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# BEST PROXIMITY POINTS FOR TRICYCLIC CONTRACTIONS

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**Abstract.** Consider a self mapping $T$ defined on the union of three subsets $A, B$ and $C$ of a metric space, $T$ is to be called a tricyclic mapping if it satisfies $T(A) \subseteq B$, $T(B) \subseteq C$ and $T(C) \subseteq A$. In this work we shall give an existence theorem of a best proximity point for tricyclic contractions in reflexive Banach spaces. The concept of tricyclic contractions is firstly introduced in this article.

**Keywords:** Best proximity point; tricyclic contractions; reflexive Banach spaces.

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## 1. Introduction

The Banach contraction mapping principle is a classical and powerful tool in nonlinear analysis which first appeared in 1922, this principle has been extended throughout the years. Let $(X,d)$ be a complete metric space and let $T$ be a contraction mapping. Then $T$ has a unique

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fixed point, i.e., the equation $Tx = x$ has a unique solution. However, let us assume that $A, B$ be two subsets of $(X,d)$ and $T : A \cup B \longrightarrow A \cup B$, $T$ is said to be cyclic provided that $T(A) \subseteq B$ and $T(B) \subseteq A$. Plainly $A \cap B \neq \varnothing$ is a necessary condition for the existence of a fixed point of $T$. Now if $A \cap B = \varnothing$, then to find an element $x \in A \cup B$ such that the $d(x,Tx) = dist(A,B)$ where $dist(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$ which is called best proximity point, is the main idea of best proximity point theorems, it can be said on the other hand, it is to study the global minimization of the real valued function $x \longmapsto d(x,Tx)$. Research on the best proximity point is an important topic in nonlinear analysis and has been studied by several authors.

In 2003, Kirk-Srinavasan-Veeramani [1] proved the following fixed point theorem for a cyclic map. Let $A$ and $B$ be nonempty, closed subsets of a complete metric space $(X,d)$. Suppose $T$ is a cyclic mapping such that

$$d(Tx,Ty) \leq kd(x,y), \text{ for some } k \in ]0,1[ \text{ and for all } (x,y) \in A \times B.$$

Then $A \cap B \neq \varnothing$ and $T$ has a unique fixed point in $A \cap B$.

In [2] Eldred and Veeramani introduced the class of cyclic contractions.

**Definition 1.1.** *Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$, a mapping $T : A \cup B \longrightarrow A \cup B$ is said to be cyclic contraction if $T$ is cyclic and if there exists $k \in (0,1)$ such that*

$$d(Tx,Ty) \leq kd(x,y) + (1-k)\, dist(A,B), \textit{ for all } (x,y) \in A \times B.$$

For a uniformly convex Banach space $X$, Eldred and Veeramani proved the following theorem.

**Theorem 1.2.** [2] *Let $A$ and $B$ be nonempty, closed and convex subsets subsets of a uniformly convex Banach space $X$ and let $T : A \cup B \longrightarrow A \cup B$ be a cyclic contraction map, for $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then there exists a unique $x \in A$ such that $x_{2n} \longrightarrow x$ and $\|x - Tx\| = dist(A,B)$.*

The next result is proven in [3]

**Theorem 1.3.** *If $T$ is a cyclic contraction defined on the union of two nonempty, bounded, closed and convex subsets of a reflexive Banach space, then $T$ has a best proximity point.*

## 2. Preliminaries

Let $A, B$ and $C$ be nonempty subsets of a metric space $(X, d)$.

A mapping $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ is said to be tricyclic provided that $T(A) \subseteq B$, $T(B) \subseteq C$ and $T(C) \subseteq A$. The following notations will be used all the way through this paper:

$D : X \times X \times X \longrightarrow [0, +\infty), (x, y, z) \longmapsto D(x, y, z) = d(x, y) + d(y, z) + d(z, x)$,

$\delta(A, B, C) = \inf\{D(x, y, z) : x \in A,\ y \in B \text{ and } z \in C\}$,

$\Delta(A, B, C) = \sup\{D(x, y, z) : x \in A,\ y \in B \text{ and } z \in C\}$,

$\Delta_{(x,y)}(C) = \sup\{D(x, y, z) : z \in C\}$ for all $x \in A$ and $y \in B$.

