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SOME FIXED POINTS PROPERTIES, STRONG AND Δ-CONVERGENCE RESULTS FOR GENERALIZED α-NONEXPANSIVE MAPPINGS IN HYPERBOLIC SPACES

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Abstract. In this paper, we introduce and study some fixed points properties and demiclosedness principle for generalized α-nonexpansive mappings in the frame work of uniformly convex hyperbolic spaces. We further establish strong and Δ-convergence theorems for Picard Normal S-iteration scheme generated by a generalized α-nonexpansive mapping in the frame work of uniformly convex hyperbolic spaces. The results obtained in this paper extend and generalize corresponding results in uniformly convex Banach spaces and many other results in this direction.

Keywords: generalized α-nonexpansive mappings; α-nonexpansive mappings; hyperbolic spaces; strong and $\triangle$-convergence theorems; Picard normal S-iteration.

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1. Introduction

Let (X,d) be a metric space and C be a nonempty closed and convex subset of X. A point $x \in C$ is called a fixed point of a nonlinear mapping $T : C \to C$, if

$$Tx = x. \tag{1.1}$$

The set of all fixed points of T is denoted by $F(T)$.

Many real life problems emanating from different disciplines such as Biology, Chemistry, Physics and so on, are modelled into mathematical equations. Over the years mathematicians have been able to express these equations in form of Equation (1.1). However, it became very tedious to get an analytic solution to Equation (1.1). Thus, researchers in this area opted for an approximate solutions. In view of this, different researchers came up with different iteration process to approximate Equation (1.1) with suitable nonlinear mappings in different domain.

The Picard iterative process

$$x_{n+1} = Tx_n, \ \forall n \in \mathbb{N}, \tag{1.2}$$

is one of the earliest iterative process used to approximate Equation (1.1), where T is a contraction mapping. Recall that a mapping $T : C \to C$ is said to be a contraction mapping if there exists $k \in (0,1)$ such that

$$d(Tx,Ty) \leq kd(x,y), \ \forall x,y \in C. \tag{1.3}$$

If $k = 1$ in (1.3), then T is called a nonexpansive mapping. In this case, the Picard iterative process fails to approximate Equation (1.1) even when the existence of the fixed point is guaranteed. To overcome this limitation, researchers in this area developed different iterative processes to approximate fixed points of nonexpansive mappings and other mappings more general than nonexpansive mappings. Among many others, are; Mann [19], Ishikawa [11], Krasnosel'skii [17], Agarwal [3], Noor [20] and so on. There are numerous papers dealing with the approximation of fixed points of nonexpansive mappings, asymptotically nonexpansive mappings, total asymptotically nonexpansive mappings in uniformly convex Banach spaces and CAT(0) spaces (for example, see [1, 2, 3, 13] and the references therein).

In 2011, Sahu [23] introduced the Normal S-iteration process in Banach space and established that the rate of convergence of the Normal S-iteration process is as fast as the Picard iteration process but faster than other fixed point iteration process that was in existence then. The Normal S-iteration process is defined as follows: Let C be a convex subset of a normed space E and $T : C \to C$ be any nonlinear mapping. For each $x_1 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ x_{n+1} = T y_n,\ n \geq 1, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$.

In time past, researchers in this area have introduced iterative processes whose rate of convergence are faster than that of the Normal S-iteration. For example, in [12], Kadioglu and Yildirim introduced Picard Normal S-iteration process and they established that the rate of convergence of the Picard Normal S-iteration process is faster than the Normal S-iteration process. The Picard Normal S-iteration process is defined as follows: Let C be a convex subset of a normed space E and $T : C \to C$ be any nonlinear mapping. For each $x_1 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1-\beta_n)x_n + \beta_n T x_n, \\ y_n = (1-\alpha_n)z_n + \alpha_n T z_n, \\ x_{n+1} = T y_n,\ n \geq 1, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$.

Remark 1.1. Clearly, if $\alpha_n = \beta_n = 0$, then iterative process (1.5) reduces to (1.2) and if $\beta_n = 0$, iterative process (1.5) reduces to (1.4).

It is worth mentioning that, in fixed point theory researchers try as much as possible to generalize existing maps. Suzuki [25] introduce a generalization of nonexpansive mapping called the Suzuki-generalized nonexpansive mapping (or mapping satisfying condition (C)). We recall form [25] that, for a nonempty subset K of a Banach space E. A mapping $T : K \to K$ is said to satisfy condition (C) (or called Suzuki-generalized nonexpansive) if for all $x, y \in K$

$$\frac{1}{2}||x - Tx|| \leq ||x - y|| \Rightarrow ||Tx - Ty|| \leq ||x - y||.$$

Suzuki [25] established the following result.

