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COMMON RANDOM FIXED POINTS UNDER QUASI CONTRACTION CONDITIONS IN SYMMETRIC SPACES

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Abstract: In this paper, we obtain some common random fixed point theorems for generalized weak contraction condition in symmetric spaces. Besides discussing special cases, we observe the usefulness of results on the setting of symmetric spaces.

Keywords: fixed point; symmetric space; random operators and quasi weak contraction.

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1. Introduction

In recent years, the study of random fixed points has attracted much attention, some of the recent literatures in random fixed point may be noted in [2, 3, 4, 5, 6, 9, 10, 11].

The study of common fixed point theorems in symmetric spaces was initiated in the work of Hicks and Rhoades [8]. Recently Beg and Abbas [5] prove some random fixed point theorem for weakly compatible random operators under generalized contractive condition in symmetric space. The purpose of this paper is to obtain some common fixed point theorems in symmetric spaces for a certain class of mappings on general setting. Theses mapping satisfy the (E.A) property which is studied for first time by Aamri and Moutawakil[1].

2. Preliminaries

Throughout this paper, (Ω,Σ) denotes a measurable space (Σ - sigma algebra).

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Definition 2.1. Let X be a non-empty set and $d: X \times X \to [0, \infty)$ a functional. Then d is called a symmetric on X if

(S1): $d(x, y) \geq 0$;

(S2): $d(x, y) = 0$; if and only if $x=y$;

(S3): $d(x, y) = d(y, x)$;

The pair (X, d) is called a symmetric space of semi–symmetric space. If we include the triangle inequality in the above definition then we get the usual definition of a metric space. However, a symmetric on X need not to be metric on X. Therefore the class of symmetric spaces is larger than metric spaces.

Let d be a symmetric on a set X. For $\varepsilon > 0$ and $x \in X$, $B(x,\varepsilon)$ denotes the spherical ball centered at x with radius ε, defined as the set $\{y \in X: d(x, y) < \varepsilon\}$.

A topology t(d) on X is given by $U \in t(d)$ if and only if for each $x \in U$, $B(x, \varepsilon) \subset U$ for some $\varepsilon > 0$. Note that

$\lim_{n\to\infty} d(x_n, x) = 0$ Iff $x_n \to x$ in the topology t(d).

Let F be a subset of X. A mapping $\xi: \Omega \to X$ is measurable if $\xi^{-1}(U) \in \Sigma$ for each open subset U of X. The mapping $T: \Omega \times F \to F$ is a random map if and only if for each fixed $x \in F$, the mapping $T(.,X): \Omega \to F$ is measurable.

A measurable mapping $\xi: \Omega \to X$ is a random fixed point of random operators $T: \Omega \times F \to F$ if and only if $T(\omega, \xi(\omega)) = \xi(\omega)$ for each $\omega \in \Omega$. We denote the set of random fixed points of a random map T by RF(T) and the set of all measurable mappings for Ω into a symmetric space by $M(\Omega, X)$.

The following two axioms are given by Wilson[11]

Definition 2.2. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in a symmetric space (X, d) and $x, y \in X$. The space X is said to satisfy the following axioms:

(W.1) $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x_n, y)$ implies $x = y$

(W.2) $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x_n, y_n)$ implies $d(y_n, x) = 0$

The following definition is essentially due to Aamri and Moutawakil[1] on a metric space.

Definition 2.3. Let X be a symmetric space. Two mappings $S, T: \Omega \times X \to X$ satisfy the property (E.A.) if there exists a sequence $\{\xi_n\}$ in $M(\Omega, X)$ such that for some $\xi \in M(\Omega, X)$

$$\lim_{n\to\infty} T(\omega, \xi_n(\omega)) = \lim_{n\to\infty} S(\omega, \xi_n(\omega)) = \xi(\omega) \text{ for every } \omega\epsilon\Omega$$

3. Common Fixed Point Theorems

Throughout this section we shall use the following notations:

1. X: = A symmetric space (X, d)
2. Φ:= A class of functions Φ: [0, ∞) → [0, ∞) satisfying.
   a. Φ is continuous and monotone non-decreasing.
   b. $\Phi(t) = 0 \iff t = 0$.
3. $M_T((\omega, x), (\omega, y)) = \max\{d(x, y),\ d(x, T(\omega, x)),\ d(y, T(\omega, y)),\ d(x, T(\omega, y)),$ $d(y, T(\omega, x))\}$
4. $M_{S,T}((\omega, x), (\omega, y)) = \max\{d(x, y),\ d(x, S(\omega, x)),\ d(y, T(\omega, y)),\ d(x, T(\omega, y)),$ $d(y, S(\omega, x))\}$

Definition 3.1. Let X be a metric space and T: Ω×X→X. Then mapping T will be called a quasi-weak contraction if

$$\psi\left(d\left(T(\omega, x), T(\omega, y)\right)\right) \leq \psi\left(M_T((\omega, x), (\omega, y))\right) - \emptyset\left(M_T((\omega, x), (\omega, y))\right)$$

for all x, y∈X, ω∈Ω where $\psi, \emptyset \in \Phi$.