**Proposition 2.1.** *Let $A, B$ and $C$ be nonempty, closed subsets of a complete metric space $(X, d)$. Suppose $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ is a tricyclic mapping such that*

$$D(Tx, Ty, Tz) \leq kD(x, y, z), \text{ for some } k \in ]0, 1[ \text{ and for all } (x, y, z) \in A \times B \times C.$$

*Then $A \cap B \cap C$ is non empty and $T$ has a unique fixed point in $A \cap B \cap C$.*

**Proof.** For any $x \in A \cup B \cup C$, we have

$$\begin{aligned} d\left(Tx, T^2x\right) &\leq D\left(Tx, T^2x, T^3x\right) \leq kD\left(x, Tx, T^2x\right) \\ d\left(T^2x, T^3x\right) &\leq D\left(T^2x, T^3x, T^4x\right) \leq kD\left(Tx, T^2x, T^3x\right) \leq k^2D\left(x, Tx, T^2x\right) \end{aligned}$$

Inductively

$$d\left(T^n x, T^{n+1}x\right) \leq k^n D\left(x, Tx, T^2x\right)$$

which implies that $(T^n x)$ is a Cauchy sequence. Hence, $(T^n x)$ converges to some point $t \in X$, nevertheless, an infinite number of terms of the sequence $(T^n x)$ lie in $A$, as in $B$ and $C$, as a result, $t \in A \cap B \cap C$. The Banach's contraction mapping principle applied to $T$ on $A \cap B \cap C$ assures the existence and the uniqueness of the fixed point.

**Example 2.2.** *Let $X$ be $\mathbb{R}^3$ endowed with its euclidean distance.*

*Suppose $A = [0, 1] \times \{0\} \times \{0\}$, $B = \{0\} \times [0, 1] \times \{0\}$ and $C = \{0\} \times \{0\} \times [0, 1]$.*

*Let $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ be defined by*

$$T(x, y, z) = \begin{cases} \left(0, \frac{x}{2}, 0\right), \ if \ (x, y, z) \in A, \\ \left(0, 0, \frac{y}{2}\right), \ if \ (x, y, z) \in B, \\ \left(\frac{z}{2}, 0, 0\right), \ if \ (x, y, z) \in C. \end{cases}$$

*Clearly, $T$ is tricyclic on $A\cup B\cup C$. Consider $(x,0,0)\in A$, $(0,y,0)\in B$ and $(0,0,z)$ , we have*

$$\begin{aligned} D(T(x,0,0),T(0,y,0),T(0,0,z)) &= \sqrt{\frac{x^2}{4}+\frac{y^2}{4}}+\sqrt{\frac{y^2}{4}+\frac{z^2}{4}}+\sqrt{\frac{x^2}{4}+\frac{z^2}{4}} \\ &= \frac{1}{2}\left(\sqrt{x^2+y^2}+\sqrt{y^2+z^2}+\sqrt{x^2+z^2}\right) \\ &= \frac{1}{2}D((x,0,0),(0,y,0),(0,0,z)). \end{aligned}$$

*It now follows from the Proposition 2.1 that $T$ has a unique fixed point.*

The concept of cyclic contraction introduced in [2] can be extended to tricyclic mappings using the map $D$.

**Definition 2.3.** *Let $A,B$ and $C$ be nonempty subsets of a metric space $(X,d)$, a mapping $T: A\cup B\cup C\longrightarrow A\cup B\cup C$ is said to be tricyclic contraction if $T$ is tricyclic and if there exists $k\in(0,1)$ such that*

$$D(Tx,Ty,Tz)\leq kD(x,y,z)+(1-k)\,\delta(A,B,C)\,,\ \textit{for all}\ (x,y,z)\in A\times B\times C.$$

*And a point $x\in A\cup B\cup C$ is said to be best proximity point for $T$ provided that*

$$D\left(x,Tx,T^2x\right)=\delta(A,B,C)\,.$$

The mapping $D$ can be looked upon as a generalized distance, and the following property is its triangular inequality

$$D(x,y,z)\leq D(x,y,t)+2d(t,z)\,,\ D(x,t,z)+2d(t,y)\ \text{and}\ D(t,y,z)+2d(t,x)\,.$$

**Definition 2.4.** *Let $a,b$ be two points of $(X,d)$ and $r>0$, the closed ball $(resp.opened)$ of focuses a and b, and of ray r, is defined by*

$$B(a,b,r)=\{x\in X: D(a,b,x)\leq (resp.\ \lneqq)\, r\}\,.$$

The ball of foci $a$ and $b$ can be empty, $d(a,b)\leq\frac{r}{2}$ is a necessary and sufficient condition for $B(a,b,r)$ to be nonempty. Let $x\in B(a,b,r)$, then $D(a,b,x)\leq r$,
which implies $x\in B(a,r)\cap B(b,r)$, consequently $B(a,b,r)$ is bounded and it is closed for the mapping $D$ is continued. Furthermore, if $X$ is a normed vector space, then $B(a,b,r)$ is convex.

