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A NEW ALGORITHM FOR VARIATIONAL INEQUALITY PROBLEMS WITH ALPHA-INVERSE STRONGLY MONOTONE MAPS AND COMMON FIXED POINTS FOR A COUNTABLE FAMILY OF RELATIVELY WEAK NONEXPANSIVE MAPS, WITH APPLICATIONS

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Abstract. Let E be a 2-uniformly convex and uniformly smooth real Banach space with dual space E^*. Let C be a nonempty closed and convex subset of E. Let $A : C \to E^*$ and $T_i : C \to E,\ i = 1,2,\cdots$, be an α-inverse strongly monotone map and a *countable family* of relatively weak nonexpansive maps, respectively. Assume that the intersection of the set of solutions of the variational inequality problem, $VI(C,A)$, and the set of common fixed points of $\{T_i\}_{i=1}^{\infty}$, $\cap_{i=1}^{\infty} F(T_i)$, is nonempty. A generalized projection algorithm is constructed and proved to converge *strongly* to some $x^* \in VI(C,A) \cap \Big(\cap_{i=1}^{\infty} F(T_i)\Big)$. Our theorem is a significant improvement of recent important results, in particular, the results of Zegeye and Shahzad (Nonlinear Anal. 70 (7) (2009), 2707-2716), Liu (Appl. Math. Mech. -Engl. Ed. 30 (7) (2009), 925-932), and Zhang *et al.* (Appl. Math. and Informatics 29 (1-2) (2011), 87-102) and a host of other results. Finally, applications of our theorem to convex optimization problems, zeros of α-inverse strongly monotone maps and complementarity problems are presented.

Keywords: convex optimization problems; generalized projection; variational inequality problems; α-inverse strongly monotone; relatively weak nonexpansive map.

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1. INTRODUCTION

Let E be a real Banach space with dual space E^*. Let C be a nonempty closed and convex subset of E and $A : C \to E^*$ be a monotone-type map. Then, we study the following problem: find $u \in C$ such that

$$\big\langle y-u, Au\big\rangle \geq 0, \text{ for all } y \in C. \tag{1}$$

This problem is called the *variational inequality* problem and has been studied extensively by various authors (see e.g., Aoyama *et al.* [4], Blum and Oettli [6], Censor *et al.* $\big($[8],[9], [10], [11]$\big)$, Chidume [12], Chidume *et al.* [17], Gibali *et al.* [19], Iiduka and Takahashi [22], Iiduka *et al.* [23], Kassay *et al.* [25], Kinderlehrer and Stampacchia [26], Lions and Stampacchia [28], Liu [29], Liu and Nashed [31], Ofoedu and Malonza [35], Osilike *et al.* [36], Reich and Sabach [37], Reich [39], Su and Xu [42], Zegeye *et al.* [48], Zhang *et al.* [51] and the references contained in them). The set of solutions of the variational inequality problem is denoted by $VI(C,A) = \{u \in C : \langle y-u, Au\rangle \geq 0, \forall\, y \in C\}$.

Variational inequality problems are connected with convex minimization problems, zeros of monotone-type maps, complementarity problems, and so on. For more on variational inequality problems and some of their applications one is refered to the classic book of Kinderlehrer and Stampacchia [26].

A map $A : C \to E^*$ is called α*-inverse strongly monotone* if there exists $\alpha > 0$ such that

$$\big\langle x-y, Ax-Ay\big\rangle \geq \alpha\|Ax-Ay\|^2, \text{ for all } x,y \in C.$$

If A is an α-inverse strongly monotone map, then it is Lipschitz with Lipschitz constant $\frac{1}{\alpha}$, i.e. $\|Ax-Ay\| \leq \frac{1}{\alpha}\|x-y\|$, for all $x,y \in C$. In the case where $E = \mathbb{R}^N$, for finding a zero of an α-inverse strongly monotone map, Golshten and Tretyakov [20] studied the following recursion formula: $x_1 = x \in \mathbb{R}^N$ and

$$x_{n+1} = x_n - \lambda_n A x_n,\ n \geq 1, \tag{2}$$

where $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence in $[0,2\alpha]$. They proved that the sequence $\{x_n\}_{n=1}^{\infty}$ generated by the recursion formula (2) converges strongly to an element of $A^{-1}0$, where

$$A^{-1}0 := \{x \in \mathbb{R}^N : Ax = 0\}.$$

In the case that $E = H$, a real Hilbert space, a well known method for solving the variational inequality problem (1) is by the use of *projection algorithm* which starts with $x_1 = x \in C$ and generates a sequence $\{x_n\}_{n=1}^{\infty}$ using the following recursion formula,

$$x_{n+1} = P_C(x_n - \lambda_n A x_n),\ n \geq 1, \tag{3}$$

where $A : C \to H$ is a monotone map and P_C is the *metric projection* of H onto C, $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of positive numbers satisfying appropriate conditions. In the case that A is α-inverse strongly monotone, Iiduka *et al.* [23] proved that the sequence $\{x_n\}_{n=1}^{\infty}$ generated by recursion formula (3) converges *weakly* to an element of $VI(C,A)$.

In real Banach spaces more general than Hilbert spaces, Alber introduced a notion of projection, $\Pi_C : E \to C$, called *generalized projection*, which is a generalization of the metric projection on a Hilbert space. This generalized projection is now a key tool in approximation methods for nonlinear operators in real Banach spaces more general than Hilbert space.

Iiduka and Takahashi [22] used the generalized projection to prove a *weak convergence* theorem for solutions of variational inequality problems under the following assumptions:
(i) E is a 2-*uniformly convex* and uniformly smooth real Banach space,
(ii) A is α-inverse strongly monotone,
(iii) $VI(C,A) \neq \emptyset$,
(iv) $\|Ay\| \leq \|Ay - Au\|$, for all $y \in C$ and $u \in VI(C,A)$ and
(v) The normalized duality map $J : E \to E^*$ is *weakly sequentially continuous*.

Chidume *et al.* [18], using the generalized projection, proved a *strong convergence* theorem for solutions of variational inequality problems assuming only conditions (i) to (iv) above in a 2-uniformly convex and uniformly smooth real Banach space.