When $\psi(t) = t$ and $\emptyset(t) = (1 - k)t$ with k ∈ (0, 1), in the definition 3.1, we recover the well known quasi-contraction due to Ciric[7].

3. Main Result

Theorem 3.1: Let X be a symmetric space and T: Ω×X→CB(X) be a quasi-contraction satisfying the property (E.A.). Then T has a unique random fixed point.

Proof: since T satisfying the (E.A.) property, then there exists a sequences $\{\xi_n\}$ in M (Ω, X) such that

$$\lim_{n\to\infty} T(\omega, \xi_n(\omega)) = \lim_{n\to\infty} \xi_n(\omega) = \xi(\omega)$$

For every ω∈Ω, for some ξ∈M (Ω, X).

Since T is a quasi-weak contraction then we get

$$\psi\left(d\left(T(\omega,\xi(\omega)),T(\omega,\xi_n(\omega))\right)\right)$$

$$\leq \psi\left(M_T\left((\omega,\xi(\omega)),(\omega,\xi_n(\omega))\right)\right)$$

$$-\ \emptyset\left(M_T\left((\omega,\xi(\omega)),(\omega,\xi_n(\omega))\right)\right) \qquad (3.1)$$

Now consider

$$\lim_{n\to\infty} M_T\left((\omega,\xi(\omega)),(\omega,\xi_n(\omega))\right)$$

$$= \lim_{n\to\infty} \max\left(\left\{d(\xi(\omega),\xi_n(\omega)), d\left(\xi(\omega),T(\omega,\xi(\omega))\right), d\left(\xi_n(\omega),T(\omega,\xi_n(\omega))\right),\right.\right.$$

$$\left.\left. d\left(\xi(\omega),T(\omega,\xi_n(\omega))\right),\ \ d\left(\xi_n(\omega),T(\omega,\xi(\omega))\right)\right\}\right)$$

$$= \max\left(\left\{d(\xi(\omega),\xi(\omega)), d\left(\xi(\omega),T(\omega,\xi(\omega))\right), d(\xi(\omega),\xi(\omega)),\right.\right.$$

$$\left.\left. d(\xi(\omega),\xi(\omega)),\ \ d\left(\xi(\omega),T(\omega,\xi(\omega))\right)\right\}\right)$$

$$= \max\left(\left\{0, d\left(\xi(\omega),T(\omega,\xi(\omega))\right), 0\ , 0, d\left(\xi(\omega),T(\omega,\xi(\omega))\right)\right\}\right)$$

$$= d\left(\xi(\omega),T(\omega,\xi(\omega))\right)$$

$$\lim_{n\to\infty} M_T\left((\omega,\xi(\omega)),(\omega,\xi_n(\omega))\right) = d\left(\xi(\omega),T(\omega,\xi(\omega))\right) \qquad (3.2)$$

Since $\psi, \emptyset \in \Phi$ then (3.1) and (3.2) implies that

$$\lim_{n\to\infty} \psi\left(d\left(T(\omega,\xi(\omega)),T(\omega,\xi_n(\omega))\right)\right) = \psi\left(d\left(T(\omega,\xi(\omega)),\xi(\omega)\right)\right)$$

$$\leq \psi\left(d\left(\xi(\omega),T(\omega,\xi(\omega))\right)\right) - \emptyset\left(d\left(\xi(\omega),T(\omega,\xi(\omega))\right)\right)$$

which is a contradiction.

Hence $T(\omega,\xi(\omega)) = \xi(\omega)$

Therefore $\xi(\omega)$ is a random fixed point of T.

