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# FIXED AND BEST PROXIMITY POINTS FOR CYCLIC WEAKLY CONTRACTION MAPPINGS

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**Abstract.** In this paper we obtain some fixed and best proximity point theorems for cyclic $(\psi, \varphi)$-weakly contraction mappings. The results obtained herein extend some recent results.

**Keywords**: Fixed point; best proximity point; cyclic weakly contraction.

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## 1. Introduction and Preliminaries

Thought this paper $\mathbb{N}$ denotes the set of naturals and $X$ a metric space $(X, d)$. Let $A$ and $B$ be nonempty subsets of a metric space $X$. A mapping $T : A \cup B \to A \cup B$ is called a cyclic mapping if $T(A) \subseteq B$ and $T(B) \subseteq A$. A point $z \in A \cup B$ is said to be fixed point of $T$ if $Tz = z$ and a best proximity point of $T$ if $d(z, Tz) = d(A, B)$, where $d(A, B) = \inf\{d(x, y) : x \in A,\ y \in B\}$. All mappings do not have fixed points. For example the mapping $T : [0, \infty) \to [0, \infty)$ defined by $Tx = 1 + x$, has no fixed points, since $x$ is never equal to $x + 1$ for any $x \in [0, \infty)$. If the fixed-point equation $Tx = x$ does not possesses a solution, it is contemplated to resolve a problem finding an element

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$x$ such that $x$ is in proximity to $Tx$ in some sense. Best proximity theorems analyze the conditions under which the optimization problem, namely $\min_{x\in A} d(x,Tx)$ has a solution [9].

Kirk et al. [7] obtained the following interesting fixed point theorem for cyclic mappings.

**Theorem 1.1.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$ and $T : A\cup B \to A\cup B$ be a cyclic mapping. Assume that there exists $\lambda \in (0,1)$ such that*

$$d(Tx,Ty) \leq \lambda d(x,y) \tag{1.1}$$

*for all $x \in A$ and $y \in B$. Then $T$ has a unique fixed point in $A\cap B$.*

The condition (1.1) entails $A\cap B$ being nonempty. Eldred and Veeramani [4] modified the condition (1.1) for the case $A\cap B = \emptyset$ as follows:

$$d(Tx,Ty) \leq \lambda d(x,y) + (1-\lambda)d(A,B) \tag{1.2}$$

for all $x \in A$ and $y \in B$, where $\lambda \in (0,1)$. The mapping $T$ satisfying condition (1.2) is called a cyclic contraction. Eldred and Veeramani [4, Th. 3.10] obtained a unique best proximity point for the mapping $T$ in a uniformly convex Banach space setting. Subsequently, a number of extensions and generalizations of their results appeared in [1, 2, 5, 10] and many others.

Recently, Al-Tagafi and Shahzad [1] introduced the notion of cyclic $\varphi$-contractions and obtained some existence results for this new class of mappings. In this paper we, extend cyclic $\varphi$-contractions and introduce the notion of cyclic $(\psi,\varphi)$-weakly contractions. Subsequently, this notion is utilized to obtain some fixed and best proximity point theorems which generalize certain results of [1], [4] and [7].

## 2. Cyclic $(\psi,\varphi)$-weakly contractions

Throughout this section $\Phi$ denotes the class of the functions $\varphi : [0,\infty) \to [0,\infty)$ satisfying:

**(a)** $\varphi$ is continuous and monotone nondecreasing,

**(b)** $\varphi(t) = 0 \Leftrightarrow t = 0$.

The function $\varphi \in \Phi$ is also known as altering distance function (see, for instance, [6]).

Now we introduce the following notion of a cyclic $(\psi, \varphi)$-weakly contraction mapping.

**Definition 2.1.** Let $A$ and $B$ be nonempty subsets of a metric space $X$ and $T : A \cup B \to A \cup B$ a cyclic mapping. The mapping $T$ will be called a cyclic $(\psi, \varphi)$-weakly contraction if, $\psi, \varphi \in \Phi$ and

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)) + \varphi(d(A, B)), \tag{2.1}$$

for all $x \in A$ and $y \in B$ (see also, [3, 8]).