Remark 1. In L_p spaces, $1 < p < \infty$, $p \neq 2$, the normalized duality map J is not weakly sequentially continuous and so the theorem of Iiduka and Takahashi [22], may not be applicable, since in this theorem J is required to be weakly sequentially continuous. The theorem of Chidume *et al.* [18] is applicable in L_p spaces, $1 < p < 2$. Consequently, the theorem of Chidume *et al.* [18] provides strong convergence theorem in L_p spaces, $1 < p < 2$.

To obtain a strong convergence theorem for finding a common element of the set of fixed points of a relatively nonexpansive map and the set of solutions of a variational inequality problem for an inverse-strongly monotone map, in a uniformly smooth and 2-*uniformly convex* Banach space by using a hybrid method Liu [30] proved the following theorem.

Theorem 1.1 (Liu [30]). *Let E be a 2-uniformly convex, uniformly smooth Banach space and C be a nonempty, closed convex subset of E. Assume that A is an operator of C into E^* that satisfies the conditions (ii) to (iv) and T is a relatively nonexpansive mapping from C into itself such that $F := F(T) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by*

$$(4)\qquad \begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ w_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)J\Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)), \\ z_n = \Pi_C w_n, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTz_n), \\ C_0 = C, \\ C_n = \{v \in C_{n-1} : \phi(v,y_n) \leq \phi(v,x_n)\}, \\ Q_n = \{v \in C : \langle x_n - v, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_n = \Pi_{C_n \cap Q_n} x_0, \ \forall \ n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy $0 \leq \alpha_n < 1$, $\limsup \alpha_n < 1$, $0 \leq \beta_n < 1$ and $\limsup \beta_n < 1$. If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a,b]$ for some a, b with $0 < a < b < \frac{c^2\alpha}{2}$, then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E.

For solving variational inequality problems and fixed points of relatively weak nonexpansive maps, Zegeye and Shahzad [49] introduced the following generalized projection algorithm:

$x_0 \in C$,

$$(5)\qquad \begin{cases} x_0 \in C \ chosen \ \ arbitrarily, \\ y_n = \Pi_C J^{-1}(Jx_n - \alpha_n A x_n), \\ z_n = T y_n, \\ H_0 = \{v \in C : \phi(v, z_0) \le \phi(v, y_0) \le \phi(v, x_0)\}, \\ H_n = \{v \in H_{n-1} \cap W_{n-1} : \phi(v, z_n) \le \phi(v, y_n) \le \phi(v, x_n)\}, \\ W_0 = C, \\ W_n = \{v \in H_{n-1} \cap W_{n-1} : \langle x_n - v, Jx_0 - Jx_n \rangle \ge 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n}(x_0), \ \forall \ n \ge 0, \end{cases}$$

and proved that the sequence $\{x_n\}_{n=1}^{\infty}$ generated by recursion formula (5) converges strongly to $\Pi_{VI(K,A)\cap F(T)} x_0$, where $\Pi_{VI(K,A)\cap F(T)}$ is the generalized projection from E onto $VI(K,A) \cap F(T)$.

Motivated by Liu [30] and Zegeye and Shahzad [49], Zhang *et al.* [51] using the following algorithm

$$(6)\qquad \begin{cases} x_0 \in C \ chosen \ \ arbitrarily, \\ w_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\ z_n = \Pi_C w_n, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T z_n), \\ C_0 = C, \\ C_n = \{v \in C_{n-1} : \phi(v, y_n) \le \phi(v, x_n)\}, \\ x_n = \Pi_{C_n} x_0, \ \forall \ n \ge 0, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1)$ such that $\limsup \beta_n < 1$ and $\liminf \alpha_n < 1$, $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a,b]$ for some a, b with $0 < a < b < \frac{c^2\alpha}{2}$, proved a strong convergence theorem in a 2-*uniformly convex* and uniformly smooth Banach space, for finding a common element of the set of variational inequality problem for an inverse strongly monotone map and the set of fixed points of a relatively weak nonexpansive map.

Remark 2. The algorithm (5) of Zegeye and Shahzad is an improvement on algorithm (4) of Liu in the sense that the sequence $\{w_n\}$ defined in algorithm (4) is dispensed with in algorithm

(5). The algorithm of Zhang *et al.* [51] (algorithm (6)) is another improvement of algorithm (4) in the sense that it dispenses with the subset Q_n defined in algorithm (4). In Zhang *et al.* [51], relatively weak nonexpansive map was called weak relatively nonexpansive map.

It is our purpose in this paper to introduce a generalized projection algorithm which is a significant improvement on algorithms (4), (5) and (6) and prove a *strong convergence* theorem for a common element for a variational inequality and a fixed point of a relatively weak nonexpansive map in a 2-uniformly convex and uniformly smooth real Banach space. Furthermore, we extend our theorem to a *countable family* of relatively weak nonexpansive maps. Finally, applications of our theorem to convex optimization problems, zeros of α-inverse strongly monotone maps and complementarity problems are presented.

2. Preliminaries

Definition 2.1. Let E be a real Banach space with dual space E^*. A map $T : E \to E$ is said to be *Lipschitz* if for each $x, y \in E$, there exists $L \geq 0$ such that $\|Tx - Ty\| \leq L\|x - y\|$.

Definition 2.2. A map $A : E \to 2^{E^*}$ is said to be *monotone* if for each $x, y \in E$, the following inequality holds: $\langle x - y, x^* - y^* \rangle \geq 0,\ \forall\ x^* \in Ax,\ y^* \in Ay$. It is called *maximal monotone* if, in addition, the graph of A, $G(A) = \{(x,y) : y \in Ax\}$, is not properly contained in the graph of any other monotone operator.

It is well known that A is maximal monotone if and only if for $(x, x^*) \in E \times E^*$, $\langle x - y, x^* - y^* \rangle \geq 0,\ \forall\ (y, y^*) \in G(A)$ implies that $x^* \in Ax$.
A map $J : E \to E^*$ defined by $J(x) := \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2,\ \|x^*\| = \|x\|,\ \forall\, x \in E\}$, is called the *normalized duality map* on E, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between the elements of E and E^*. The following are some properties of the normalized duality map which will be needed in the sequel (see e.g., Ibaraki and Takahashi [21]).

- If E is uniformly convex, then J is one-to-one and onto.
- If E is uniformly smooth, then J is single-valued.