Uniqueness: To prove the uniqueness, we suppose that T has two distinct points $\xi_1(\omega)$ and $\xi_2(\omega)$ then

$$M_T\left((\omega,\xi_1(\omega)),(\omega,\xi_2(\omega))\right)$$

$$= \max\left(\left\{d(\xi_1(\omega),\xi_2(\omega)), d\left(\xi_1(\omega),T(\omega,\xi_1(\omega))\right), d\left(\xi_2(\omega),T(\omega,\xi_2(\omega))\right),\right.\right.$$

$$\left.\left. d\left(\xi_1(\omega),T(\omega,\xi_2(\omega))\right), d\left(\xi_2(\omega),T(\omega,\xi_1(\omega))\right)\right\}\right)$$

$$= \max\Big(\big\{d\big(\xi_1(\omega),\xi_2(\omega)\big), d\big(\xi_1(\omega),\xi_1(\omega)\big), d\big(\xi_2(\omega),\xi_2(\omega)\big),$$
$$d\big(\xi_1(\omega),\xi_2(\omega)\big), d\big(\xi_2(\omega),\xi_1(\omega)\big)\big\}\Big) \tag{3.3}$$
$$= d\big(\xi_1(\omega),\xi_2(\omega)\big)$$

Since $\psi, \emptyset \in \Phi$ then

$$\psi\Big(d\big(\xi_1(\omega),\xi_2(\omega)\big)\Big) = \psi\Big(d\Big(T\big(\omega,\xi_1(\omega)\big), T\big(\omega,\xi_2(\omega)\big)\Big)\Big)$$
$$\leq \psi M_T\Big(\big(\omega,\xi_1(\omega)\big),\big(\omega,\xi_2(\omega)\big)\Big) - \emptyset\Big(M_T\Big(\big(\omega,\xi_1(\omega)\big),\big(\omega,\xi_2(\omega)\big)\Big)\Big)$$
$$= \psi\Big(d\big(\xi_1(\omega),\xi_2(\omega)\big)\Big) - \emptyset\Big(d\big(\xi_1(\omega),\xi_2(\omega)\big)\Big)$$

i.e. $\psi\Big(d\big(\xi_1(\omega),\xi_2(\omega)\big)\Big) \leq \psi\Big(d\big(\xi_1(\omega),\xi_2(\omega)\big)\Big) - \emptyset\Big(d\big(\xi_1(\omega),\xi_2(\omega)\big)\Big)$

which is a contradiction

so $\quad d\big(\xi_1(\omega),\xi_2(\omega)\big) = 0.$

$\xi_1(\omega) = \xi_2(\omega)$ for every $\omega\in\Omega$.

Therefore $\xi(\omega)$ is a common unique random fixed point of T.

Theorem 3.2: Let X be a symmetric space and S, T: $\Omega\times X\to CB(X)$ be mappings satisfying the modified property (E.A.) such that

$$\psi\, d\,\big(S(\omega,x),T(\omega,y)\big) \leq \psi\Big(M_{S,T}\big((\omega,x),(\omega,y)\big)\Big) - \emptyset\Big(M_{S,T}\big((\omega,x),(\omega,y)\big)\Big)$$

for all x, $y\in X$, $\omega\in\Omega$ where $\psi, \emptyset \in \Phi$. Then T and S have a unique common random fixed point.

Proof: Since S and T satisfies the (E.A.) property then there exists a sequences $\{\xi_n\}$ in M(Ω, X) such that

$$\lim_{n\to\infty} T\big(\omega,\xi_n(\omega)\big) = \lim_{n\to\infty} S\big(\omega,\xi_n(\omega)\big) = \lim_{n\to\infty} \xi_n(\omega) = \xi(\omega)$$

For every$\omega\in\Omega$, for some $\xi\in$ M (Ω, X) since S and T is a quasi-weak contraction then we get

$$d\,\Big(S\big(\omega,\xi(\omega)\big), T\big(\omega,\xi_n(\omega)\big)\Big)$$
$$\leq \psi\Big(M_{S,T}\Big(\big(\omega,\xi(\omega)\big),\big(\omega,\xi_n(\omega)\big)\Big)\Big) - \emptyset\Big(M_{S,T}\Big(\big(\omega,\xi(\omega)\big),\big(\omega,\xi_n(\omega)\big)\Big)\Big)$$

Now consider

$$\lim_{n\to\infty} M_{S,T}\Big(\big(\omega,\xi(\omega)\big),\big(\omega,\xi_n(\omega)\big)\Big)$$
$$= \lim_{n\to\infty} \max\Big(\Big\{d\big(\xi(\omega),\xi_n(\omega)\big), d\Big(\xi(\omega),S\big(\omega,\xi(\omega)\big)\Big), d\Big(\xi_n(\omega),T\big(\omega,\xi_n(\omega)\big)\Big),$$
$$d\Big(\xi(\omega),T\big(\omega,\xi_n(\omega)\big)\Big),\ d\Big(\xi_n(\omega),S\big(\omega,\xi(\omega)\big)\Big)\Big\}\Big)$$