**Remark 2.2.** We remark that:

**1.** A cyclic $\varphi$-contraction is cyclic $(\psi, \varphi)$-weakly contraction with $\psi(t) = t$ for $t \geq 0$.

**2.** A cyclic contraction is cyclic $(\psi, \varphi)$-weakly contraction with $\psi(t) = t$, $\varphi(t) = (1 - \lambda)t$ for $t \geq 0$ and $\lambda \in (0, 1)$.

Recall that, a Banach space $X$ is said to be:

**(a)** uniformly convex if there exists a strictly increasing function $\delta : (0, 2] \to [0, 1]$ such that the following implication holds for all $x, y, p \in X$, $R > 0$ and $r \in [0, 2R]$:

$$\left.\begin{array}{l} \|x - p\| \leq R \\ \|y - p\| \leq R \\ \|x - y\| \geq r \end{array}\right\} \Rightarrow \left\| \frac{x + y}{2} - p \right\| \leq \left(1 - \delta\left(\frac{r}{R}\right)\right) R;$$

**(b)** strictly convex if the following implication holds for all $x, y, p \in X$ and $R > 0$:

$$\left.\begin{array}{c} \|x - p\| \leq R \\ \|y - p\| \leq R \\ x \neq y \end{array}\right\} \Rightarrow \left\| \frac{x + y}{2} - p \right\| < R.$$

We begin with the following lemma.

**Lemma 2.3.** *Let $A$ and $B$ be nonempty subsets of a metric space $X$ and $T : A \cup B \to A \cup B$ a cyclic $(\psi, \varphi)$-weakly contraction mapping. For $x_0 \in A \cup B$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then for all $x \in A$ and $y \in B$,*

**(i)** *$\varphi(d(A, B)) \leq \varphi(d(x, y))$;*

**(ii)** *$d(Tx,Ty) \leq d(x,y)$; and*

**(iii)** *$d(x_{n+2},x_{n+1}) = d(Tx_{n+1},Tx_n) \leq d(x_{n+1},x_n)$ for each $n \geq 0$.*

**Proof.** **(i)** Since $d(A,B) = d(x,y)$ for all $x \in A$ and $y \in B$ and $\varphi \in \Phi$, we have $\varphi(d(A,B)) \leq \varphi(d(x,y))$.

**(ii)** Since $T$ is a cyclic $(\psi,\varphi)$-weakly contraction, we have

$$\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \varphi(d(x,y)) + \varphi(d(A,B))$$

for all $x \in A$ and $y \in B$.

From (i) $\varphi(d(A,B)) \leq \varphi(d(x,y))$, hence

$$\psi(d(Tx,Ty)) \leq \psi(d(x,y)).$$

Since $\varphi \in \Phi$, it follows that $d(Tx,Ty) \leq d(x,y)$.

**(iii)** Since $T$ is a cyclic $(\psi,\varphi)$-weakly contraction, we have

$$\begin{aligned}\psi(d(x_{n+2},x_{n+1})) &= \psi(d(Tx_{n+1},Tx_n))\\ &\leq \psi(d(x_{n+1},x_n)) - \varphi(d(x_{n+1},x_n)) + \varphi(d(A,B))\end{aligned}$$

for all $n \geq 0$. Using (i) and (ii), we get

$$\psi(d(x_{n+2},x_{n+1})) = \psi(d(Tx_{n+1},Tx_n)) \leq \psi(d(x_{n+1},x_n)).$$

Now since $\psi \in \Phi$, it follows that

$$d(x_{n+2},x_{n+1}) = d(Tx_{n+1},Tx_n) \leq d(x_{n+1},x_n).$$

**Theorem 2.4.** *Let $A$ and $B$ be nonempty subsets of a metric space $X$ and $T : A \cup B \to A \cup B$ a cyclic $(\psi,\varphi)$-weakly contraction mapping. For $x_0 \in A \cup B$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then $\lim_{n\to\infty} d(x_n,Tx_n) = d(A,B)$.*

**Proof.** It follows from Lemma 2.3 (iii) that $\{d(x_n,x_{n+1})\}$ is a decreasing sequence. Thus $\lim_{n\to\infty} d(x_n,x_{n+1}) = r_0$ for some $r_0 \geq d(A,B)$. If $d(x_{n_0},x_{n_0+1}) = 0$ for some $n_0 \geq 1$ then we are done. Assume that $d(x_n, x_{n+1}) > 0$ for each $n \geq 1$. Since $T$ is a cyclic $(\psi, \varphi)$-weakly contraction, we have

$$\psi(d(x_{n+1}, x_{n+2})) \leq \psi(d(x_n, x_{n+1})) - \varphi(d(x_n, x_{n+1})) + \varphi(d(A, B)) \tag{2.2}$$

for each $n \geq 1$.