- In particular, if a Banach space E is uniformly smooth and uniformly convex, the dual space is also uniformly smooth and uniformly convex. Hence, the normalized duality map J on E and the normalized duality map J_* on its dual space E^*, are both uniformly continuous on bounded sets, and $J_* = J^{-1}$.

The *modulus of convexity* of a space E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1,\, \varepsilon = \|x-y\|\right\}.$$

The space E is *uniformly convex* if $\delta_E(\varepsilon) > 0$ for every $\varepsilon \in (0,2]$. If there exist a constant $c > 0$ and a real number $p > 1$ such that $\delta_E(\varepsilon) \geq c\varepsilon^p$, then E is said to be *p-uniformly convex*. Typical examples of such spaces are the L_p, ℓ_p and Sobolev spaces, W_p^m, for $1 < p < \infty$, where

$$L_p \ (or\ l_p)\ or\ W_p^m \text{ is } \begin{cases} p-\text{uniformly convex}, & \text{if} \quad 2 \leq p < \infty; \\ 2-\text{uniformly convex}, & \text{if} \quad 1 < p \leq 2. \end{cases}$$

Let $S := \{z \in E : \|z\| = 1\}$. A space E is said to have a *Gâteaux differentiable norm* if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{7}$$

exists for all $x, y \in S$ and is said to have a *uniformly Gâteaux differentiable norm* if for each $y \in S$, limit (7) exists and is attained uniformly, for $x \in S$. The space E is said to have a *Fréchet differentiable norm* if, for each $x \in S$, limit (7) exists and is attained uniformly for $y \in S$.

Definition 2.3. Let E be a real normed space of dimension ≥ 2. The *modulus of smoothness* of E, $\rho_E : [0,\infty) \to [0,\infty)$, is defined by $\rho_E(\tau) := \sup\left\{\frac{\|x+y\|+\|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau,\ \tau > 0\right\}$. The space E is called *smooth* if $\rho_E(\tau) > 0$, $\forall\, \tau > 0$ and is called *uniformly smooth* if $\lim\limits_{t\to 0^+} \dfrac{\rho_E(t)}{t} = 0$.

In the sequel, we shall need the following definitions and results. Let E be a smooth real Banach space with dual space E^*. The function $\phi : E \times E \to \mathbb{R}$, defined by,

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall\, x, y \in E, \tag{8}$$

where J is the normalized duality mapping from E into E^* will play a central role in what follows. It was introduced by Alber and has been studied by Alber [2], Alber and Guerre-Delabriere [3], Chidume *et al.* [14], Chidume *et al.* [15], Chidume and Idu [16], Chidume *et* *al.* [13], Kamimura and Takahashi [24], Reich [38], Takahashi and Zembayashi [44], Takahashi and Zembayashi [45], Zegeye [47] and a host of other authors.

If $E = H$, a real Hilbert space, equation (8) reduces to $\phi(x,y) = \|x-y\|^2$, $\forall x, y \in H$. It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2, \ \forall x, y \in E. \tag{9}$$

Define a map $V : E \times E^* \to \mathbb{R}$ by $V(x,x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2$. Then, it is easy to see that

$$V(x,x^*) = \phi(x, J^{-1}(x^*)), \ \forall \ x \in E, \ x^* \in E^*. \tag{10}$$

Let C be a nonempty closed and convex subset of a smooth, strictly convex and reflexive real Banach space E. The *generalized projection* map introduced by Alber [1], is a map $\Pi_C : E \to C$, such that for any $x \in E$, there corresponds a unique element $x_0 := \Pi_C(x) \in C$ such that $\phi(x_0, x) = \min_{y \in C} \phi(y,x)$. We remark that if $E = H$ is a real Hilbert space, the generalized projection Π_C coincides with the metric projection from H onto C.

Definition 2.4. Let C be a nonempty closed and convex subset of E and let $T : C \to E$ be a map. A point $x^* \in C$ is called a *fixed point* of T if $Tx^* = x^*$. The set of fixed points of T will be denoted by $F(T)$. A point $p \in C$ is said to be an *asymptotic fixed point* of T if C contains a sequence $\{x_n\}_{n=1}^{\infty}$ which converges *weakly* to p and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$.

Definition 2.5. A map $T : C \to E$ is said to be *relatively nonexpansive*, if the following conditions hold (see e.g., Butnariu *et al.* [7], Reich [40] and Matsushita and Takahashi ([32], [33])):

(1) $F(T) \neq \emptyset$,
(2) $\phi(p, Tx) \leq \phi(p,x)$, $\forall \, x \in C$ and $p \in F(T)$,
(3) $\widehat{F}(T) = F(T)$.

Definition 2.6. A point $p \in C$ is said to be a *strong asymptotic fixed point* of T if C contains a sequence $\{x_n\}_{n=1}^{\infty}$ which converges *strongly* to p and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ (see e.g., Reich [40] and Matsushita and Takahashi [32]). The set of *strongly* asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$.

Definition 2.7. A map $T : C \to E$ is said to be *relatively weak nonexpansive*, if the following conditions hold (see e.g., Zegeye and Shahzad [49]):

(1) $F(T) \neq \emptyset$,
(2) $\phi(p,Tx) \leq \phi(p,x)$, $\forall\, x \in C$ and $p \in F(T)$,
(3) $\widetilde{F}(T) = F(T)$

If E is strictly convex and reflexive real Banach space and $A : E \to E^*$ is a continuous monotone map with $A^{-1}(0) \neq \varnothing$, it is known that $J_r := (J+rA)^{-1}J$, for $r > 0$, is relatively weak nonexpansive (see e.g., Kohasaka [27]). Clearly, every relatively nonexpansive map is relatively weak nonexpansive. Let $T : C \to E$ be a map, we have that $F(T) \subset \widetilde{F}(T) \subset \widehat{F}(T)$. It follows that for any relatively nonexpansive map $F(T) = \widetilde{F}(T) = \widehat{F}(T)$.

An example of a relatively weak nonexpansive map which is not relatively nonexpansive is given in Zhang *et al.* [51].

Let E be a real Banach space with dual space E^*. A map $A : C \to E^*$ is said to be *hemicontinuous* if for each $x,y \in C$, a map $F : [0,1] \to E^*$ defined by $F(t) := A(tx+(1-t)y)$ is continuous with respect to the weak topology of E^*. Let $N_C(v)$ denote the normal cone for C at a point $v \in C$, that is

$$N_C(v) := \{w^* \in E^* : \langle v-z, w^*\rangle \geq 0,\ \forall\, z \in C\}.$$

The following lemmas will also be needed in the sequel.