Let $x,y\in B(a,b,r)$ and $\lambda\in[0,1]$.

$$\begin{aligned} D(\lambda x+(1-\lambda)y,a,b) &\leq \lambda\|x-a\|+(1-\lambda)\|y-a\|+\lambda\|x-b\|+ \\ &\quad (1-\lambda)\|y-b\|+\lambda\|a-b\|+(1-\lambda)\|a-b\| \\ &= \lambda D(x,a,b)+(1-\lambda)D(y,a,b) \\ &\leq r. \end{aligned}$$

**Example 2.5.** *Let $X$ be $\mathbb{R}^2$ along with its Euclidean norm, then $B((0,0),(0,1),4)$ is the ellipse of focuses $(0,0)$ and $(0,1)$. If $d_2((x,y),(x',y'))=d(x,x')+d(y,y')$ then $B((0,1),(1,0),4)$ is a "generalized" lozenge, and*

$$B((0,1),(1,0),4)=B((0,0),(1,1),4).$$

*Two balls of distinct foci may be equal.*

## 3. Main results

We first give a useful approximation result in the form of a lemma

**Lemma 3.1.** *Let $A,B$ and $C$ be nonempty subsets of a metric space $(X,d)$.*
*Let $T:A\cup B\cup C\longrightarrow A\cup B\cup C$ be a tricyclic contraction map, then starting with any $x_0$ in $A\cup B\cup C$ we have $D\left(x_n,Tx_n,T^2x_n\right)\underset{n\longrightarrow\infty}{\longrightarrow}\delta(A,B,C)$ where $x_{n+1}=Tx_n$, $n=0,1...$*

**Proof.** Let $n\in\mathbb{N}^3\backslash\{1,2\}$, we have

$$\begin{aligned} D\left(x_n,Tx_n,T^2x_n\right) &\leq kD(x_{n-1},x_n,x_{n+1})+(1-k)\delta(A,B,C) \\ &\leq k(kD(x_{n-2},x_{n-1},x_n)+(1-k)\delta(A,B,C))+(1-k)\delta(A,B,C) \\ &= k^2D(x_{n-2},x_{n-1},x_n)+\left(1-k^2\right)\delta(A,B,C) \\ &\leq k^2(kD(x_{n-3},x_{n-2},x_{n-1})+(1-k)\delta(A,B,C))+\left(1-k^2\right)\delta(A,B,C) \\ &= k^3D(x_{n-3},x_{n-2},x_{n-1})+\left(1-k^3\right)\delta(A,B,C). \end{aligned}$$

Inductively, we have

$$\delta(A,B,C)\leq D\left(x_n,Tx_n,T^2x_n\right)\leq k^nD(x_0,x_1,x_2)+(1-k^n)\delta(A,B,C).$$

As a consequence, $D\left(x_n, Tx_n, T^2x_n\right) \underset{n\longrightarrow\infty}{\longrightarrow} \delta(A,B,C)$.

Afterwards we give a simple existence result from which we can deduce an existence result of best proximity point for tricyclic contractions when one of the sets is boundedly compact.

**Proposition 3.2.** *Let $A,B$ and $C$ be nonempty subsets of a metric space $(X,d)$. Let $T : A\cup B\cup C \longrightarrow A\cup B\cup C$ be a tricyclic contraction, continuous on $A$, let $x_0 \in A$, and define $x_{n+1} = Tx_n$. Suppose $(x_{3n})$ has a convergent subsequence in $A$. Then there exists $x \in A$ such that*

$$D\left(x, Tx, T^2x\right) = \delta(A,B,C).$$

**Proof.** Suppose $\left(x_{3n_k}\right)$ converges to $x \in A$, we have

$$\begin{aligned} \delta(A,B,C) &\le D\left(x_{3n_k-1}, x, Tx\right) \\ &\le D\left(x_{3n_k-1}, x_{3n_k}, Tx\right) + 2d\left(x_{3n_k}, x\right) \\ &\le D\left(x_{3n_k-1}, x_{3n_k}, x_{3n_k+1}\right) + 2d\left(x_{3n_k+1}, Tx\right) + 2d\left(x_{3n_k}, x\right) \end{aligned}$$