$$= \max\left(\left\{d\big(\xi(\omega),\xi(\omega)\big), d\left(\xi(\omega),S\big(\omega,\xi(\omega)\big)\right), d\big(\xi(\omega),\xi(\omega)\big),\right.\right.$$

$$\left.\left. d\big(\xi(\omega),\xi(\omega)\big),\ \ d\left(\xi(\omega),S\big(\omega,\xi(\omega)\big)\right)\right\}\right)$$

$$= \max\left(\left\{0, d\left(\xi(\omega),S\big(\omega,\xi(\omega)\big)\right), 0, 0,\ \ d\left(\xi(\omega),S\big(\omega,\xi(\omega)\big)\right)\right\}\right)$$

$$= d\left(\xi(\omega),S\big(\omega,\xi(\omega)\big)\right)$$

$\therefore\ \psi,\emptyset \in \Phi$ then implies that

$$\lim_{n\to\infty} \psi\, d\left(S\big(\omega,\xi(\omega)\big), T\big(\omega,\xi_n(\omega)\big)\right) = \psi\left(d\left(S\big(\omega,\xi(\omega)\big),\xi(\omega)\right)\right)$$

$$\leq \psi\left(d\left(\xi(\omega),S\big(\omega,\xi(\omega)\big)\right)\right) - \emptyset\left(d\left(\xi(\omega),S\big(\omega,\xi(\omega)\big)\right)\right)$$

which is a contradiction

so $S\big(\omega,\xi(\omega)\big) = \xi(\omega)$

Therefore $\xi(\omega)$ is a fixed point of S.

Similarly, we have proved that $\xi(\omega)$ is a fixed point of T.

Uniqueness: To prove the uniqueness, we suppose that S and T has two distinct points $\xi_1(\omega)$ and $\xi_2(\omega)$ then

$$M_{S,T}\left(\big(\omega,\xi_1(\omega)\big),\big(\omega,\xi_2(\omega)\big)\right)$$

$$\leq \max\left\{d\big(\xi_1(\omega),\xi_2(\omega)\big), d\left(\xi_1(\omega),S\big(\omega,\xi_1(\omega)\big)\right), d\left(\xi_2(\omega),T\big(\omega,\xi_2(\omega)\big)\right),\right.$$

$$\left. d\left(\xi_1(\omega),T\big(\omega,\xi_2(\omega)\big)\right),\ \ d\left(\xi_2(\omega),S\big(\omega,\xi_1(\omega)\big)\right)\right\}$$

$$M_{S,T}\left(\big(\omega,\xi_1(\omega)\big),\big(\omega,\xi_2(\omega)\big)\right)$$

$$\leq \max\{d\big(\xi_1(\omega),\xi_2(\omega)\big), d\big(\xi_1(\omega),\xi_1(\omega)\big), d\big(\xi_2(\omega),\xi_2(\omega)\big),$$

$$d\big(\xi_1(\omega),\xi_2(\omega)\big),\ \ d\big(\xi_2(\omega),\xi_1(\omega)\big)\}$$

$$= d\big(\xi_1(\omega),\xi_2(\omega)\big)$$

Since $\psi,\emptyset \in \Phi$ then

$$\psi\, d\big(\xi_1(\omega),\xi_2(\omega)\big) \leq \psi\left(d\left(S\big(\omega,\xi_1(\omega)\big),T\big(\omega,\xi_2(\omega)\big)\right)\right)$$

$$\leq \psi\left(M_{S,T}\big(\omega,\xi_1(\omega)\big),\big(\omega,\xi_2(\omega)\big)\right) - \emptyset\left(M_{S,T}\big(\omega,\xi_1(\omega)\big),\big(\omega,\xi_2(\omega)\big)\right)$$

$$\leq \psi\left(d\big(\xi_1(\omega),\xi_2(\omega)\big)\right) - \emptyset\left(d\big(\xi_1(\omega),\xi_2(\omega)\big)\right)$$

i. e.

$$\psi\, d\big(\xi_1(\omega), \xi_2(\omega)\big) \leq \psi\Big(d\big(\xi_1(\omega), \xi_2(\omega)\big)\Big) - \emptyset\Big(d\big(\xi_1(\omega), \xi_2(\omega)\big)\Big)$$

which is a contradiction

so $d\big(\xi_1(\omega), \xi_2(\omega)\big) = 0$

$\Rightarrow \xi_1(\omega) = \xi_2(\omega)$ for every $\omega \in \Omega$.

Therefore, S and T have a common random unique fixed point.

Conflict of Interests

The authors declare that there is no conflict of interests.

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