Lemma 2.8 (Alber [1])**.** *Let C be a nonempty closed and convex subset of a smooth, strictly convex and reflexive real Banach space, E. Then,*

$$\phi(y,\Pi_C(x)) + \phi(\Pi_C(x),x) \leq \phi(y,x),\ \textit{for all}\ \ x \in E,\ \ y \in C.$$

Lemma 2.9 (Alber [2])**.** *Let E be a reflexive strictly convex and smooth Banach space with E^* as its dual. Then,*

$$V(x,x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x,x^*+y^*), \tag{11}$$

for all $x \in E$ *and* $x^*, y^* \in E^*$.

Lemma 2.10 (Alber [2]). *Let* C *be a nonempty closed and convex subset of a smooth real Banach space* E, $x \in E$ *and* $x_0 \in C$. *Then,* $x_0 := \Pi_C x$ *if and only if*

$$\langle y - x_0, Jx_0 - Jx \rangle \geq 0, \text{ for all } y \in C.$$

Lemma 2.11 (Zegeye and Shahzad [50]). *Let* C *be a nonempty closed and convex subset of a real reflexive, strictly convex and smooth Banach space* E. *If* $A : C \to E^*$ *is a continuous monotone mapping, then* $VI(C,A)$ *is closed and convex.*

Lemma 2.12 (Nilsrakoo and Saejung [34]). *Let* C *be a closed convex subset of a uniformly convex and uniformly smooth Banach space* E *and let* $\{T_i : C \to E\}_{i=1}^{\infty}$ *be a sequence of mappings such that* $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$, $\phi(p, T_i x) \leq \phi(p, x)$, $\forall\, x \in C$ *and* $p \in \cap_{i=1}^{\infty} F(T_i)$, $i \in \mathbb{N}$. *Suppose that* $\{\alpha_i\}_{i=1}^{\infty}$ *is a sequence in* $(0,1)$ *such that* $\sum_{i=1}^{\infty} \alpha_i = 1$ *and* $T : C \to E$ *is defined by* $Tx = J^{-1}\Big(\sum_{i=1}^{\infty} \alpha_i J T_i x\Big)$ *for each* $x \in C$. *Let* $\{x_n\}$ *be a bounded sequence in* C. *Then, the following are equivalent:* (a) $x_n - Tx_n \to 0$, (b) $x_n - T_i x_n \to 0$, *for each* $i \in \mathbb{N}$. *In particular,* $F(T) = \cap_{i=1}^{\infty} F(T_i)$.

3. MAIN RESULTS

In theorem 3.1 below, the map A is assumed to satisfy the following condition,

$$\|Ay\| \leq \|Ay - Au\|, \text{ for all } y \in C \text{ and } u \in VI(C,A). \tag{12}$$

We now prove the following theorem.

Theorem 3.1. *Let* E *be a uniformly smooth and* 2*-uniformly convex real Banach space with dual space* E^*. *Let* C *be a nonempty closed and convex subset of* E, *let* $A : C \to E^*$ *be an* α*-inverse strongly monotone map and let* $T : C \to E$ *be a relatively weak nonexpansive map. Assume that* $W := F(T) \cap VI(C,A) \neq \emptyset$. *For arbitrary* $x_1 \in C$, *let the sequence* $\{x_n\}_{n=1}^{\infty}$ *be iteratively defined by*

$$
\begin{cases}
x_1 \in C := C_1, \\
u_n = \Pi_C J^{-1}(Jx_n - \lambda A x_n), \\
y_n = T u_n, \\
C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_1 \ \ \forall \ \ n \geq 1,
\end{cases}
\tag{13}
$$

where Π_C is the generalized projection of E onto C, $J : E \to E^*$ is the normalized duality map, $\lambda \in \left(0, \frac{\alpha}{2L}\right)$ and $L > 0$ denotes a Lipschitz constant of J^{-1}. Then, the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W$.

Proof. Our method of proof will follow some of the ideas used in Chidume *et al.* [18]. The proof will be divided into 5 steps.

Step 1: $\Pi_{C_{n+1}}$ is well defined.

It suffices to show that C_{n+1} is closed and convex for all $n \geq 1$. The proof follows by induction. Since $C_1 := C$ is closed and convex, C_n is closed and convex for some $n \geq 1$ and $\phi(z, y_n) \leq \phi(z, x_n)$ if and only if $\langle z, Jx_n - Jy_n \rangle - \|x_n\|^2 + \|y_n\|^2 \leq 0$. Therefore,

$$C_{n+1} = \{z \in C_n : \gamma(z) \leq 0\},$$

where $\gamma(z) := \langle z, Jx_n - Jy_n \rangle - \|x_n\|^2 + \|y_n\|^2$, is closed and convex, for all $n \geq 1$. Hence, $\Pi_{C_{n+1}}$ is well defined.

Step 2: $x_n \to x^* \in C$ as $n \to \infty$.

Let $u \in C_n$, for all $n \geq 1$. Since $x_n = \Pi_{C_n} x_1$ and by applying lemma 2.8, we have that

$$\phi(x_n, x_1) = \phi(\Pi_{C_n} x_1, x_1) \leq \phi(u, x_1),$$

which yields that $\{\phi(x_n, x_1)\}_{n=1}^{\infty}$ is bounded; it follows by inequality (9) that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded. Furthermore, for each $n \in \mathbb{N}$, $x_n = \Pi_{C_n} x_1$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, again by lemma 2.8, we obtain that

$$\phi(x_n, x_1) \leq \phi(x_{n+1}, x_n) + \phi(x_n, x_1) \leq \phi(x_{n+1}, x_1). \tag{14}$$

Thus, $\{\phi(x_n,x_1)\}_{n=1}^{\infty}$ converges. Now, for $m > n$, $x_m = \Pi_{C_m}x_1 \in C_m \subset C_n$, applying lemma 2.8, we get that

$$\phi(x_m,x_n) = \phi(x_m,\Pi_{C_n}x_1) \leq \phi(x_m,x_1) - \phi(x_n,x_1) \to 0 \text{ as } n,m \to \infty. \tag{15}$$

From a result of Kamimura and Takahashi, [24], we obtain that

$$\|x_m - x_n\| \to 0 \text{ as } n,m \to \infty, \tag{16}$$

and so $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in C. Since C is closed, it follows that there exists $x^* \in C$ such that

$$x_n \to x^* \text{ as } n \to \infty. \tag{17}$$

Step 3: $W \subset C_n$, for each $n \in \mathbb{N}$.