But $d\left(x_{3n_k}, x\right) \underset{k\longrightarrow\infty}{\longrightarrow} 0$, and, since $T$ is continuous on $A$, $d\left(x_{3n_k+1}, Tx\right) \underset{k\longrightarrow\infty}{\longrightarrow} 0$. The Lemma 3.1 ensures that

$$D\left(x_{3n_k-1}, x_{3n_k}, x_{3n_k+1}\right) \underset{k\longrightarrow\infty}{\longrightarrow} \delta(A,B,C).$$

Thus

$$D\left(x_{3n_k-1}, x, Tx\right) \underset{k\longrightarrow\infty}{\longrightarrow} \delta(A,B,C).$$

Since

$$\begin{aligned} D\left(x_{3n_k}, Tx, T^2x\right) - \delta(A,B,C) &\le k\left(D\left(x_{3n_k-1}, x, Tx\right) - \delta(A,B,C)\right) \\ &\le D\left(x_{3n_k-1}, x, Tx\right) - \delta(A,B,C). \end{aligned}$$

Hence $D\left(x_{3n_k}, Tx, T^2x\right) \le D\left(x_{3n_k-1}, x, Tx\right)$. Consequently

$$\delta(A,B,C) \le D\left(x_{3n_k}, Tx, T^2x\right) \le D\left(x_{3n_k-1}, x, Tx\right) \underset{k\longrightarrow\infty}{\longrightarrow} \delta(A,B,C)$$

This means $D\left(x, Tx, T^2x\right) = \delta(A,B,C)$.

**Corollary 3.3.** *Let $A,B$ and $C$ be nonempty subsets of a metric space $(X,d)$. Let $T : A\cup B\cup C \longrightarrow A\cup B\cup C$ be a tricyclic contraction map, continuous. If either $A,B$ or $C$ are boundedly compact, then there exists exists $x \in A\cup B\cup C$ such that $D\left(x, Tx, T^2x\right) = \delta(A,B,C)$.*

The coming proposition leads to an existence result when $X$ is a normed linear space and the dimension of the span of one of the sets is finite.

**Proposition 3.4.** *Let $A,B$ and $C$ be nonempty subsets of a complete metric space $(X,d)$, let $T: A\cup B\cup C \longrightarrow A\cup B\cup C$ be a tricyclic contraction map. Then for any $x_0$ in $A\cup B\cup C$ and $x_{n+1} = Tx_n$, $n = 0,1...$, the sequences $(x_{3n}), (x_{3n+1})$ and $(x_{3n+2})$ are bounded.*

**Proof.** Suppose $x_0 \in A$ (the proof when $x_0$ is in $B$ or $C$ is similiar ), then, by the lemma 3.1, $D(x_{3n}, x_{3n+1}, x_{3n+2})$ converges to $\delta(A,B,C)$. It is enough then to prove that $(x_{3n+2})$ is bounded.

Suppose the contrary, then there exists $N$ such that

$$d\left(T^4x_0, T^{3N+2}x_0\right) > \frac{M}{2} \quad \text{where} \quad M > \frac{D\left(x_0, Tx_0, T^4x\right)}{1/k^3 - 1} + \delta(A,B,C).$$

Let $N_0 = \min\left\{N : d\left(T^4x_0, T^{3N+2}x_0\right) > \frac{M}{2}\right\}$, then

$$d\left(T^4x_0, T^{3N_0+2}x_0\right) > \frac{M}{2} \quad \text{and} \quad d\left(T^4x_0, T^{3N_0-1}x_0\right) \le \frac{M}{2},$$

Therefore

$$D\left(T^3x_0, T^4x_0, T^{3N_0+2}x_0\right) \ge 2d\left(T^4x_0, T^{3N_0+2}x_0\right) > M.$$