Theorem 1.2 ([25]). *Let K be a nonempty convex subset of a Banach space E and $T : K \to K$ be a mapping satisfying condition (C). Assume also that either of the following holds:*

(i) K is compact;

(ii) K is weakly compact and E has Opial property.

Then T has a fixed point.

Aoyama and Kohsaka [4] introduced another type of generalized nonexpansive mapping called α-nonexpansive mapping.

Definition 1.3. Let E be a Banach space and C be a nonempty closed and convex subset of E. A mapping $T : C \to C$ is said to be α-nonexpansive if for all $x, y \in C$ and $\alpha < 1$,

$$||Tx - Ty||^2 \leq \alpha||Tx - y||^2 + \alpha||Ty - x||^2 + (1 - 2\alpha)||x - y||^2.$$

They obtained the following result.

Theorem 1.4 ([4]). *Let C be a nonempty convex subset of a uniformly convex Banach space E and $T : C \to C$ be an α-nonexpansive mapping. Then $F(T)$ is nonempty if and only if there exists $x \in C$ such that $\{T^n(x)\}$ is bounded.*

The Suzuki-generalized nonexpansive mapping and the α-nonexpansive mapping raised the following natural question.

Question: Does there exists a class of mapping, which contain both the Suzuki-generalized nonexpansive mapping and α-nonexpansive mapping.

The question was partially answered in affirmation by Pant and Shukla in [21]. Indeed, they introduce and studied the generalized α-nonexpansive mapping in Banach space.

Definition 1.5. Let E be a Banach space and C be a nonempty subset of E. A mapping $T : C \to C$ is said to be a generalized α-nonexpansive if for all $x, y \in C$ there exists $\alpha \in [0,1)$ such that

$$\frac{1}{2}||x - Tx|| \leq ||x - y|| \Rightarrow ||Tx - Ty|| \leq \alpha||Tx - y|| + \alpha||Ty - x|| + (1 - 2\alpha)||x - y||.$$

Example 1.6 ([21])**.** Let $X = \{(0,0), (2,0), (0,4), (4,0), (4,5), (5,4)\}$ be a subset of $\mathbb{R}^2$. Define a norm $\|\cdot\|$ on X by $\|(x_1, x_2)\| = |x_1| + |x_2|$. Then $(X, \|\cdot\|)$ is a Banach space. Define a mapping $T : X \to X$ by

$$T : \begin{pmatrix} (0,0), (2,0), (0,4), (4,0), (4,5), (5,4) \\ (0,0), (0,0), (0,0), (2,0), (4,0), (0,4) \end{pmatrix}, \tag{1.6}$$

It was established in [21] that T is a generalized α-nonexpansive mapping for $\alpha \geq \frac{1}{5}$, but is neither a Suzuki-generalize nonexpansive nor an α-nonexpansive mapping. Furthermore, existence and convergence results were established in [21].

Remark 1.7.

(i) It is well-known that nonexpansive mappings satisfy condition C. However, the converse of this statement is not always true (see [25]).

(ii) Clearly, if $\alpha = 0$, then α-nonexpansive mapping reduces to a nonexpansive mapping.

(iii) Also, if $\alpha = 0$, then generalized α-nonexpansive mapping reduces to Suzuki-generalized nonexpansive mapping.

Beside the nonlinear mappings involved in the study of fixed point theory, the role played by the spaces involved is also very important. It is known in literature that Banach spaces have been studied extensively. This is because of the fact that Banach spaces always have convex structures. However, metric spaces do not naturally enjoy this structure. Therefore the need to introduce convex structures to it arises. The notion of convex metric spaces was first introduced by Takahashi [26] who studied the fixed point theory for nonexpansive mappings in the settings of convex metric spaces. Since then, several attempts have been made to introduce different convex structures on metric spaces. An example of a metric space with a convex structure is the hyperbolic space. Different convex structures have been introduced to hyperbolic spaces resulting to different definitions of hyperbolic spaces (see [8, 15, 22]). Although the class of hyperbolic spaces defined by Kohlenbach [15] is slightly restrictive than the class of hyperbolic spaces introduced in [8], it is however, more general than the class of hyperbolic spaces introduced in [22]. Moreover, it is well-known that Banach spaces are examples of hyperbolic spaces introduced in [15]. Some other examples of this class of hyperbolic spaces includes CAT(0) spaces, Hadamard manifords, Hilbert ball with the hyperbolic metric, Catesian products of Hilbert balls and $\mathbb{R}$-trees. The reader should please see [8, 9, 15, 22] for more discussion and examples of hyperbolic spaces.