The proof is by induction. Clearly, $W \subset C_1 = C$. Assume that $W \subset C_n$, for some $n \geq 1$. Let $u \in W$ be arbitrary. Using lemma 2.8, lemma 2.9, the fact that T is relatively nonexpansive and the recursion formula (13), we have that

$$\begin{aligned}
\phi(u,y_n) &= \phi(u,Tu_n) \\
&\leq \phi(u,u_n) = \phi(u,\Pi_C J^{-1}(Jx_n - \lambda Ax_n)) \\
&\leq \phi(u,J^{-1}(Jx_n - \lambda Ax_n)) = V(u,Jx_n - \lambda Ax_n) \\
&\leq V(u,Jx_n) - 2\lambda\langle J^{-1}(Jx_n - \lambda Ax_n) - u, Ax_n\rangle \\
&= \phi(u,x_n) - 2\lambda\langle x_n - u, Ax_n\rangle \\
&- 2\lambda\langle J^{-1}(Jx_n - \lambda Ax_n) - x_n, Ax_n\rangle. \qquad (18)
\end{aligned}$$

Since A is α-inverse strongly monotone and $u \in VI(C,A)$, we obtain that

$$\begin{aligned}
-2\lambda\langle x_n - u, Ax_n\rangle &= -2\lambda\langle x_n - u, Ax_n - Au\rangle - 2\lambda\langle x_n - u, Au\rangle \\
&\leq -2\alpha\lambda\|Ax_n - Au\|^2. \qquad (19)
\end{aligned}$$

Furthermore, applying the fact that $J^{-1}: E^* \to E$ is Lipschitz with constant denoted by $L > 0$ (see e.g., Xu [46]) and condition (12), we obtain that

$$\begin{aligned} -2\lambda\langle J^{-1}(Jx_n - \lambda Ax_n) - x_n, Ax_n\rangle &= -2\lambda\langle J^{-1}(Jx_n - \lambda Ax_n) - J^{-1}(x_n), Ax_n\rangle \\ &\leq 2\lambda\|J^{-1}(Jx_n - \lambda Ax_n) - J^{-1}(Jx_n)\|\|Ax_n\| \\ &\leq 2\lambda^2 L\|Ax_n\|^2 \leq 2\lambda^2 L\|Ax_n - Au\|^2 \\ &\leq \alpha\lambda\|Ax_n - Au\|^2. \end{aligned} \tag{20}$$

Using inequalities (18), (19) and (20), we have that

$$\phi(u, y_n) \leq \phi(u, u_n) \leq \phi(u, x_n) - \alpha\lambda\|Ax_n - Au\|^2. \tag{21}$$

Consequently, we obtain that

$$\begin{aligned} \phi(u, y_n) &\leq \phi(u, u_n) \leq \phi(u, x_n) - \alpha\lambda\|Ax_n - Au\|^2 \\ &\leq \phi(u, x_n), \end{aligned} \tag{22}$$

and so

$$\phi(u, y_n) \leq \phi(u, u_n) \leq \phi(u, x_n). \tag{23}$$

Therefore, $u \in C_{n+1}$ and so $W \subset C_{n+1}$. Hence, $W \subset C_n, \ \forall\, n \geq 1$.

Step 4: $u_n \to x^* \in F(T)$.

By using condition (15), inequality (23) and the fact that $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we have that

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) \to 0 \text{ as } n \to \infty. \tag{24}$$

This implies by Kamimura and Takahashi [24] that

$$\|x_{n+1} - y_n\| \to 0 \text{ as } n \to \infty, \tag{25}$$

and

$$\|x_{n+1}-u_n\| \to 0 \text{ as } n \to \infty. \tag{26}$$

Applying conditions (16) and (26), we observe that

$$\|u_n-x_n\| \leq \|u_n-x_{n+1}\|+\|x_{n+1}-x_n\| \to 0 \text{ as } n \to \infty. \tag{27}$$

The fact that J is norm-to-norm uniformly continuous on bounded subsets of E gives that

$$\|Ju_n-Jx_n\| \to 0 \text{ as } n \to \infty. \tag{28}$$

Moreover, by conditions (17) and (27),

$$u_n \to x^* \text{ as } n \to \infty, \tag{29}$$

and by conditions (25) and (26),

$$\|y_n-u_n\| = \|Tu_n-u_n\| \leq \|y_n-x_{n+1}\|+\|x_{n+1}-u_n\| \to 0 \text{ as } n \to \infty. \tag{30}$$

Since T is relatively nonexpansive map, we obtain that $x^* \in F(T)$.

Step 5: $x_n \to x^* \in VI(C,A)$.

Let $S \subset E \times E^*$ be a map defined as follows:

$$Sv = \begin{cases} Av+N_C(v) & if \;\; v \in C, \\ \emptyset & if \;\; v \notin C. \end{cases} \tag{31}$$

From a result of Rockafellar [41], we have that S is maximal monotone and $S^{-1}0 = VI(C,A)$. Let $(v,w) \in G(S)$. Therefore, $w \in Sv = Av+N_C(v)$, and so, we obtain that $w-Av \in N_C(v)$. Since $u_n = \Pi_C J^{-1}(Jx_n-\lambda Ax_n) \in C$, we have that $\langle v-u_n, w-Av\rangle \geq 0$. Moreover, applying lemma 2.10 and the recursion formula again, it is easy to see that

$$\langle v-u_n, Ju_n-(Jx_n-\lambda Ax_n)\rangle \geq 0, \tag{32}$$

and thus $\langle v-u_n, \frac{Jx_n-Ju_n}{\lambda} - Ax_n\rangle \le 0$.

