *and* S *have a common critical point in X, (i.e. there exists* $\overline{x}\in X$ *such that* $T\overline{x}=S\overline{x}=\{\overline{x}\}$*.)*

Proof. Let $\gamma_{\alpha i}=\inf\limits_{x\in X}\varphi_{\alpha i}(x)$ and set $\gamma=\inf\limits_{(\alpha,i)}\gamma_{\alpha i}$. Since h is right locally bounded from above, there exits $\lambda>0$ such that $a=\sup h([\gamma,\gamma+\lambda])<\infty$. It follows that there exists $b>0$ such that $K(s,t)\leq b$ for all $s,t\in[0,a]$.

Let $(x_0,y_0)\in G_f$ such that $\gamma\leq\varphi_{\alpha i}(y_0)\leq\gamma+\lambda$ for all $(\alpha,i)\in\Lambda\times I$ and define $Y_0\subset Y$ by

$$Y_0=\left\{y\in fx_0;\forall(\alpha,i),\ \varphi_{\alpha i}(y)\leq\varphi_{\alpha i}(y_0)\right\}.$$

By the hypothesis, for all $(u,v)\in Tx_0\times Sx_0$ there exist $(w,z)\in fu\times fv$ such that

$$\varphi_{\alpha i}(z)\leq\varphi_{\alpha i}(y_0)\quad\text{and}\quad\varphi_{\alpha i}(w)\leq\varphi_{\alpha i}(y_0)$$

for all $(\alpha,i)\in\Lambda\times I$, that is $w,z\in Y_0$, then we obtain

$$\varphi_{\alpha i}(z),\varphi_{\alpha i}(w)\in[0,a]\Rightarrow h(\varphi_{\alpha i}(z)),h(\varphi_{\alpha i}(w))\le a$$

for all $(\alpha,i)\in\Lambda\times I$, which implies that

$$K(h(\varphi_{\alpha i}(z)),h(\varphi_{\alpha i}(w)))\le b,$$

for all $(\alpha,i)\in\Lambda\times I$. Since $y\mapsto b\varphi_{\alpha i}(y)$ is also lower semi-continuous function we can set

$$M_0^{\alpha i}=\left\{(x,y)\in G_f;\max\{c_\alpha\rho_\alpha(x-x_0),c_i\delta_i(y-y_0)\}\le b[\varphi_{\alpha i}(y_0)-\varphi_{\alpha i}(y)]\right\}$$

using Lemma 2.5., $\left(M_0^{\alpha i},\preceq\right)$ has a minimal element, say $(\bar{x},\bar{y})$. from (3.1) we get

$$\begin{cases}\max\{c_\alpha\rho_\alpha(\bar{x}-\bar{u}),c_i\delta_i(\bar{y}-\overline{w})\}\le b[\varphi_{\alpha i}(\bar{y})-\varphi_{\alpha i}(\bar{z})]\\ \max\{c_\alpha\rho_\alpha(\bar{x}-\bar{v}),c_i\delta_i(\bar{y}-\bar{z})\}\le b[\varphi_{\alpha i}(\bar{y})-\varphi_{\alpha i}(\overline{w})]\end{cases}$$

Now, for (α,i) fixed, if $\varphi_{\alpha i}(\overline{w})\le\varphi_{\alpha i}(\bar{z})$ we get

$$\max\{c_\alpha\rho_\alpha(\bar{x}-\bar{u}),c_i\delta_i(\bar{y}-\overline{w})\}\le b[\varphi_{\alpha i}(\bar{y})-\varphi_{\alpha i}(\overline{w})]\Leftrightarrow(\bar{u},\overline{w})\preceq(\bar{x},\bar{y})$$

which implies that $\bar{u}=\bar{x}$ and $\bar{y}=\overline{w}$ then $\bar{x}=\bar{v}$, hence $\{\bar{x}\}=T\bar{x}\cap S\bar{x}$.

The same conclusion holds in the case $\varphi_{\alpha i}(\overline{w})\ge\varphi_{\alpha i}(\bar{z})$.