Now by Lemma 2.3 (i) and (2.2), we have

$$\varphi(d(A, B)) \leq \varphi(d(x_n, x_{n+1})) \leq \psi(d(x_n, x_{n+1})) - \psi(d(x_{n+1}, x_{n+2})) + (A, B). \tag{2.3}$$

Since $\psi, \varphi \in \Phi$ and $d(x_n, x_{n+1}) \geq r_0 \geq d(A, B)$, it follows from (2.3) that

$$\lim_{n\to\infty} \varphi(d(x_n, x_{n+1})) = \varphi(r_0) = \varphi(d(A, B))$$

for each $n \geq 1$. Since $\varphi \in \Phi$, $r_0 = d(A, B)$.

In view of Remark 2.2 (1) and (2), Proposition 3.1 of [4] and Theorem 3 of [1] are special cases of Theorem 2.4.

**Theorem 2.5.** *Let $A$ and $B$ be nonempty subsets of a metric space $X$ and $T : A \cup B \to A \cup B$ a cyclic $(\psi, \varphi)$-weakly contraction mapping. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. If $\{x_{2n}\}$ has a convergent subsequence in $A$, then there exists a point $z \in A$ such that $d(z, Tz) = d(A, B)$.*

**Proof.** Let $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ such that $\lim_{k\to\infty} x_{2n_k} = z$. Since

$$d(A, B) \leq d(z, x_{2n_k-1}) \leq d(z, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1})$$

for each $k \geq 1$, it follows from Theorem 2.4 that $\lim_{k\to\infty} d(x_{2n_k}, x_{2n_k-1}) = d(A, B)$. Since

$$d(A, B) \leq d(x_{2n_k}, Tz) = d(x_{2n_k-1}, z)$$

for each $k \geq 1$, it follows that $d(z, Tz) = d(A, B)$.

In view of Remark 2.2 (2), Proposition 3.2 of [4] is a special case of Theorem 2.5.

**Corollary 2.6.** [1, Theorem 4]. *Let $A$ and $B$ be nonempty subsets of a metric space $X$ and $T : A \cup B \to A \cup B$ a cyclic $\varphi$-weakly contraction mapping. For $x_0 \in A$, define*

*$x_{n+1} := Tx_n$ for each $n \geq 0$. If $\{x_{2n}\}$ has a convergent subsequence in $A$, then there exists a point $z \in A$ such that $d(z, Tz) = d(A, B)$.*

**Proof.** It comes from Theorem 2.5, when $\varphi(t) = t$.

**Lemma 2.7.** *Let $A$ and $B$ be nonempty subsets of a uniformly convex Banach space $X$ such that $A$ is convex. Let $T : A \cup B \to A \cup B$ be a cyclic $(\psi, \varphi)$-weakly contraction mapping. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then*

$$\lim_{n\to\infty} \|x_{2n+2} - x_{2n}\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|x_{2n+3} - x_{2n+1}\| = 0.$$

**Proof.** Suppose that $\lim_{n\to\infty} \|x_{2n+2} - x_{2n}\| > 0$. Then there exists $\varepsilon_0 > 0$ such that for each $k \geq 1$, there is an $n_k \geq k$ satisfying

$$\|x_{2n_k+2} - x_{2n_k}\| \geq \varepsilon_0. \tag{2.4}$$

Choose $0 < \gamma < 1$ such that $\frac{\varepsilon_0}{\gamma} > d(A, B)$ and choose $\varepsilon$ such that

$$0 < \varepsilon < \min \left\{ \frac{\varepsilon_0}{\gamma} - d(A, B), \frac{d(A, B)\delta(\gamma)}{1 - \delta(\gamma)} \right\}.$$