Now,

$$\begin{aligned}
\langle v-u_n, w\rangle &\ge \langle v-u_n, Av\rangle \\
&\ge \langle v-u_n, Av\rangle + \Big\langle v-u_n, \frac{Jx_n-Ju_n}{\lambda} - Ax_n\Big\rangle \\
&\ge \langle v-u_n, Av-Au_n\rangle + \langle v-u_n, Au_n-Ax_n\rangle \\
&\quad + \Big\langle v-u_n, \frac{Jx_n-Ju_n}{\lambda}\Big\rangle \\
&\ge -\|v-u_n\|\frac{\|u_n-x_n\|}{\alpha} - \|v-u_n\|\frac{\|Ju_n-Jx_n\|}{\lambda} \\
&\ge -M\Big(\frac{\|u_n-x_n\|}{\alpha} + \frac{\|Ju_n-Jx_n\|}{\lambda}\Big),
\end{aligned}$$

where $M=\sup\{\|v-u_n\| : n\ge 1\}$. It follows from conditions (27), (28) and (29) that $\langle v - x^*, w\rangle \ge 0$. Since S is maximal monotone, we obtain that $x^*\in S^{-1}0 = VI(C,A)$. Hence, $x^*\in W$, and this completes the proof. □

4. STRONG CONVERGENCE THEOREMS FOR COUNTABLE FAMILIES

Lemma 4.1. *Let C be a closed convex subset of a uniformly convex and uniformly smooth real Banach space E and let $T_i : C\to E$, $i=1,2,\cdots$, be a countable family of relatively weak nonexpansive maps. Assume that $\cap_{i=1}^{\infty}F(T_i)\ne\emptyset$ and $\{\alpha_i\}_{i=1}^{\infty}$ is a sequence in $(0,1)$ such that $\sum_{i=1}^{\infty}\alpha_i = 1$. Let the map $T : C\to E$ be defined by*

$$Tx = J^{-1}\Big(\sum_{i=1}^{\infty}\alpha_i JT_i x\Big),$$

for each $x\in C$. Then, T is relatively weak nonexpansive and $F(T)=\cap_{i=1}^{\infty}F(T_i)$.

Proof. By lemma 2.12, we have that $F(T)=\cap_{i=1}^{\infty}F(T_i)$. It now remains to prove that T is relatively weak nonexpansive, i.e.

(1) $F(T)\ne\emptyset$,

(2) $\phi(p,Tx)\le\phi(p,x)$, $\forall\, x\in C$ and $p\in F(T)$,

(3) $\widetilde{F}(T)=F(T)$.

Condition (1) follows from lemma 2.12 since $F(T)=\cap_{i=1}^{\infty}F(T_i)\ne\emptyset$.

Let $p \in F(T) = \cap_{i=1}^{\infty} F(T_i)$ and $x \in C$. Since T_i, $i = 1, 2, \cdots$, is relatively weak nonexpansive, we obtain that

$$\begin{aligned}
\phi(p, Tx) &= \phi\left(p, J^{-1}\Big(\sum_{i=1}^{\infty} \alpha_i J T_i x\Big)\right) \\
&= \|p\|^2 - 2\left\langle p, \sum_{i=1}^{\infty} \alpha_i J T_i x \right\rangle + \left\| \sum_{i=1}^{\infty} \alpha_i J T_i x \right\|^2 \\
&\leq \sum_{i=1}^{\infty} \alpha_i \|p\|^2 - 2 \sum_{i=1}^{\infty} \alpha_i \langle p, J T_i x\rangle + \sum_{i=1}^{\infty} \alpha_i \|T_i x\|^2 \\
&= \sum_{i=1}^{\infty} \alpha_i \phi(p, T_i x) \leq \sum_{i=1}^{\infty} \alpha_i \phi(p, x) \\
&= \phi(p, x).
\end{aligned}$$

Therefore, condition (2) holds. For condition (3), we need to show that $\widetilde{F}(T) = F(T) = \cap_{i=1}^{\infty} F(T_i)$. Now, by applying lemma 2.12 and the fact that T_i is relatively weak nonexpansive, for each $i \in \mathbb{N}$, yield that
$p \in \widetilde{F}(T)$ if and only if there is a sequence $\{x_n\}_{n=1}^{\infty}$ in C and a point $p \in C$ such that $x_n \to p$ and $x_n - Tx_n \to 0$ if and only if there is a sequence $\{x_n\}_{n=1}^{\infty}$ in C and a point $p \in C$ such that $x_n \to p$ and $x_n - T_i x_n \to 0$, for each $i = 1, 2, \cdots$, if and only if $p \in \widetilde{F}(T_i) = F(T_i)$, for each $i = 1, 2, \cdots$, if and only if $p \in \cap_{i=1}^{\infty} \widetilde{F}(T_i) = \cap_{i=1}^{\infty} F(T_i) = F(T)$. Therefore, $\widetilde{F}(T) = \cap_{i=1}^{\infty} F(T_i) = F(T)$. Hence, T is relatively weak nonexpansive. □

Remark 3. Lemma 4.1 extends theorem 3.2 of Nilsrakoo and Saejung [34] from the class of relatively nonexpansive maps to the more general class of relatively weak nonexpansive maps.

We now prove the following strong convergence theorem.

Theorem 4.2. *Let E be a uniformly smooth and 2-uniformly convex real Banach space with dual space E^*. Let C be a nonempty closed and convex subset of E, let $A : C \to E^*$ be an α-inverse strongly monotone map and let $T_i : C \to E$, for $i = 1, 2, \cdots$, be a countable family of relatively weak nonexpansive maps. Assume that $W := \cap_{i=1}^{\infty} F(T_i) \cap VI(C, A) \neq \emptyset$. For arbitrary $x_1 \in C$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be iteratively defined by*

$$
\left\{
\begin{aligned}
&x_1 \in C := C_1,\\
&u_n = \Pi_C J^{-1}(Jx_n - \lambda A x_n),\\
&y_n = J^{-1}\Big(\sum_{i=1}^{\infty} \alpha_i J T_i u_n\Big),\\
&C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \phi(v, x_n)\},\\
&x_{n+1} = \Pi_{C_{n+1}} x_1, \ \forall \ n \ge 1,
\end{aligned}
\right. \tag{33}
$$

where Π_C is the generalized projection of E onto C, $J : E \to E^*$ is the normalized duality map, $\lambda \in \Big(0, \frac{\alpha}{2L}\Big)$, $L > 0$ denotes a Lipschitz constant of J^{-1} and $\{\alpha_i\}_{i=1}^{\infty}$ is a sequence in $(0,1)$ such that $\sum_{i=1}^{\infty} \alpha_i = 1$. Then, the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W$.