Let $K(s,t)=1$ for all $s,t\in[0,\infty)$ and $X=Y$, then we get the following,

Corollary 3.2. *Let $\left(X,\{\rho\}_{\alpha\in\Lambda}\right)$ be a complete Hausdorff locally convex space and let $T,S:X\to2^X$ and $f:X\to2^X$ be nonempty set-valued maps.*

Assume that for every $\alpha\in\Lambda$ there exist $c_\alpha>0$ and lower semi-continuous functions $\varphi_\alpha:X\to[0,\infty)$ satisfies for each $(x,y)\in G_f$ and $(u,v)\in Tx\times Sx$ there exist $(w,z)\in fu\times fv$ such that:

$$\begin{cases}c_\alpha\rho_\alpha(x-u)\le\varphi_\alpha(y)-\varphi_\alpha(z)\\ c_\alpha\rho_\alpha(x-v)\le\varphi_\alpha(y)-\varphi_\alpha(w)\end{cases}\tag{3.2}$$

Then T and S have a common critical point in X.

If T, S and f are only single valued maps, we have the next result.

Corollary 3.3. *Let $\left(X,\{\rho\}_{\alpha\in\Lambda}\right)$ and $\left(Y,\{\delta_i\}_{i\in I}\right)$ be two complete Hausdorff locally convex spaces. Let $T,S:X\to X$ and $f:X\to Y$ be single valued maps.*

Assume that for every $(\alpha,i)\in\Lambda\times I$ there exist $c_\alpha, c_i>0$ and lower semi-continuous functions $\varphi_{\alpha i}:Y\to[0,\infty)$ satisfies for each $x\in X$ such that:

$$\begin{cases}\max\{c_\alpha\rho_\alpha(x-Tx),c_i\delta_i(fx-f(Tx))\}\le K(h(\varphi_{\alpha i}(fx)),h(\varphi_{\alpha i}(f(Sx))))[\varphi_{\alpha i}(fx)-\varphi_{\alpha i}(f(Sx))]\\ \max\{c_\alpha\rho_\alpha(x-Sx),c_i\delta_i(fx-f(Sx))\}\le K(h(\varphi_{\alpha i}(fx)),h(\varphi_{\alpha i}(f(Tx))))[\varphi_{\alpha i}(fx)-\varphi_{\alpha i}(f(Tx))]\end{cases} \tag{3.3}$$

Then T and S have a common fixed point in X.

Example 3.4. *Let $\mathbb{R}^\omega$ be the space of all real valued sequences endowed by the family $(\rho_k)_{k\in\mathbb{N}}$ of semi-norms where $\rho_k(x)=|x_k|$ of a real valued sequence $x=(x_n)$. It is known that $(\mathbb{R}^\omega;\{\rho_k\}_k)$ is a Hausdorff complete locally convex space.*

In this example we use the following data $X=Y=\mathbb{R}^\omega$, $f(x)=x$, $h(t)=t$, $\varphi_k(x)=|x_k|$, $c_k=\frac{1}{4}$ and

$$K(s,t)=\begin{cases}2, & \text{if } (s,t)\in[0,1]\times[0,1]\\ 1, & \text{otherwise.}\end{cases}$$

$$Tx=\begin{cases}({x_1}^2,{x_2}^2,\ldots,{x_k}^2,\ldots), & \text{if } x_i\in[0,1]\ (\forall i\in\mathbb{N})\\ \left(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2},\ldots\right), & \text{if } x_i\in]1,\infty)\ (\forall i\in\mathbb{N})\\ 0, & \text{otherwise}\end{cases}$$

$$Sx=\begin{cases}({x_1}^3,{x_2}^3,\ldots,{x_k}^3,\ldots), & \text{if } x_i\in[0,1]\ (\forall i\in\mathbb{N})\\ (\frac{1}{3},\frac{1}{3},\ldots,\frac{1}{3},\ldots), & \text{if } x_i\in]1,\infty)\ (\forall i\in\mathbb{N})\\ 0, & \text{otherwise}\end{cases}$$

Note that for each $x\in[0,1]$

$$\begin{cases}\frac{1}{4}\left(x-x^2\right)\le 2\left(x-x^3\right)\\ \frac{1}{4}\left(x-x^3\right)\le 2\left(x-x^2\right)\end{cases}$$

and for each $x\in(1,\infty)$

$$\begin{cases}\frac{1}{4}\left(x-\frac{1}{2}\right)\le\left(x-\frac{1}{3}\right)\\ \frac{1}{4}\left(x-\frac{1}{3}\right)\le\left(x-\frac{1}{2}\right)\end{cases}$$

we get for all $x\in X$,

$$\begin{cases} c_k\rho_k\left(x-Tx\right)\leq K\left(\varphi_k\left(x\right),\varphi_k\left(Sx\right)\right)\left[\varphi_k\left(x\right)-\varphi_k\left(Sx\right)\right] \\ c_k\rho_k\left(x-Sx\right)\leq K\left(\varphi_k\left(x\right),\varphi_k\left(Tx\right)\right)\left[\varphi_k\left(x\right)-\varphi_k\left(Tx\right)\right] \end{cases}$$