It is worth mentioning that, as far as we know, no work has been done on fixed point problems for generalized α-nonexpansive mappings in convex metric spaces. Therefore, it is necessary to extend results on fixed point problems for generalized α-nonexpansive mappings from the frame work of Banach spaces to the settings of hyperbolic spaces, since the class of hyperbolic spaces generalizes the class of Banach spaces.

Motivated by all these facts, we introduce and study some fixed points properties and demicloseness principle for generalized α-nonexpansive mapping in uniformly convex hyperbolic spaces introduced by Kohlenbach [15], and establish both strong and Δ-convergence theorems for approximating fixed point of this class of mappings using the Picard Normal S-iteration. Thus, the results obtained in this paper extend and generalize corresponding results in uniformly convex Banach spaces and many other results in this direction.

2. **Preliminaries**

Throughout this paper, we carry out all our study in the frame work of hyperbolic space introduced by Kohlenbach [15].

Definition 2.1. A hyperbolic space (X,d,W) is a metric space (X,d) together with a convex mapping $W : X^2 \times [0,1] \to X$ satisfying

(1) $d(u,W(x,y,\alpha)) \leq \alpha d(u,x) + (1-\alpha)d(u,y)$;

(2) $d(W(x,y,\alpha),W(x,y,\beta)) = |\alpha-\beta| d(x,y)$;

(3) $W(x,y,\alpha) = W(y,x,1-\alpha)$;

(4) $d(W(x,z,\alpha),W(y,w,\alpha)) \leq (1-\alpha)d(x,y) + \alpha d(z,w)$;

for all $w,x,y,z \in X$ and $\alpha,\beta \in [0,1]$.

Example 2.2. [24] Let X be a real Banach space which is equipped with norm $||.||$. Define the function

$d : X^2 \to [0,\infty)$ by

$$d(x,y) = ||x-y||$$

as a metric on X. Then, we have that (X,d,W) is a hyperbolic space with mapping $W : X^2 \times [0,1] \to X$ defined by $W(x,y,\alpha) = (1-\alpha)x + \alpha y$.

Definition 2.3. [24] Let X be a hyperbolic space with a mapping $W : X^2 \times [0,1] \to X$.

(i) A nonempty subset C of X is said to be convex if $W(x,y,\alpha) \in C$ for all $x,y \in C$ and $\alpha \in [0,1]$.

(ii) X is said to be uniformly convex if for any $r > 0$ and $\varepsilon \in (0,2]$, there exists a $\delta \in (0,1]$ such that for all $x,y,z \in X$

$$d(W(x,y,\frac{1}{2}),z) \leq (1-\delta)r,$$

provided $d(x,z) \leq r, d(y,z) \leq r$ and $d(x,y) \geq \varepsilon r$.

(iii) A map $\eta : (0,\infty) \times (0,2] \to (0,1]$ which provides such a $\delta = \eta(r,\varepsilon)$ for a given $r > 0$ and $\varepsilon \in (0,2]$ is known as a modulus of uniform convexity of X. The mapping η is said to be monotone, if it decreases with r (for a fixed ε).

Definition 2.4. Let C be a nonempty subset of a metric space X and $\{x_n\}$ be any bounded sequence in C. For $x \in X$, let $r(\cdot,\{x_n\}) : X \to [0,\infty)$ be a continuous functional defined by

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x_n,x).$$

The asymptotic radius $r(C,\{x_n\})$ of $\{x_n\}$ with respect to C is given by

$$r(C,\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in C\}.$$

A point $x \in C$ is said to be an asymptotic center of the sequence $\{x_n\}$ with respect to $C \subseteq X$ if

$$r(x,\{x_n\}) = \inf\{r(y,\{x_n\}) : y \in C\}.$$

The set of all asymptotic centers of $\{x_n\}$ with respect to C is denoted by $A(C,\{x_n\})$. If the asymptotic radius and the asymptotic center are taken with respect to X, then we simply denote them by $r(\{x_n\})$ and $A(\{x_n\})$ respectively.

It is well-known that in uniformly convex Banach spaces and CAT(0) spaces, bounded sequences have unique asymptotic center with respect to closed convex subsets.

Definition 2.5. [14]. A sequence $\{x_n\}$ in X is said to $\triangle$-converge to $x \in X$, if x is the unique asymptotic center of $\{x_{n_k}\}$ for every subsequence $\{x_{nk}\}$ of $\{x_n\}$. In this case, we write $\triangle$-$\lim\limits_{n\to\infty} x_n = x$.