Proof. We observe from lemma 4.1 that the map $T : C \to E$ defined by $Tu_n := J^{-1}\Big(\sum_{i=1}^{\infty} \alpha_i J T_i u_n\Big)$ is relatively weak nonexpansive and $F(T) = \cap_{i=1}^{\infty} F(T_i)$. It follows by theorem 3.1 that the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W := F(T) \cap VI(C, A)$. $\square$

5. Applications

Corollary 5.1. *Let $E = L_p, \ell_p$ and W_m^p, $1 < p \le 2$. Let C be a nonempty closed and convex subset of E and let $A : C \to E^*$ be an α-inverse strongly monotone map. Let $T_i : C \to E$, for $i = 1, 2, \cdots$, be a countable family of relatively weak nonexpansive maps. Assume that $W := \cap_{i=1}^{\infty} F(T_i) \cap VI(C, A) \ne \emptyset$. For arbitrary $x_1 \in C$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be iteratively defined by*

$$
\left\{
\begin{aligned}
&x_1 \in C := C_1,\\
&u_n = \Pi_C J^{-1}(Jx_n - \lambda A x_n),\\
&y_n = J^{-1}\Big(\sum_{i=1}^{\infty} \alpha_i J T_i u_n\Big),\\
&C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \phi(v, x_n)\},\\
&x_{n+1} = \Pi_{C_{n+1}} x_1, \ \forall \ n \ge 1,
\end{aligned}
\right. \tag{34}
$$

where Π_C is the generalized projection of E onto C, $J : E \to E^*$ be the normalized duality map, $\lambda \in \left(0, \frac{\alpha}{2L}\right)$, $L > 0$ denotes a Lipschitz constant of J^{-1} and $\{\alpha_i\}_{i=1}^{\infty}$ is a sequence in $(0,1)$ such that $\sum_{i=1}^{\infty} \alpha_i = 1$. Then, the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W$.

Proof. We observe that E is 2-uniformly convex and uniformly smooth and from lemma 4.1 the map $T : C \to E$ defined by $Tu_n := J^{-1}\Big(\sum_{i=1}^{\infty} \alpha_i JT_i u_n\Big)$ is relatively weak nonexpansive and $F(T) = \cap_{i=1}^{\infty} F(T_i)$. It follows from theorem 3.1 that the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W := F(T) \cap VI(C,A)$. □

We now consider further applications. We state the theorems. Proofs of the theorems follow as proofs of similar applications given in Chidume *et al.* [18], and Iiduka and Takahashi [22]. However, we sketch the details here for completeness.

5.1. Approximating a zero of an α-inverse strongly monotone map.

Theorem 5.2. *Let E be a 2-uniformly convex and uniformly smooth real Banach space with dual space E^*. Let $A : E \to E^*$ be an α-inverse strongly monotone map and let $T_i : E \to E$, $i = 1, 2, \cdots$, be a countable family of relatively weak nonexpansive maps. Assume that $W := \cap_{i=1}^{\infty} F(T_i) \cap A^{-1}0 \neq \emptyset$, where $A^{-1}0 = \{u \in E : Au = 0\} \neq \emptyset$. For arbitrary $x_1 \in E$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be iteratively defined by*

$$
(35) \qquad \begin{cases}
x_1 \in E := C_1, \\
u_n = J^{-1}(Jx_n - \lambda A x_n), \\
y_n = J^{-1}\Big(\sum_{i=1}^{\infty} \alpha_i JT_i u_n\Big), \\
C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \phi(v, x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_1, \ \forall\, n \ge 1,
\end{cases}
$$

where $J : E \to E^*$ is the normalized duality map, $\lambda \in \left(0, \frac{\alpha}{2L}\right)$, $L > 0$ denotes a Lipschitz constant of J^{-1} and $\{\alpha_i\}_{i=1}^{\infty}$ is a sequence in $(0,1)$ such that $\sum_{i=1}^{\infty} \alpha_i = 1$. Then, the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W := \cap_{i=1}^{\infty} F(T_i) \cap A^{-1}0$.

Proof. We observe from lemma 4.1 that the map $T : E \to E$ defined by $Tu_n := J^{-1}\Big(\sum_{i=1}^{\infty} \alpha_i JT_i u_n\Big)$ is relatively weak nonexpansive and $F(T) = \cap_{i=1}^{\infty} F(T_i)$. By setting $C_1 = E$ and $\Pi_E = I$ in theorem 3.1, we observe that

$$u_n = J^{-1}(Jx_n - \lambda Ax_n) = \Pi_E J^{-1}(Jx_n - \lambda Ax_n),\ n \geq 1. \tag{36}$$

Also, $VI(E,A) = A^{-1}0$ and $\|Ay\| = \|Ay - 0\| = \|Ay - Au\|$, for all $y \in E$, $u \in A^{-1}0$. It follows from theorem 3.1 that $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge *strongly* to some $x^* \in W := F(T) \cap A^{-1}0$. □

5.2. Approximating a solution of complementarity problem. Let C be a nonempty closed and convex subset of E and $A : C \to E^*$ be a map. Let the polar in E^* be defined by the set $C^* = \{y^* \in E^* : \langle x, y^* \rangle \geq 0, \text{ for all } x \in C\}$. Then, we study the following problem: find $u \in C$ such that $Au \in C^*$ and $\big\langle u, Au \big\rangle = 0$. This problem is called the *complementarity* problem (see e.g., Blum and Oettli [6]). The set of solutions of the complementarity problem will be denoted by $K(C,A)$.