Hence, all assumptions of Corollary 3.3. are satisfied, then T and S have a common fixed point. Note that $\overline{x}=1$ is a common fixed point for T and S.

As a direct consequence of Theorem 3.2., we have the next corollary.

Corollary 3.5. *Let $\left(X,\{\rho\}_{\alpha\in\Lambda}\right)$ be a complete Hausdorff locally convex space. Let $T,S:X\rightarrow 2^X$ and $f:X\rightarrow 2^X$ be nonempty set-valued maps.*

Assume that for every $\alpha\in\Lambda$ there exist lower semi-continuous functions $\varphi_\alpha:X\rightarrow[0,\infty)$ and functions $H_\alpha,L_\alpha:X\rightarrow[0,\infty)$ such that for some $\varepsilon>0$

$$\begin{cases} \sup\left\{H_\alpha\left(x\right):x\in X,\,\varphi_\alpha\left(x\right)\leq\inf_{z\in X}\varphi_\alpha\left(z\right)+\varepsilon\right\}<\infty \\ \sup\left\{L_\alpha\left(x\right):x\in X,\,\varphi_\alpha\left(x\right)\leq\inf_{z\in X}\varphi_\alpha\left(z\right)+\varepsilon\right\}<\infty \end{cases}$$

Assume that for each $(x,y)\in G_f$ and $(u,v)\in Tx\times Sx$ there exist $(w,z)\in fu\times fv$ such that:

$$\begin{cases} \rho_\alpha\left(x-u\right)\leq H_\alpha\left(x\right)\left[\varphi_\alpha\left(y\right)-\varphi_\alpha\left(z\right)\right] \\ \rho_\alpha\left(x-v\right)\leq L_\alpha\left(x\right)\left[\varphi_\alpha\left(y\right)-\varphi_\alpha\left(w\right)\right] \end{cases} \tag{3.4}$$

Then T and S have a common critical point in X.

Proof. Let α be fixed and set

$$X_\alpha=\left\{x\in X;\,\varphi_\alpha\left(x\right)\leq\inf_{z\in X}\varphi_\alpha\left(z\right)+\varepsilon\right\}$$

since φ_α is lower semi-continuous X_α is nonempty closed subset of X, then it is complete subset of X, put $c_\alpha=\sup_{x\in X_1}\left\{H_\alpha\left(x\right),L_\alpha\left(x\right)\right\}$.

It is clear that if $c_\alpha=0$, then all points in X_1 are critical points of T and S. Assume that $c_\alpha>0$, then we get by (3.4)

$$\begin{cases} \rho_\alpha\left(x-u\right)\leq c_\alpha\left[\varphi_{\alpha i}\left(y\right)-\varphi_{\alpha i}\left(z\right)\right] \\ \rho_\alpha\left(x-v\right)\leq c_\alpha\left[\varphi_{\alpha i}\left(y\right)-\varphi_{\alpha i}\left(w\right)\right] \end{cases}$$

for all $x \in X_1$, which implies

$$\begin{cases} c_\alpha^{-1}\rho_\alpha(x-u) \leq \varphi_{\alpha i}(y) - \varphi_{\alpha i}(z) \\ c_\alpha^{-1}\rho_\alpha(x-v) \leq \varphi_{\alpha i}(y) - \varphi_{\alpha i}(w) \end{cases}$$

using Corollary 3.2, there exits $\bar{x} \in X$ such that $\{\bar{x}\} = T\bar{x} \cap S\bar{x}$.

In the sequel, we assume that $T \neq S$, i.e. there exits $x \in X$ such that $Tx \cap Sx = \emptyset$.