The tricyclic contraction property of $T$,

$$M < D\left(T^3x_0, T^4x_0, T^{3N_0+2}x_0\right) \le k^3 D\left(x_0, Tx_0, T^{3N_0-1}x_0\right) + \left(1-k^3\right)\delta(A,B,C),$$

Hence

$$\begin{aligned} \frac{M-\delta(A,B,C)}{k^3} + \delta(A,B,C) &\le d(x_0,Tx_0) + d\left(Tx_0, T^{3N_0-1}x_0\right) + d\left(T^{3N_0-1}x_0, x_0\right) \\ &\le d(x_0,Tx_0) + d\left(Tx_0, T^4x_0\right) + d\left(T^4x_0, T^{3N_0-1}x_0\right) \\ &\quad + d\left(T^{3N_0-1}x_0, T^4x_0\right) + d\left(T^4x_0, x_0\right) \\ &\le D\left(x_0, Tx_0, T^4x\right) + M. \end{aligned}$$

Thus, $M \le \frac{D\left(x_0,Tx_0,T^4x\right)}{1/k^3-1} + \delta(A,B,C)$, which is a contradiction.

**Corollary 3.5.** *Let $A,B$ and $C$ be nonempty, closed subsets of a normed linear space $X$, let $T: A\cup B\cup C \longrightarrow A\cup B\cup C$ be a tricyclic contraction map. If either the span of $A$, the span of $B$ or the span of $C$ is a finite dimensional subspace of $X$, then there exists exists $x \in A\cup B\cup C$ such that $D\left(x, Tx, T^2x\right) = \delta(A,B,C)$.*

**Proof.** Assume that the span of $A$ is a finite dimensional subspace of $X$. By the previous proposition 3.4, we have, $(x_{3n})$ is bounded. The Bolzano-Weierstrass theorem assures that the sequence $(x_{3n})$ has a convergent subsequence, $x_{3n_k} \underset{k\longrightarrow\infty}{\longrightarrow} x \in A$, since $A$ is closed, and by using the proposition 3.2, we conclude that $D\left(x, Tx, T^2x\right) = \delta(A,B,C)$.

**Definition 3.6.** [4] *A normed linear space $X$ is said to have the property $(C)$ if every bounded decreasing net of nonempty, closed and convex subsets of $X$ has a nonempty intersection.*

For example, a reflexive Banach space has the property $(C)$, so does every weakly compact convex subset of a Banach.

We now proceed to our main result of this paper.

**Theorem 3.7.** *Let $A, B$ and $C$ be nonempty, closed, bounded and convex subsets of a reflexive Banach space $X$, let $T : A\cup B\cup C \longrightarrow A\cup B\cup C$ be a tricyclic contraction map. Then $T$ has a best proximity point.*

**Proof.** Define

$$\Sigma = \left\{ (E,F,G) \subset (A,B,C) \Big/ \begin{array}{c} E,F \text{ and } G \text{ are nonempty, bounded, closed} \\ \text{and convex with } T \text{ is tricyclic on } E\cup F\cup G. \end{array} \right\}.$$

$\Sigma$ is nonempty for $(A,B,C) \in \Sigma$ and is partially ordered by the reverse inclusion, that is,

$$(E_1,F_1,G_1) \le (E_2,F_2,G_2) \Longleftrightarrow (E_2,F_2,G_2) \subset (E_1,F_1,G_1).$$

Let $(E_i,F_i,G_i)_{i\in I}$ be an increasing chain of $\Sigma$, i.e. $(E_i)_{i\in I}$ , $(F_i)_{i\in I}$ and $(G_i)_{i\in I}$ are decreasing, since $X$ is a reflexive Banach space, $\underset{i\in I}{\cap} E_i$ , $\underset{i\in I}{\cap} F_i$ and $\underset{i\in I}{\cap} G_i$ are nonempty, bounded and closed. But

$$(E_i,F_i,G_i)_{i\in I} \le \left( \underset{i\in I}{\cap} E_i , \underset{i\in I}{\cap} F_i, \underset{i\in I}{\cap} G_i \right), \text{ for all } i \text{ in } I.$$

Every increasing chain in $\Sigma$ is bounded above, so by using Zorn's lemma we obtain a maximal element, say $(H,I,J) \in \Sigma$. We have

$$\left(\overline{co}\left(T\left(J\right)\right), \overline{co}\left(T\left(H\right)\right), \overline{co}\left(T\left(I\right)\right)\right) \subset (H,I,J)$$