Remark 2.6. [16]. We note that $\triangle$-convergence coincides with the usually weak convergence known in Banach spaces with the usual Opial property.

Lemma 2.7. [18] *Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset C of X.*

Lemma 2.8. [7] *Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and let $\{x_n\}$ be a bounded sequence in X with $A(\{x_n\}) = \{x\}$. Suppose $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$ with $A(\{x_{n_k}\}) = \{x_1\}$ and $\{d(x_n, x_1)\}$ converges, then $x = x_1$.*

Lemma 2.9. [13] *Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x^* \in X$ and $\{t_n\}$ be a sequence in $[a,b]$ for some $a, b \in (0,1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that* $\limsup_{n\to\infty} d(x_n, x^*) \leq c$, $\limsup_{n\to\infty} d(y_n, x^*) \leq c$ *and* $\lim_{n\to\infty} d(W(x_n, y_n, t_n), x^*) = c$, *for some* $c > 0$. *Then* $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Definition 2.10. Let C be a nonempty subset of a hyperbolic space X and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is called a Fejér monotone sequence with respect to C if for all $x \in C$ and $n \geq 1$,

$$d(x_{n+1}, x) \leq d(x_n, x).$$

Proposition 2.11. [10] *Let $\{x_n\}$ be a sequence in X and C be a nonempty subset of X. Suppose that $T : C \to C$ is any nonlinear mapping and the sequence $\{x_n\}$ is Fejer monotone with respect to C, then we have the following:*

(i) $\{x_n\}$ *is bounded.*

(ii) The sequence $\{d(x_n, x^*)\}$ *is decreasing and converges for all* $x^* \in F(T)$*.*

(iii) $\lim_{n\to\infty} d(x_n, F(T))$ *exists.*

Lemma 2.12 ([21])**.** *Let* C *be a nonempty subset of a hyperbolic space* X*. Let* $T : C \to C$ *be a generalized* α*-nonexpansive mapping and* $F(T) \neq \emptyset$*, then* T *is quasi-nonexpansive.*

Lemma 2.13 ([21])**.** *Let* C *be a nonempty subset of a hyperbolic space* X*. Let* $T : C \to C$ *be a generalized* α*-nonexpansive mapping, then for all* $x, y \in C$*,* $||x - Ty|| \leq \frac{(3+\alpha)}{(1-\alpha)}||x - Tx|| + ||x - y||$*.*

3. Main Results

We recall that in a metric space X, a mapping $T : C \subseteq X \to C$ is said to be generalized α-nonexpansive if for all $x, y \in C$ there exists $\alpha \in [0, 1)$ such that

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq \alpha d(Tx, y) + \alpha d(Ty, x) + (1 - 2\alpha)d(x, y).$$

3.1. Fixed Points Properties for Generalized α-Nonexpansive Mappings.

Theorem 3.1. *Let* C *be a nonempty closed and convex subset of a hyperbolic space* X*. Let* $T : C \to C$ *be a generalized* α*-nonexpansive mapping and* $F(T) \neq \emptyset$*, then* $F(T)$ *is closed and convex.*

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ such that $\{x_n\}$ converges to some $y \in C$. We show that $y \in F(T)$. Since $\frac{1}{2}d(x_n, Tx_n) = \frac{1}{2}d(Tx_n, Tx_n) = 0 \leq d(x_n, y)$, by definition of T, we obtain

$$\begin{aligned}
d(x_n, Ty) &\leq \alpha d(Tx_n, y) + \alpha d(Ty, x_n) + (1 - 2\alpha)d(x_n, y) \\
&\Longrightarrow (1 - \alpha)d(x_n, Ty) \leq \alpha d(x_n, y) + (1 - 2\alpha)d(x_n, y) \\
&\Longrightarrow d(x_n, Ty) \leq d(x_n, y),
\end{aligned}$$

Since $\lim_{n\to\infty} d(x_n, y) = 0$, then by sandwich theorem, we obtain

$$\lim_{n\to\infty} d(x_n, Ty) = 0.$$

By the uniqueness of limit, we have that

$$Ty = y.$$

Hence, $F(T)$ is closed.