Theorem 3.6. *Let $(X, \{\rho\}_{\alpha\in\Lambda})$ be a complete Hausdorff locally convex space and let $T, S :$ $X \to 2^X$ be two set-valued maps.*

Assume that for every $\alpha \in \Lambda$ there exist $c_\alpha > 0$ and lower semi-continuous functions $\varphi_\alpha :$ $X \to [0,\infty)$ satisfies for each $x \in X$ and $(u,v) \in Tx \times Sx$ such that:

(1) $c_\alpha\rho_\alpha(u-v) \leq \varphi_\alpha(u) - \varphi_\alpha(v)$;

(2) $\varphi_\alpha(u) \leq \varphi_\alpha(x)$.

Then T and S have a common critical point in X.

Proof. Let $\varepsilon > 0$, $\gamma_\alpha = \inf_{x\in X} \varphi_\alpha(x)$ and $\varphi_0 = \inf_{\alpha\in\Lambda} \gamma_\alpha$. Define X_0 as follows

$$X_0 = \{x \in X; (\forall\alpha \in \Lambda)\ \varphi_\alpha(x) \leq \varphi_0 + \varepsilon\}$$

which is complete subset of X since φ_α is lower semi-continuous function. Let define a partial order on X_0 by

$$x \preccurlyeq y \Leftrightarrow c_\alpha\rho_\alpha(x-y) \leq \varphi_\alpha(x) - \varphi_\alpha(y) \quad (\forall\alpha \in \Lambda)$$

it is clear that if $x \in X_0$ we get $\varphi_\alpha(x) \leq \varphi_0 + \varepsilon$ and hence using conditions (1) and (2) we get

$$\varphi_\alpha(v) \leq \varphi_\alpha(u) \leq \varphi_\alpha(x) \leq \varphi_0 + \varepsilon$$

where $(u,v) \in Tx \times Sx$, that is, T and S maps X_0 to 2^{X_0}.

Theorem 2.4. shows that $(X_0, \preccurlyeq)$ has a maximal element, say $\bar{x}$. Then

$$\bar{u} \preccurlyeq \bar{x} \quad \text{and} \quad \bar{v} \preccurlyeq \bar{x}$$

which is equivalent to

$$c_\alpha\rho_\alpha(\bar{u}-\bar{x}) \leq \varphi_\alpha(\bar{u}) - \varphi_\alpha(\bar{x}) \quad (\forall\alpha \in \Lambda) \tag{3.5}$$

again, the condition (2) implies that

$$\varphi_\alpha(\overline{u}) = \varphi_\alpha(\overline{x})$$

so by (3.5) we get $\overline{u} = \overline{x}$. By condition (1) we obtain

$$c_\alpha \rho_\alpha(\overline{u} - \overline{v}) \leq \varphi_\alpha(\overline{v}) - \varphi_\alpha(\overline{v}) \quad (\forall \alpha \in \Lambda)$$

and since $\overline{u} = \overline{x}$ we get $\overline{x} \preccurlyeq \overline{v}$ i.e. $\overline{x} = \overline{v}$. Which completes the proof.

4. Application to Probabilistic Normed Spaces.

Definition 4.1.[16] *A probabilistic normed space is a triple $(X, F, \min)$, where X is a linear space, $F = \{F_x :\ x \in X\}$ is a family of distribution functions satisfying:*

(1) $F_x(0) = 0$ *for all* $x \in X$

(2) $F_x(t) = 1$ *for all* $t > 0 \Leftrightarrow x = 0$

(3) $F_{\alpha x}(t) = F_x(\frac{t}{|\alpha|})$, $\mathbb{C}$ *or* $\mathbb{R}$, $\alpha \neq 0, \forall x \in X$

(4) $F_{x+y}(s+t) \geq \min(F_x(s), F_y(t)), \forall x, y \in X, \forall t, s \geq 0.$

The topology in X is defined by the system of neighborhoods of $0 \in X$

$$U(0, \varepsilon, \lambda) = \{x \in X;\ F_x(\varepsilon) > 1 - \lambda, \varepsilon > 0, \lambda \in (0,1)\}$$

This is a locally convex Hausdorff topology, called the (ε, λ)-topology [11, 7].