By Theorem 2.4, there exist $N_1$ and $N_2$ such that

$$\|x_{2n_k+2} - x_{2n_k+1}\| \leq d(A, B) + \varepsilon \quad \text{and} \quad \|x_{2n_k+1} - x_{2n_k}\| \leq d(A, B) + \varepsilon \tag{2.5}$$

for all $n_k \geq N_1, N_2$. Let $N := \max\{N_1, N_2\}$. It follows from (2.4), (2.5) and the uniform convexity of $X$ that

$$\left\| \frac{x_{2n_k+2} + x_{2n_k}}{2} - x_{2n_k+1} \right\| \leq \left( 1 - \delta \left( \frac{\varepsilon_0}{d(A, B) + \varepsilon} \right) \right) (d(A, B) + \varepsilon)$$

for all $n_k \geq N$. As $\frac{x_{2n_k+2} + x_{2n_k}}{2} \in A$, the choice of $\varepsilon$ and the fact that $\delta$ is strictly increasing imply that

$$\left\| \frac{x_{2n_k+2} + x_{2n_k}}{2} - x_{2n_k+1} \right\| < d(A, B),$$

for all $n_k \geq N$, a contradiction. Therefore $\lim_{n\to\infty} \|x_{2n+2} - x_{2n}\| = 0$. Similarly we can show that $\lim_{n\to\infty} \|x_{2n+3} - x_{2n+1}\| = 0$.

**Theorem 2.8.** *Let $A$ and $B$ be nonempty subsets of a uniformly convex Banach space $X$ such that $A$ is convex. Let $T : A \cup B \to A \cup B$ be a cyclic $(\psi, \varphi)$-weakly contraction* *mapping. For $x_0 \in A$ define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then for each $\varepsilon > 0$, there exists a positive integer $N_0$ such that for all $m > n \geq N_0$*

$$\|x_{2m} - x_{2n+1}\| < d(A,B) + \varepsilon.$$

**Proof.** Suppose the contrary. Then there exists $\varepsilon_0 > 0$ such that for each $k \geq 1$, there exist $m_k > n_k \geq k$ satisfying

$$\|x_{2m_k} - x_{2n_k+1}\| \geq d(A,B) + \varepsilon_0 \ \text{ and } \ \|x_{2(m_k-1)} - x_{2n_k+1}\| < d(A,B) + \varepsilon_0. \tag{2.6}$$

By the triangle inequality and (2.6), we have

$$\begin{aligned} d(A,B) + \varepsilon_0 &\leq \|x_{2m_k} - x_{2n_k+1}\| \\ &\leq \|x_{2m_k} - x_{2(m_k-1)}\| + \|x_{2(m_k-1)} - x_{2n_k+1}\| \\ &< \|x_{2m_k} - x_{2(m_k-1)}\| + d(A,B) + \varepsilon_0. \end{aligned}$$

Making $k \to \infty$ and using Lemma 2.7, we get

$$\lim_{k\to\infty} \|x_{2m_k} - x_{2n_k+1}\| = d(A,B) + \varepsilon_0. \tag{2.7}$$

Since $T$ is a cyclic $(\psi, \varphi)$-weakly contraction, by Lemma 2.3 (i) and (ii), and the triangle inequality, we obtain

$$\begin{aligned} \psi(\|x_{2m_k} - x_{2n_k+1}\|) &\leq \psi(\|x_{2m_k} - x_{2m_k+2}\|) + \psi(\|x_{2m_k+2} - x_{2m_k+3}\|) + \psi(\|x_{2m_k+3} - x_{2n_k+1}\|) \\ &\leq \psi(\|x_{2m_k} - x_{2m_k+2}\|) + \psi(\|x_{2m_k+1} - x_{2m_k+2}\|) + \psi(\|x_{2m_k+3} - x_{2n_k+1}\|) \\ &\leq \psi(\|x_{2m_k} - x_{2m_k+2}\|) + \psi(\|x_{2m_k} - x_{2m_k+1}\|) \\ &- \varphi(\|x_{2m_k} - x_{2m_k+1}\|) + \varphi(d(A,B)) + \psi(\|x_{2m_k+3} - x_{2n_k+1}\|) \\ &\leq \psi(\|x_{2m_k} - x_{2m_k+2}\|) + \psi(\|x_{2m_k} - x_{2m_k+1}\|) + \psi(\|x_{2m_k+3} - x_{2n_k+1}\|). \end{aligned} \tag{2.8}$$