Hence

$$T\left(\overline{co}\left(T\left(J\right)\right)\right) \subset T\left(H\right) \subset \overline{co}\left(T\left(H\right)\right).$$

Thus $T$ is cyclic on $\overline{co}(T(J)) \cup \overline{co}(T(H)) \cup \overline{co}(T(I))$, it now follows from the maximality of $(H,I,J)$

$$\overline{co}(T(J)) = H,\ \overline{co}(T(H)) = I \text{ et } \overline{co}(T(I)) = J.$$

Let $x \in H$, $y \in I$, for all $z \in J$, we have

$$\begin{aligned} D(Tx,Ty,Tz) &\leq kD(x,y,z) + (1-k)\,\delta(A,B,C) \\ &\leq k\Delta(H,I,J) + (1-k)\,\delta(A,B,C) = \Lambda. \end{aligned}$$

Then

$$Tz \in B(Tx,Ty,\Lambda),\ \text{for all } z \in J.$$

Therefore $T(J) \subset B(Tx,Ty,\Lambda)$ which is nonempty, closed, bounded and convex. Thus

$$H = \overline{co}(T(J)) \subset B(Tx,Ty,\Lambda).$$

So

$$\Delta_{(Tx,Ty)}(H) \leq \Lambda.$$

Put

$$\begin{aligned} E &= \left\{(x,y) \in H \times I : \Delta_{(x,y)}(J) \leq \Lambda\right\}, \\ F &= \left\{(y,z) \in I \times J : \Delta_{(y,z)}(H) \leq \Lambda\right\}, \\ G &= \left\{(z,x) \in J \times H : \Delta_{(z,x)}(I) \leq \Lambda\right\}. \end{aligned}$$

Note that $E$, $F$ and $G$ are nonempty, bounded, closed and convex, indeed

$$E = \bigcap_{z \in J} \Psi_z^{-1}([0,\Lambda]) \ \text{ where } \Psi_z : H \times I \longrightarrow \mathbb{R}_+;\ (x,y) \longmapsto D(x,y,z).$$

Furthermore, Let $(x_1,y_1),(x_2,y_2)\in E$ and $\lambda\in[0,1]$, then
$\lambda(x_1,y_1)+(1-\lambda)(x_2,y_2)=(\lambda x_1+(1-\lambda)x_2,\lambda y_1+(1-\lambda)y_2)\in E$, indeed, let $z\in J$

$$\begin{aligned}
D(z,\lambda x_1+(1-\lambda)x_2,\lambda y_1+(1-\lambda)y_2) &= \|z-\lambda x_1-(1-\lambda)x_2\|+\|z-\lambda y_1-(1-\lambda)y_2\| \\
&\quad +\|\lambda x_1+(1-\lambda)x_2-\lambda y_1-(1-\lambda)y_2\| \\
&\leq \lambda\|z-x_1\|+(1-\lambda)\|z-x_2\|+\lambda\|z-y_1\|+(1-\lambda)\|z-y_2\| \\
&\quad +\lambda\|x_1-y_1\|+(1-\lambda)\|x_2-y_2\| \\
&= \lambda D(x_1,y_1,z)+(1-\lambda)D(x_2,y_2,z) \\
&\leq \max\{D(x_1,y_1,z),D(x_2,y_2,z)\} \\
&\leq \max\left\{\Delta_{(x_1,y_1)}(J),\Delta_{(x_2,y_2)}(J)\right\}\leq\Lambda.
\end{aligned}$$

Which implies that $E$ is convex in $X\times X$. Define

$$\begin{aligned}
\widetilde{T}\ :(A\times B)\cup(B\times C)\cup(C\times A) &\longrightarrow (A\times B)\cup(B\times C)\cup(C\times A) \\
(x,y) &\longmapsto \widetilde{T}(x,y)=(Tx,Ty)
\end{aligned}$$