Theorem 5.3. *Let E be a 2-uniformly convex and uniformly smooth real Banach space with dual space E^*. Let C be a nonempty closed and convex subset of E, and let $A : C \to E^*$ be an α-inverse strongly monotone map and A satisfies the following condition, $\|Ay\| \leq \|Ay - Au\|$, for all $y \in C$ and $u \in K(C,A)$. Let $T_i : C \to E$, $i = 1,2,\cdots$, be a countable family of relatively weak nonexpansive maps. Assume that $W := \cap_{i=1}^{\infty} F(T_i) \cap K(C,A) \neq \emptyset$. For arbitrary $x_1 \in C$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be iteratively defined by*

$$\begin{cases} x_1 \in C := C_1, \\ u_n = \Pi_C J^{-1}(Jx_n - \lambda Ax_n), \\ y_n = J^{-1}\Big(\sum_{i=1}^{\infty} \alpha_i JT_i u_n\Big), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1,\ \forall\, n \geq 1, \end{cases} \tag{37}$$

where Π_C is the generalized projection of E onto C, $J : E \to E^*$ is the normalized duality map, $\lambda \in \left(0, \frac{\alpha}{2L}\right)$, $L > 0$ denotes a Lipschitz constant of J^{-1} and $\{\alpha_i\}_{i=1}^{\infty}$ is a sequence in $(0,1)$ such that $\sum_{i=1}^{\infty} \alpha_i = 1$. Then, the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W$.

Proof. We observe from lemma 4.1 that the map $T : C \to E$ defined by $Tu_n := J^{-1}\Big(\sum_{i=1}^{\infty} \alpha_i JT_i u_n\Big)$ is relatively weak nonexpansive and $F(T) = \cap_{i=1}^{\infty} F(T_i)$. From lemma 7.1.1 of Takahashi [43], we obtain that $VI(C,A) = K(C,A)$. It follows from theorem 3.1 that the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W := F(T) \cap K(C,A) \neq \emptyset$. $\square$

5.3. Approximating a minimizer of a continuously Fr*é*chet differentiable convex functional.

Lemma 5.4 (Baillon and Haddad [5], see also Iiduka and Takahashi [22])**.** *Let E be a Banach space, f is a continuously Fréchet differentiable, convex functional on E and let ∇f denote the gradient of f. If ∇f is $\frac{1}{\alpha}$-Lipschitz continuous, then ∇f is α-inverse strongly monotone.*

Theorem 5.5. *Let E be a 2-uniformly convex and uniformly smooth real Banach space with dual space E^*. Let C be a nonempty closed and convex subset of E and let $T_i : C \to E$, for $i = 1, 2, \cdots$, be a countable family of relatively weak nonexpansive maps. Let $f : E \to \mathbb{R}$ be a map satisfying the following conditions:*

(1) f is a continuously Fréchet differentiable convex functional on E and ∇f is a $\frac{1}{\alpha}$-Lipschitz map;

(2) $K = arg\min_{y \in C} f(y) = \{x^* \in C : f(x^*) = \min_{y \in C} f(y)\} \neq \emptyset$;

(3) $\|\nabla f|_C(y)\| \leq \|\nabla f|_C(y) - \nabla f|_C(u)\|$, for all $y \in C$ and $u \in K$.

Assume that $W := \cap_{i=1}^{\infty} F(T_i) \cap K \neq \emptyset$. *For arbitrary* $x_1 \in C$, *let the sequence* $\{x_n\}_{n=1}^{\infty}$ *be iteratively defined by*

$$
\begin{cases}
x_1 \in C := C_1, \\
u_n = \Pi_C J^{-1}(Jx_n - \lambda \nabla f|_C x_n), \\
y_n = J^{-1}\Big(\sum_{i=1}^{\infty} \alpha_i J T_i u_n\Big), \\
C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_1 \ \ \forall \ n \geq 1,
\end{cases} \tag{38}
$$

where Π_C *is the generalized projection of* E *onto* C, $J : E \to E^*$ *is the normalized duality map,* $\lambda \in \left(0, \frac{\alpha}{2L}\right)$, $L > 0$ *denotes a Lipschitz constant of* J^{-1} *and* $\{\alpha_i\}_{i=1}^{\infty}$ *is a sequence in* $(0,1)$ *such that* $\sum_{i=1}^{\infty} \alpha_i = 1$. *Then, the sequences* $\{x_n\}_{n=1}^{\infty}$ *and* $\{u_n\}_{n=1}^{\infty}$ *converge strongly to some* $x^* \in W$.

Proof. We observe from lemma 4.1 that the map $T : C \to E$ defined by $Tu_n := J^{-1}\Big(\sum_{i=1}^{\infty} \alpha_i J T_i u_n\Big)$ is relatively weak nonexpansive and $F(T) = \cap_{i=1}^{\infty} F(T_i)$. Using condition (1) of theorem 5.5, it follows from lemma 5.4 that $\nabla f|_C$ is an α-inverse strongly monotone map of C into E^*. Since f is differentiable and convex, we have, as in Chidume *et al.* [18] and Iiduka and Takahashi [22], that $VI(C, \nabla f|_C) = K = arg \min_{y \in C} f(y)$. By applying theorem 3.1, we obtain that the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W := F(T) \cap K \neq \emptyset$. □

5.4. Analytical representations of duality maps in L_p, l_p and W_m^p spaces, $1 < p < \infty$. The analytical representations of duality maps are known in a number of Banach spaces. In particular, they are known in L_p, l_p and W_m^p, $1 < p < \infty$, (see e.g., Alber and Ryazantseva [2], p. 36).

Remark 4. We make the following remarks.

(1) The generalized projection algorithm (13) studied in theorem 3.1 is much simpler than the algorithms (4), (5) and (6) studied in theorems of Liu [30], Zegeye and Shahzad [49], and Zhang *et al.* [51], respectively. The algorithm (13) contains much less equations to compute. Furthermore, algorithm (5) has one iteration parameter α_n; each of algorithm (4) and (6) has 3 iterations parameters: β_n, λ_n and α_n. These are to be computed at each

step of the iteration process. The iteration parameter in the algorithm (13) of theorem 3.1 is *one* fixed arbitrary constant $\lambda \in \left(0, \frac{\alpha}{2L}\right)$, which is to be computed once and then used at each step of the iteration process. Consequently, this makes algorithm (13) more efficient and attractive than any of the algorithms (4), (5), and (6).

(2) Theorem 4.2 is an extension of theorem 3.1 from the case where T is a relatively weak nonexpansive map to the case where the T is replaced by *a countable family* of relatively weak nonexpansive maps. Consequently, theorem 4.2 further extends Theorems of Liu [30], Zegeye and Shahzad [49] and Zhang *et al.* [51] to *countable families* of relatively weak nonexpansive maps.

Conflict of Interests

The authors declare that there is no conflict of interests.

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