Next, we show that $F(T)$ is convex. Let $x, y \in F(T)$, since $\frac{1}{2}d(x,Tx) = \frac{1}{2}d(Tx,Tx) = 0 \leq d(x,z)$ and $\frac{1}{2}d(y,Ty) = \frac{1}{2}d(Ty,Ty) = 0 \leq d(y,z)$, by definition of T, we obtain

$$\begin{aligned} d(x,Tz) &\leq \alpha d(Tx,z) + \alpha d(Tz,x) + (1-2\alpha)d(x,z) \\ &\Rightarrow (1-\alpha)d(x,Tz) \leq \alpha d(x,z) + (1-2\alpha)d(x,z) \\ &\Rightarrow d(x,Tz) \leq d(x,z). \end{aligned} \tag{3.1}$$

Using similar argument, we have

$$d(y,Tz) \leq d(y,z). \tag{3.2}$$

Let $z = W(x,y,\beta)$, for $\beta \in [0,1]$, then from (3.1) and (3.2), we obtain

$$\begin{aligned} d(x,y) &\leq d(x,Tz) + d(Tz,y) \\ &\leq d(x,z) + d(z,y) \qquad (3.3) \\ &= d(x,W(x,y,\beta)) + d(W(x,y,\beta),y) \\ &\leq (1-\beta)d(x,x) + \beta d(x,y) + (1-\beta)d(x,y) + \beta d(y,y) \\ &= d(x,y). \end{aligned}$$

Hence, we conclude that (3.1) and (3.2) are $d(x,Tz) = d(x,z)$ and $d(y,Tz) = d(y,z)$ respectively. Because if $d(x,Tz) < d(x,z)$ or $d(y,Tz) < d(y,z)$, then the inequality in (3.3) becomes strictly less than, which therefore gives us a contradiction, that is, $d(x,y) < d(x,y)$. Hence, we have that $Tz = z$. Thus, $W(x,y,\beta) \in F(T)$, which implies that $F(T)$ is convex. □

In view of Remark 1.7, we have the following corollaries.

Corollary 3.2. *Let C be a nonempty closed and convex subset of a hyperbolic space X. Let $T : C \to C$ be a nonexpansive mapping and $F(T) \neq \emptyset$, then $F(T)$ is closed and convex.*

Corollary 3.3. *Let C be a nonempty closed and convex subset of a hyperbolic space X. Let $T : C \to C$ be a Suzuki-generalized nonexpansive mapping and $F(T) \neq \emptyset$, then $F(T)$ is closed and convex.*

Next, we establish the demiclosedness principle for generalized α-nonexpansive mappings in hyperbolic spaces.

Theorem 3.4. *Let C be a nonempty closed and convex subset of a complete hyperbolic space X with monotone modulus of uniform convexity η. Let $T : C \to C$ be a generalized α-nonexpansive mapping and $\{x_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and Δ-$\lim_{n\to\infty} x_n = x$. Then $x \in F(T)$.*

Proof. Since $\{x_n\}$ is a bounded sequence in X, we have from Lemma 2.7 that $\{x_n\}$ has a unique asymptotic center in C. Also, since $\Delta - \lim_{n\to\infty} x_n = x$, we have that $A(\{x_n\}) = \{x\}$. Using Lemma 2.13 and the hypothesis that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, we have

$$d(x_n, Tx) \leq \frac{(3+\alpha)}{(1-\alpha)} d(x_n, Tx_n) + d(x_n, x),$$

taking $\limsup_{n\to\infty}$, we have

$$\begin{aligned}\limsup_{n\to\infty} d(x_n, Tx) &\leq \frac{(3+\alpha)}{(1-\alpha)} \limsup_{n\to\infty} d(x_n, Tx_n) + \limsup_{n\to\infty} d(x_n, x) \\ &= \limsup_{n\to\infty} d(x_n, x).\end{aligned}$$

By the uniqueness of asymptotic center, we obtain that $Tx = x$. Hence $x \in F(T)$. □

3.2. **Strong and Δ-Convergence Theorems for α-Nonexpansive Mappings.** We now study iterative process (1.5) in hyperbolic spaces. Let C be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X and $T : C \to C$ be a generalized α-nonexpansive mapping. The sequence $\{x_n\}$ is defined recursively as follows:

$$\begin{cases} x_1 \in C, \\ z_n = W(x_n, Tx_n, \beta_n), \\ y_n = W(z_n, Tz_n, \gamma_n), \\ x_{n+1} = W(Ty_n, 0, 0), \quad n \geq 1, \end{cases} \tag{3.4}$$

where $\{\gamma_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. We now state and prove the following lemmas which will be needed in the proof of our main theorems. In the course of establishing our result, we note that for all $x^* \in F(T)$, we have