Since $\psi \in \Phi$, (2.8) implies that

$$\begin{aligned} \|x_{2m_k} - x_{2n_k+1}\| &\leq \|x_{2m_k} - x_{2m_k+2}\| + \|x_{2m_k} - x_{2m_k+1}\| - \varphi(\|x_{2m_k} - x_{2m_k+1}\|) \\ &+ \varphi(d(A,B)) + \|x_{2m_k+3} - x_{2n_k+1}\| \\ &\leq \|x_{2m_k} - x_{2m_k+2}\| + \|x_{2m_k} - x_{2m_k+1}\| + \|x_{2m_k+3} - x_{2n_k+1}\|. \end{aligned}$$

Making $k \to \infty$ and using (2.7) and Lemma 2.7, we get

$$\begin{aligned} d(A,B)+\varepsilon_0 &\leq d(A,B)+\varepsilon_0 - \lim_{k\to\infty}\varphi(\|x_{2m_k}-x_{2m_k+1}\|)+\varphi(d(A,B)) \\ &\leq d(A,B)+\varepsilon_0. \end{aligned}$$

Hence

$$\lim_{k\to\infty}\varphi(\|x_{2m_k}-x_{2m_k+1}\|)=\varphi(d(A,B)). \tag{2.9}$$

Since $\varphi \in \Phi$, by (2.6) and (2.9)

$$\begin{aligned} \varphi(d(A,B)+\varepsilon_0) &\leq \lim_{k\to\infty}\varphi(\|x_{2m_k}-x_{2m_k+1}\|) \\ &= \varphi(d(A,B)) < \varphi(d(A,B)+\varepsilon_0), \end{aligned}$$

a contradiction and hence the Theorem.

**Theorem 2.9.** *Let $A$ and $B$ be nonempty subsets of a uniformly convex Banach space $X$ such that $A$ is closed. Let $T : A\cup B \to A\cup B$ be cyclic $(\psi,\varphi)$-weakly contraction mapping. For $x_0 \in A$ define $x_{n+1} := Tx_n$ for each $n \geq 0$. If $d(A,B)=0$, then $T$ has a unique fixed point $z \in A\cap B$.*

**Proof.** Let $\varepsilon > 0$ be given. By Theorem 2.4, there exists $N_1$ such that

$$\|x_{2n}-x_{2n+1}\| < \varepsilon$$

for all $n \geq N_1$. By Theorem 2.8, there exists $N_2$ such that

$$\|x_{2m}-x_{2n+1}\| < \varepsilon$$

for all $m > n \geq N_2$. Let $N := \max\{N_1, N_2\}$. Then

$$\|x_{2m}-x_{2n}\| \leq \|x_{2m}-x_{2n+1}\|+\|x_{2n+1}-x_{2n}\| < 2\varepsilon$$

for all $m > n \geq N$. Thus $\{x_{2n}\}$ is a Cauchy sequence in $A$. Since $X$ is complete and $A$ is closed, it follows that $x_{2n} \to z \in A$ as $n \to \infty$. Now by Theorem 2.5, we have $d(z,Tz)=d(A,B)=0$, and $z$ is a fixed point of $T$. The uniqueness of fixed point follows easily.

**Corollary 2.10.**[1, Theorem 6]. *Let $A$ and $B$ be nonempty subsets of a uniformly convex Banach space $X$ such that $A$ is closed. Let $T : A\cup B \to A\cup B$ be cyclic $\varphi$-weakly* *contraction mapping. For $x_0 \in A$ define $x_{n+1} := Tx_n$ for each $n \geq 0$. If $d(A,B) = 0$, then $T$ has a unique fixed point $z \in A \cap B$.*

**Proof.** It comes from Theorem 2.9, when $\psi(t) = t$.

**Theorem 2.11.** *Let $A$ and $B$ be nonempty subsets of a uniformly convex Banach space $X$ such that $A$ is closed and convex. Let $T : A \cup B \to A \cup B$ be cyclic $(\psi, \varphi)$-weakly contraction mapping. For $x_0 \in A$ define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then $\{x_{2n}\} \in A$ and $\{x_{2n+1}\} \in B$ are Cauchy sequences.*