To see this we define for each $\lambda \in (0,1)$

$$\rho_\lambda(x) = \sup\{t \in \mathbb{R};\ F_x(t) \leq 1 - \lambda\} \tag{4.1}$$

From properties (1)-(4) of F_x one can verify that ρ_λ is a seminorm defining a topology on X which coincides with the (ε, λ)-topology. In particular, we have

$$F_x(\rho_\lambda(x)) \leq 1 - \lambda,\ \forall x \in X,\ \forall \lambda \in (0,1) \tag{4.2}$$

and

$$\rho_\lambda(x) < \varepsilon \Leftrightarrow F_x(\varepsilon) > 1 - \lambda \tag{4.3}$$

for more details see [20, 11, 7].

Definition 4.2.

(1) We say that $\{x_n\}_{n\in\mathbb{N}}$*, sequence of elements from a probabilistic normed space* $(X,F,\min)$ *is a Cauchy sequence if for every* $\varepsilon>0$ *and* $\lambda\in(0,1)$ *there exists an* $n_0=n_0(\varepsilon,\lambda)$ *such that*

$$\forall n\geq n_0,\forall m\in\mathbb{N},F_{x_{n+m}-x_n}(\varepsilon)>1-\lambda.$$

(2) A probabilistic normed space $(X,F,\min)$ *is said to be complete if every Cauchy sequence in* X *converges in* X*.*

We define probabilistic Caristi-type map for single valued and set valued mappings as follows.

Definition 4.3.

(1) A self mapping T *in* $(X,F,\min)$ *is said to be probabilistic Caristi-type mapping on* X *if there is a lower semi-continuous function* $\varphi_\lambda: X\to[0,\infty)$ *satisfying the condition*

$$\varphi_\lambda(x)-\varphi_\lambda(Tx)<t\Rightarrow F_{x-Tx}(t)>1-\lambda,\forall x\in X,\forall\lambda\in(0,1). \tag{4.4}$$

(2) Let $T,S:X\to 2^X$ *be two set-valued maps in* $(X,F,\min)$*. We say that* T *and* S *satisfy the common Caristi condition (CC)-property, if there is a lower semi-continuous function* $\varphi_\lambda: X\to[0,\infty)$ *satisfying for each* $\lambda\in(0,1)$*,* $x\in X$ *and* $(u,v)\in Tx\times Sx$*,*

$$\varphi_\lambda(u)-\varphi_\lambda(v)<t\Rightarrow F_{u-v}(t)>1-\lambda \tag{4.5}$$

Lemma 4.4. *Every probabilistic Caristi-type mapping on probabilistic normed space* $(X,F,\min)$ *is Caristi-type mapping in the corresponding locally convex space* $(X,\{\rho_\lambda\})$

Proof. Suppose the contrary, that there exist $\lambda\in(0,1)$ and $x\in X$ such that

$$\varphi_\lambda(x)-\varphi_\lambda(Tx)<\rho_\lambda(x-Tx)$$

Then, there exists $\lambda_0>0$ such that

$$\varphi_\lambda(x)-\varphi_\lambda(Tx)\leq\lambda_0\leq 1-F_{x-Tx}(t)$$

so $1-F_{x-Tx}(t)<1-\lambda_0$ which contradicts condition (1).

Therefore,

$$\rho_\lambda(x-Tx)\leq\varphi_\lambda(x)-\varphi_\lambda(Tx).$$

Theorem 4.5. *Let $(X,F,\min)$ be a complete probabilistic normed space and $T: X \to X$ a probabilistic Caristi-type mapping. Then T has a fixed point.*

Proof. Using Lemma 4.4. and Theorem 2.4.

Lemma 4.6. *Let $(X,F,\min)$ be a probabilistic normed space. Then $(X,F,\min)$ is complete if and only if the locally convex space $(X,\{\rho_\lambda\})$ is complete.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$, be Cauchy sequence in $(X,F,\min)$, there exists an $n_0 = n_0(\varepsilon,\lambda)$

$$\forall n \geq n_0, \forall m \in \mathbb{N}, F_{x_{n+m}-x_n}(\varepsilon) > 1-\lambda$$

for evrey $\varepsilon > 0$ and $\forall\lambda \in (0,1)$ by (4.3)

$$\rho_\lambda(x_{n+m}-x_n) < \varepsilon \Leftrightarrow F_{x_{n+m}-x_n}(\varepsilon) > 1-\lambda$$

then $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy sequence in $(X,\{\rho_\lambda\})$.

Therefore is convergent as a result of $(X,F,\min)$ is complete.