$$\begin{aligned}
&\frac{1}{2}d(x^*,Tx^*) = \frac{1}{2}d(x^*,x^*) \leq d(x^*,z_n),\\
&\frac{1}{2}d(x^*,Tx^*) = \frac{1}{2}d(x^*,x^*) \leq d(x^*,x_n) \quad \text{and}\\
&\frac{1}{2}d(x^*,Tx^*) = \frac{1}{2}d(x^*,x^*) \leq d(x^*,y_n),
\end{aligned} \tag{3.5}$$

which by the definition of T, implies

$$\begin{aligned}
&d(Tx^*,Tz_n) \leq \alpha d(Tx^*,z_n) + \alpha d(Tz_n,x^*) + (1-2\alpha)d(x^*,z_n)\\
&d(Tx^*,Tx_n) \leq \alpha d(Tx^*,x_n) + \alpha d(Tx_n,x^*) + (1-2\alpha)d(x^*,x_n) \quad \text{and}\\
&d(Tx^*,Ty_n) \leq \alpha d(Tx^*,y_n) + \alpha d(Ty_n,x^*) + (1-2\alpha)d(x^*,y_n).
\end{aligned} \tag{3.6}$$

Lemma 3.5. *Let C be a nonempty closed and convex subset of a hyperbolic space X. Let $T: C \to C$ be a generalized α-nonexpansive mapping and $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by (3.4), where $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$, then the following hold:*

(i) $\{x_n\}$ *is bounded.*

(ii) $\lim_{n\to\infty} d(x_n,x^*)$ *exists for all* $x^* \in F(T)$.

(iii) $\lim_{n\to\infty} d(x_n,F(T))$ *exists.*

Proof. Let $x^* \in F(T)$, then from (3.6) and Lemma 2.12, we obtain

$$\begin{aligned}
&d(Tx^*,Tz_n) \leq d(x^*,z_n)\\
&d(Tx^*,Tx_n) \leq d(x^*,x_n),\\
&d(Tx^*,Ty_n) \leq d(x^*,y_n).
\end{aligned} \tag{3.7}$$

Again, from (3.4) and (3.7), we have

$$\begin{aligned} d(z_n,x^*) &= d(W(x_n,Tx_n,\beta_n),x^*) \\ &\leq (1-\beta_n)d(x_n,x^*)+\beta_n d(Tx_n,x^*) \\ &\leq (1-\beta_n)d(x_n,x^*)+\beta_n d(x_n,x^*) \\ &= d(x_n,x^*). \end{aligned} \tag{3.8}$$

From (3.4), (3.7) and (3.8), we obtain

$$\begin{aligned} d(y_n,x^*) &= d(W(z_n,Tz_n,\gamma_n),x^*) \\ &\leq (1-\gamma_n)d(z_n,x^*)+\gamma_n d(Tz_n,x^*) \\ &\leq (1-\gamma_n)d(z_n,x^*)+\gamma_n d(z_n,x^*) \\ &= d(z_n,x^*) \qquad (3.9) \\ &\leq d(x_n,x^*). \end{aligned}$$

From (3.4), (3.7) and (3.9), we obtain

$$\begin{aligned} d(x_{n+1},x^*) &= d(W(Ty_n,0,0),x^*) \\ &\leq d(Ty_n,x^*) \\ &\leq (y_n,x^*) \qquad (3.10) \\ &\leq d(x_n,x^*), \end{aligned}$$

which implies that $d(x_{n+1},x^*) \leq d(x_n,x^*)$ for all $x^* \in F(T)$. Hence, $\{x_n\}$ is Fejer monotone with respect to $F(T)$ and by Proposition 2.11, $\{x_n\}$ is bounded, $\lim_{n\to\infty} d(x_n,x^*)$ exists for all $x^* \in F(T)$ and $\lim_{n\to\infty} d(x_n,F(T))$ exists. □

Lemma 3.6. *Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and C be a nonempty closed and convex subset of X. Let $T : C \to C$ be a generalized α-nonexpansive mapping and $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by (3.4), where $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$, then* $\lim_{n\to\infty} d(x_n,Tx_n) = 0$.

Proof. From Lemma 3.5, we have that $\lim_{n\to\infty} d(x_n,x^*)$ exists for all $x^* \in F(T)$. Suppose that $\lim_{n\to\infty} d(x_n,x^*) = c$. If we take c = 0, then we are done. Thus, we consider the case where $c > 0$.