**Proof.** If $d(A,B) = 0$, the result follows from Theorem 2.9. So assume that $d(A,B) > 0$. Suppose that the sequence $\{x_{2n}\}$ is not Cauchy. Then there exists $\varepsilon_0 > 0$ such that for each $k \geq 1$, there exist $m_k > n_k \geq k$ satisfying

$$\|x_{2m_k} - x_{2n_k}\| \geq \varepsilon_0. \tag{2.10}$$

Choose $0 < \gamma < 1$ such that $\dfrac{\varepsilon_0}{\gamma} > d(A,B)$ and choose $\varepsilon$ such that

$$0 < \varepsilon < \min\left\{\frac{\varepsilon_0}{\gamma} - d(A,B), \frac{d(A,B)\delta(\gamma)}{1-\delta(\gamma)}\right\}.$$

By Theorem 2.4, there exists $N_1$ such that

$$\|x_{2n_k} - x_{2n_k+1}\| < d(A,B) + \varepsilon. \tag{2.11}$$

for all $n_k \geq N_1$. By Theorem 2.8, there exists $N_2$ such that

$$\|x_{2m_k} - x_{2n_k+1}\| < d(A,B) + \varepsilon. \tag{2.12}$$

for all $n_k \geq N_2$. Let $N := \max\{N_1, N_2\}$. It follows from (2.11), (2.12) and the uniform convexity of $X$ that

$$\left\|\frac{x_{2n_k+2} + x_{2n_k}}{2} - x_{2n_k+1}\right\| \leq \left(1 - \delta\left(\frac{\varepsilon_0}{d(A,B)+\varepsilon}\right)\right)(d(A,B)+\varepsilon)$$

for all $n_k \geq N$. The choice of $\varepsilon$ and the fact that $\delta$ is strictly increasing imply that

$$\left\|\frac{x_{2n_k+2} + x_{2n_k}}{2} - x_{2n_k+1}\right\| < d(A,B),$$

for all $n_k \geq N$, a contradiction. Thus $\{x_{2n}\}$ is a Cauchy sequence in $A$. Similarly, we can show that $\{x_{2n+1}\}$ is a Cauchy sequence in $B$.

**Theorem 2.12.** *Let $A$ and $B$ be nonempty subsets of a uniformly convex Banach space $X$ such that $A$ is closed and convex. Let $T : A \cup B \to A \cup B$ be cyclic $(\psi, \varphi)$-weakly contraction mapping. For $x_0 \in A$ define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then there exists a unique $z \in A$ such that $x_{2n} \to z$, $T^2 z = z$ and $\|z - Tz\| = d(A, B)$.*

**Proof.** By Theorem 2.11, $\{x_{2n}\}$ is a Cauchy sequence in $A$ and hence $x_{2n} \to z \in A$ as $n \to \infty$. By Theorem 2.5, $\|z - Tz\| = d(A, B)$. To show that $z$ is unique we assume that there exists a $y \in A$ such that $\|y - Ty\| = d(A, B)$ with $T^2 y = y$. By Lemma 2.3 (i) and (ii), we have

$$\|Ty - z\| = \|Ty - T^2 z\| \leq \|y - Tz\| \text{ and } \|Tz - y\| = \|Tz - T^2 y\| \leq \|z - Ty\|.$$

Thus $\|Tz - y\| = \|z - Ty\|$. In fact $\|z - Ty\| = d(A, B)$; otherwise $\|z - Ty\| > d(A, B)$ and since $T$ is cyclic $(\psi, \varphi)$-weakly contraction, it follows that

$$\begin{aligned} \psi(\|Tz - y\|) &= \psi(\|Tz - T^2 y\|) \\ &\leq \psi(\|z - Ty\|) - \varphi(\|z - Ty\|) + \varphi(d(A, B)) \\ &< \psi(\|z - Ty\|) - \varphi(A, B) + \varphi(A, B) \\ &= \psi(\|z - Ty\|) = \psi(Tz - y\|), \end{aligned}$$

a contradiction. Thus $\|z - Ty\| = d(A, B) = \|y - Tz\|$. Now by convexity of $A$ and $X$

$$0 < \left\| \frac{y + z}{2} - Ty \right\| = \left\| \frac{y - Ty}{2} + \frac{z - Ty}{2} \right\| < d(A, B),$$

a contradiction. Thus $y = z$.

In view of Remark 2.2 (1), Theorem 8 of [1] is a special case of Theorem 2.12.

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