Since $T$ is tricyclic on $A\cup B\cup C$, $\widetilde{T}$ is tricyclic on $(A\times B)\cup(B\times C)\cup(C\times A)$.
Let $(x,y)\in H\times I$, $\widetilde{T}(x,y)=(Tx,Ty)\in F$, then $\widetilde{T}(H\times I)\subset F$. So $\widetilde{T}$ is tricyclic on $E\cup F\cup G$.
Furthermore $(H\times I,I\times J,J\times H)$ is maximal in

$$\widetilde{\Sigma}=\left\{\begin{array}{c} ((E\times F),(F\times G),(G\times E))\subset((A\times B),(B\times C),(C\times A))\ \text{such that :} \\ (E\times F),(F\times G)\ \text{and}\ (G\times E)\ \text{are nonempty, bounded, closed} \\ \text{and convex with}\ \widetilde{T}\ \text{is tricyclic on}\ (E\times F)\cup(F\times G)\cup(G\times E)\ . \end{array}\right\}.$$

$\widetilde{\Sigma}$ is partially ordered by

$$((E_1\times F_1),(F_1\times G_1),(G_1\times E_1))\ \widetilde{\lesssim}\ ((E_2\times F_2),(F_2\times G_2),(G_2\times E_2))\Longleftrightarrow$$

$$((E_2\times F_2),(F_2\times G_2),(G_2\times E_2))\subset((E_1\times F_1),(F_1\times G_1),(G_1\times E_1)).$$

So,

$$E=H\times I,\ F=I\times J\ \text{and}\ G=J\times H.$$

Inductively, for all $(x,y) \in H \times I$

$$\begin{aligned}
\Delta_{(x,y)}(J) \leq \Lambda \implies & \Delta_{(x,y)}(J) - k\Delta(H,I,J) \leq (1-k)\,\delta(A,B,C) \\
\implies & (1-k)\Delta(H,I,J) \leq (1-k)\,\delta(A,B,C) \\
\implies & \Delta(H,I,J) \leq \delta(A,B,C).
\end{aligned}$$

For all $(p,q,r) \in (H,I,J)$, we have

$$\delta(A,B,C) \leq D\left(p,Tp,T^2p\right), D\left(T^2q,q,Tq\right), D\left(Tr,T^2r,r\right) \leq \Delta(H,I,J) = \delta(A,B,C),$$

which brings an end to the proof of the theorem.

**Example 3.8.** *Let $X$ be $\mathbb{R}^2$ normed by the norm $\|(x,y)\| = |x|+|y|$, let $A = [-2,-1]\times\{0\}$, $B = \{0\}\times[1,2]$ and $C = [1,2]\times\{0\}$, then $\delta(A,B,C) = D((-1,0),(0,1),(1,0)) = 6$.*
*Put $T : A\cup B\cup C \longrightarrow A\cup B\cup C$ such that*

$$\begin{aligned}
T(x,0) &= \left(0, \frac{-x+1}{2}\right) \; if \; x \in [-2,-1], \\
T(0,y) &= \left(\frac{y+1}{2}, 0\right) \; if \; y \in [1,2], \\
T(z,0) &= \left(-\frac{z+1}{2}, 0\right) \; if \; z \in [1,2].
\end{aligned}$$

*Then $T$ is a tricyclic contraction and $(-1,0)$ is a best proximity point. Indeed,*

$$\begin{aligned}
D(T(x,0),T(0,y),T(z,0)) &= \left(\frac{y+1}{2}+\frac{-x+1}{2}\right)+\left(\frac{y+1}{2}+\frac{z+1}{2}\right)+\left(\frac{z+1}{2}+\frac{-x+1}{2}\right) \\
&= \frac{1}{2}D((x,0),(0,y),(z,0)) + (1-\frac{1}{2})\delta(A,B,C)
\end{aligned}$$

*And $D\left((-1,0),T(-1,0),T^2(-1,0)\right) = D((-1,0),(0,1),(1,0)) = \delta(A,B,C)$.*

## Conflict of Interests

*The authors declare that there is no conflict of interests.*

## REFERENCES

[1] Kirk, WA, Srinivasan, PS, Veeramani, P: Fixed points for mappings satisfying cyclic contractive conditions. Fixed Point Theory 4 (1) (2003), 79-86.

[2] Eldred, A.A., Veeramani, P. Existence and convergence of best proximity points. J. Math. Anal. Appl. 323(2) (2006), 1001-1006..

[3] Gabeleh and Shahzad: Some new results on cyclic relatively nonexpansive mappings in convexmetric spaces. J. Inequal. Appl. 2014 (2014), Article ID 350.

[4] Takahashi, W: A convexity in metric space and nonexpansive mappings. Kodai Math. Semin. Rep. 22 (1970), 142-149.