Next Theorem is an immediate application from the common fixed point results of the previous section.

Theorem 4.7. *Let $(X,F,\min)$ be a complete probabilistic normed space and $T,S: X \to 2^X$ two set-valued maps satisfy the (CC)-property for a family of lower semi-continuous functions $\varphi_\lambda : X \to [0,\infty)$.*
Then T and S have a common critical point in X provided that $\varphi_\lambda(u) \leq \varphi_\lambda(x)$ for all $\lambda \in (0,1)$.

Proof. Let $\rho_\lambda(x) = \sup\{t \in \mathbb{R};\ F_x(t) \leq 1-\lambda\}$, then we have for each $x \in X$ and $(u,v) \in Tx \times Sx$

$$\rho_\lambda(u-v) \leq \varphi_\lambda(u) - \varphi_\lambda(u)$$

Hence, T and S satisfy all conditions of Theorem 3.6. with $(X,\{\rho_\lambda\})$ corresponding to $(X,F,\min)$, so T and S have a common critical point in X.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] M. Abbas, B. Ali, and S. Romaguera, Multivalued Caristis type mappings in fuzzy metric spaces and a characterization of fuzzy metric completeness, Filomat, 29 (6) (2015), 1217–1222.

[2] J.S. Bae, E.W. Cho, and S.H. Yeom, A generalization of the Caristi-Kirk fixed point theorem and its applications to mapping theorems, J. Korean Math. Soc, 31 (1994), 29–48.

[3] S. Banach, Sur les oprations dans les ensembles abstraits et leur application aux quations intgrales, Fund. Math. 3(1) (1922), 133–181.

[4] H. Brézis and F.E. Browder, A general principle on ordered sets in nonlinear functional analysis, Adv. Math. 21(3) (1976), 355–364.

[5] J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, Trans. Amer. Math. Soc. 215 (1976), 241–251.

[6] J. Caristi and W. A. Kirk, Geometric fixed point theory and inwardness conditions, The Geometry of Metric and Linear Spaces, (1975), 74–83.

[7] J. Chi, C. Yan, M. Zhang, and B. Ju, On locally convex probabilistic normed spaces, J. Inequal. Appl. 2016(1) (2016), 319.

[8] D. Downing and W. A. Kirk, A generalization of Caristi's theorem with applications to nonlinear mapping theory, Pac. J. Math. 69(2) (1977), 339–346.

[9] I. Ekeland, On the variational principle, J. Math. Anal. Appl. 47(2) (1974), 324–353.

[10] Y. Feng and S. Liu, Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings, J. Math. Anal. Appl. 317(1) (2006), 103–112.

[11] O. Hadic and N. Sad, On the (ε,λ)-topology of probabilistic locally convex spaces, Glasnik matematicki, 13(33) (1978), 293–297.

[12] G. Isac, Sur l'existence de l'optimum de Pareto, Riv. Mat. Univ. Parma (4) 9 (1983), 303–325.

[13] G. Isac, Un theoreme de point fixe de type Caristi dans les espaces localement convexes, Review Res. Fac. Sci. Univ. Novi Sad (Math. Ser.), 15 (1985), 31–42.

[14] W.A. Kirk, Caristi's fixed point theorem and metric convexity, In Colloquium Mathematicae, volume 36, pages 81–86. Inst. Math. Polish Acad. Sci. 1976.

[15] S. Lazaiz, K. Chaira, M. Aamri, and El-M. Marhrani, Pseudo-metric space and fixed point theorem, Fixed Point Theory Appl. 2017 (2017), Article ID 3.

[16] B. Schweizer and A. Sklar, Probabilistic metric spaces, Courier Corporation, 2011.

[17] J. Siegel, A new proof of Caristi's fixed point theorem, Proc. Amer. Math. Soc. 66 (1977), 54–56.

[18] F. Treves, Locally convex spaces and linear partial differential equations, volume 146. Springer Science and Business Media, 2012.

[19] M. Turinici, Functional type Caristi-Kirk theorems, Libertas Math. 25 (2005), 1–12.

[20] H. D. Vuong, A fixed point theorem for nonexpansive mappings in locally convex spaces, Vietnam J. Math. 34 (2) (2006), 149–155.

[21] J.D. Weston, A characterization of metric completeness, Proc. Amer. Math. Soc. 64(1) (1977), 186–188.