Since

$$\frac{1}{2}d(x^*,Tx^*) = 0 \leq d(x^*,x_n),$$

we obtain from the definition of T that

$$\begin{aligned} d(x^*,Tx_n) = d(Tx^*,Tx_n) &\leq \alpha_n d(Tx^*,x_n) + \alpha_n d(Tx_n,x^*) + (1-2\alpha_n)d(x^*,x_n) \\ &\Rightarrow (1-\alpha_n)d(x^*,Tx_n) \leq (1-\alpha_n)d(x_n,x^*) \\ &\Rightarrow d(x^*,Tx_n) \leq d(x_n,x^*). \end{aligned}$$

Thus,

$$\limsup_{n\to\infty} d(Tx_n,x^*) \leq c.$$

From (3.8), we have

$$d(z_n,x^*) \leq d(x_n,x^*),$$

which implies that

$$\limsup_{n\to\infty} d(z_n,x^*) \leq c. \tag{3.11}$$

From (3.9) and (3.10), we have

$$d(x_{n+1},x^*) \leq d(z_n,x^*).$$

Thus, taking $\liminf_{n\to\infty}$, we have that

$$c \leq \liminf_{n\to\infty} d(z_n,x^*). \tag{3.12}$$

From (3.11) and (3.12), we obtain that $\lim_{n\to\infty} d(z_n,x^*) = c$. That is,

$$\lim_{n\to\infty} d(W(x_n,Tx_n,\beta_n),x^*) = c.$$

Thus, by Lemma 2.9, we have

$$\lim_{n\to\infty} d(x_n,Tx_n) = 0.$$

□

Theorem 3.7. *Let C be a nonempty closed and convex subset of a complete hyperbolic space X with monotone modulus of uniform convexity η. Let $T : C \to C$ be a generalized α-nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is defined by (3.4), then $\{x_n\}$ Δ-converges to a fixed point of T.*

Proof. Let $W_\Delta(x_n) := \cup A(\{u_n\})$, where the union is taken over all subsequence $\{u_n\}$ of $\{x_n\}$. We now show that $W_\Delta(x_n) \subset F(T)$ and that $W_\Delta(x_n)$ contains only one point.
Let $u \in W_\Delta(x_n)$, then by Lemma 3.5, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. This implies from Lemma 2.7 that we can find a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v$, for some $v \in C$. By Lemma 3.6, we have that $\lim_{n\to\infty} d(v_n, Tv_n) = 0$, which together with Theorem 3.4 gives that $v \in F(T)$. Therefore, $\{d(u_n, v)\}$ converges and by Lemma 2.8, we have that $v = u \in F(T)$. Hence, $W_\Delta(x_n) \subset F(T)$.
Next, we show that $W_\Delta(x_n)$ contains only one point. Let $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ be arbitrary subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. Then by Lemma 3.5, we have that $\{d(x_n, u)\}$ converges, since $u \in F(T)$. Thus, by Lemma 2.8, we have that $u = x \in F(T)$. Hence, $W_\Delta(x_n) = \{x\}$. Therefore, $\{x_n\}$ $\triangle$-converges to a common fixed point of T. $\square$

Theorem 3.8. *Let C be a nonempty closed and convex subset of a complete hyperbolic space X with monotone modulus of uniform convexity η. Let $T : C \to C$ be a generalized α-nonexpansive mapping with $F(T) \neq \emptyset$ and the sequence $\{x_n\}$ be generated by (3.4). Then the sequence $\{x_n\}$ converges strongly to some fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(x_n, F(T)) = \inf_{x\in F(T)} d(x_n, x)$.*

Proof. Suppose that $\{x_n\}$ converges to a fixed point, say x^* of T. Then $\lim_{n\to\infty} d(x_n, x^*) = 0$, and since $0 \leq d(x_n, F(T)) \leq d(x_n, x^*)$, it follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Therefore, $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.
Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. From Lemma 3.5, we have that $\lim_{n\to\infty} d(x_n, F(T))$ exists and so, it follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Suppose that $\{x_{n_k}\}$ is any arbitrary subsequence of $\{x_n\}$ and $\{p_k\}$ a sequence in $F(T)$ such that for all $n \geq 1$,

$$d(x_{n_k}, p_k) < \frac{1}{2^k}.$$

From (3.10), we obtain that

$$d(x_{n+1}, p_k) \leq d(x_{n_k}, p_k) < \frac{1}{2^k}.$$

Thus,

$$\begin{aligned} d(p_{k+1}, p_k) &\leq d(p_{k+1}, x_{n+1}) + d(x_{n+1}, p_k) \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}. \end{aligned}$$

This shows that $\{p_k\}$ is a Cauchy sequence in $F(T)$. Also, by Lemma 3.1, we have that $F(T)$ is closed. Thus, $\{p_k\}$ is a convergent sequence in $F(T)$ and say it converges to $q \in F(T)$. Therefore, since

$$d(x_{n_k}, q) \leq d(x_{n_k}, p_k) + d(p_k, q) \to 0 \text{ as } n \to \infty,$$

we have $\lim_{n\to\infty} d(x_{n_k}, q) = 0$ and so $\{x_{n_k}\}$ converges strongly to $q \in F(T)$. Since, $\lim_{n\to\infty} d(x_n, q)$ exists, it follows that $\{x_n\}$ converges strongly to q. □

Theorem 3.9. *Let C be a nonempty closed and convex subset of a complete hyperbolic space X with monotone modulus of uniform convexity η. Let $T : C \to C$ be a generalized α-nonexpansive mapping with $F(T) \neq \emptyset$ and the sequence $\{x_n\}$ be generated by (3.4). Suppose that there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0,\infty)$ such that $f(d(x, F(T)) \leq d(x, Tx)$ for all $x \in C$. Then the sequence $\{x_n\}$ converges strongly to $x^* \in F(T)$.*

Proof. From Lemma 3.5, we have $\lim_{n\to\infty} d(x_n, F(T))$ exist and by Lemma 3.6, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Using the fact that $f(d(x, F(T)) \leq d(x, Tx)$ for all $x \in C$, we have that $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since f is nondecreasing with $f(0) = 0$ and $f(t) > 0$ for $t \in (0,\infty)$, it then follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Hence, by Theorem 3.8 $\{x_n\}$ converges strongly to $x^* \in F(T)$. □

In view of Remark 1.1, be letting $\beta_n = 0$, (3.4) becomes:

$$\begin{cases} x_1 \in C, \\ y_n = W(x_n, Tx_n, \gamma_n), \\ x_{n+1} = W(Tx_n, 0, 0), \quad n \geq 1, \end{cases} \tag{3.13}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$. Thus, the following corollaries hold.

Corollary 3.10. *Let C be a nonempty closed and convex subset of a complete hyperbolic space X with monotone modulus of uniform convexity η. Let $T : C \to C$ be a generalized α-nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is the sequence defined by (3.13), then $\{x_n\}$ Δ-converges to a fixed point of T.*

Corollary 3.11. *Let C be a nonempty closed and convex subset of a complete hyperbolic space X with monotone modulus of uniform convexity η. Let $T : C \to C$ be a generalized α-nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is the sequence defined by (3.13), then $\{x_n\}$ converges strongly to some fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(x_n, F(T)) = \inf_{x\in F(T)} d(x_n, x)$.*

Corollary 3.12. *Let C be a nonempty closed and convex subset of a complete hyperbolic space X with monotone modulus of uniform convexity η. Let $T : C \to C$ be a generalized α-nonexpansive mapping with $F(T) \neq \emptyset$ and the sequence $\{x_n\}$ be defined by (3.13). Suppose that there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0,\infty)$ such that $f(d(x, F(T)) \leq d(x, Tx)$ for all $x \in C$. Then the sequence $\{x_n\}$ converges strongly to $x^* \in F(T)$.*

In view of Remark 1.7, the following corollaries also hold.

Corollary 3.13. *Let C be a nonempty closed and convex subset of a complete hyperbolic space X with monotone modulus of uniform convexity η. Let $T : C \to C$ be a Suzuki-generalized nonexpansive mapping with $F(T) \neq \emptyset$. Let the sequence $\{x_n\}$ be defined by (3.4), then $\{x_n\}$ Δ-converges to a fixed point of T.*

Corollary 3.14. *Let C be a nonempty closed and convex subset of a complete hyperbolic space X with monotone modulus of uniform convexity η. Let $T : C \to C$ be a Suzuki-generalized nonexpansive mapping with $F(T) \neq \emptyset$. Let the sequence $\{x_n\}$ be defined by (3.4), then $\{x_n\}$ converges strongly to some fixed point of T if and only if $\liminf_{n\to} d(x_n, F(T)) = 0$, where $d(x_n, F(T)) = \inf_{x\in F(T)} d(x_n, x)$.*

Corollary 3.15. *Let C be a nonempty closed and convex subset of a complete hyperbolic space X with monotone modulus of uniform convexity η. Let $T : C \to C$ be a Suzuki-generalized nonexpansive mapping with $F(T) \neq \emptyset$ and the sequence $\{x_n\}$ be defined by (3.4). Suppose that there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0,\infty)$ such that $f(d(x,F(T)) \leq d(x,Tx)$ for all $x \in C$. Then the sequence $\{x_n\}$ converges strongly to $x^* \in F(T)$.*

Conflict of Interest

The authors declare that there is no conflict of interests.

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