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FIXED POINT THEOREMS FOR CONVEX CONTRACTIONS ON CONE 2-METRIC SPACE OVER BANACH ALGEBRA

P. RAMA BHADRA MURTHY*, M. RANGAMMA

Department of Mathematics, Osmania University, Telangana, India-500007

Abstract. In 2007, B.Singh, S Jain and P Bhagat introduced cone 2-metric space and proved some fixed point theorems of certain contractive mappings while Tao Wang, Jiangdong Yin, and Qi Yan introduced cone 2-metric spaces over Banach algebra and established some existence and uniqueness theorems of fixed points for some contractive mappings. In this paper, we extended some results of Istraţescu's convex contractions to cone 2-metric space over Banach algebra and presented two fixed point theorems. Examples are given showing the significance of our results.

Keywords: fixed point theorems; convex contractions; cone 2-metric space over Banach algebra.

2010 AMS Subject Classification: 47H10, 54H25.

1. Introduction

In 1962, Gähler [6] introduced 2-metric space with area of triangle as an underlying example. He showed that the 2-metric is not a continuous function of its variables as in the case of usual metric. Also a convergent sequence in 2-metric space need not be a Cauchy sequence [13].

*Corresponding author
E-mail address: badri2502@gmail.com
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Among many other generalizations of an ordinary metric, 2-metric has not been known to be topologically equivalent to an ordinary metric and also there was no easy relationship between the results obtained in 2-metric spaces and ordinary metric spaces.

In 2007, Huang and Zhang [8] introduced cone metric space generalizing ordinary metric space by replacing the codomain with the ordered Banach space. They defined the convergence and Cauchyness in such space and presented some fixed point theorems for normal cones. Rezapour and Hamlbarani has comeup with the same results for non-normal cones showing the existence of non-normal cones. In 2012, B.Singh et.al., [14] introduced cone 2-metric space generalizing cone metric space and 2-metric space. Also see ([3], [10], [15]). In 2013, Liu and Xu [11] introduced cone metric spaces over Banach algebras by replacing the Banach space E with the Banach algebra $\mathscr{A}$. They proved fixed point theorems of generalized Lipschitz mappings with weaker conditions on generalized Lipschitz constant by means of spectral radius. In 2014, Xu and Radenovic [17] proved the results in cone metric spaces over Banach algebras without assumption of normality. For results in cone metric spaces over Banach algebras, see ([4],[5],[7],[12],[16]).

Recently, Mohammad A.Alghamdi et.al.,[1] introduced the notion of convex contractions in cone metric spaces and proved some fixed point theorems in the new class of cone convex contractions for non-normal cones. Also see [2]. It is to be noted that convex contractions in metric spaces was given by Istraţescu [9]. In this paper, we have presented cone convex contractions on cone 2-metric space over Banach algebras and proved some fixed theorems. Examples are given showing the significance of our results in cone 2-metric space over Banach algebra.

Throughout this paper, let $\mathbb{N}, \mathbb{N}_0, \mathbb{R}^+$ denote the set of natural numbers, whole numbers and non-negative real numbers respectively.

2. **Preliminaries**

Let $\mathscr{A}$ always be a real Banach Algebra *i.e.*, an operation of multiplication is defined on $\mathscr{A}$ as follows: (For all $x, y, z \in \mathscr{A}, \alpha \in \mathbb{R}$)

(i) $(xy)z = x(yz)$

(ii) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$

(iii) $\alpha(xy) = (\alpha x)y = x(\alpha y)$

(iv) $\|xy\| \leqslant \|x\|\|y\|$

We shall assume that a Banach Algebra always has a unit (multiplicative identity) e satisfying $ex = xe = x$ for all $x \in \mathscr{A}$. An element $x \in \mathscr{A}$ is said to be an invertible if there is an element $y \in \mathscr{A}$ (called inverse of x) such that $xy = yx = e$. The inverse of x is denoted by x^{-1}.

Proposition 2.1. *([11]) Let $\mathscr{A}$ be a Banach Algebra with a unit e and $x \in \mathscr{A}$. If the spectral radius $r(x)$ of x is less than 1, i.e.,*

$$r(x) = \lim_{x \to a} \|x^n\|^{\frac{1}{n}} < 1$$

then $e - x$ is invertible and $(e-x)^{-1} = \sum_{i=0}^{\infty} x^i$.

Remark 2.1. *([11])*

(i) For all $x \in A$, we have $r(x) \leqslant \|x\|$.

(ii) If $r(k) < 1$ then $\|k^n\| \longrightarrow 0 (n \longrightarrow \infty)$.

Definition 2.1. *([11]) Let $\mathscr{A}$ be a real Banach Algebra and θ be the zero element of $\mathscr{A}$. If $\mathscr{P}$ is a subset of $\mathscr{A}$, then $\mathscr{P}$ is called a cone if*

(1) $\mathscr{P}$ is closed, non-empty and $\mathscr{P} \neq \{\theta\}$

(2) $a, b \in \mathbb{R}^+, x, y \in \mathscr{P} \Rightarrow ax + by \in \mathscr{P}$

(3) $x, y \in \mathscr{P} \Rightarrow xy \in \mathscr{P}$ and

(4) $x \in \mathscr{P}$ and $-x \in \mathscr{P} \Rightarrow x = \theta$.

Definition 2.2. *Let X be a non-empty set and $d : X^3 \longrightarrow \mathscr{A}$ be a map satisfying the following conditions:*

(1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x,y,z) \neq \theta$.

(2) If atleast two of the three points x, y, z are equal then $d(x,y,z) = \theta$.

(3) $d(x,y,z) = d(p(x,y,z))$ for all $x, y, z \in X$ and for all permutations $p(x,y,z)$.

(4) $d(x,y,z) \preceq d(x,y,w) + d(x,w,z) + d(w,y,z)$ for all $x, y, z, w \in X$.

Then d is called cone 2-metric and (X, d) is called cone 2-metric space over Banach algebra.

Definition 2.3. *Let $\{x_n\}$ be a sequence in cone 2-metric space over Banach algebra (X,d). Then*

(1) *$\{x_n\}$ is said to be convergent to $x \in X$ if $\lim_{n\to\infty} d(x_n,x,a) = \theta$ for all $a \in X$.*
(2) *$\{x_n\}$ is said to be a Cauchy sequence in X if $\lim_{m,n\to\infty} d(x_n,x_m,a) = \theta$ for all $a \in X$.*
(3) *(X,d) is said to be complete if every cauchy sequence is convergent in X.*

Remark 2.2 ([1])**.**

(1) *$c \in int\mathscr{P} \Leftrightarrow [-c,c]$ is a neighborhood of θ in norm topology in Banach space E.*
(2) *$[-c,c] = (c-\mathscr{P}) \cap (\mathscr{P}-c)$; $\quad int\mathscr{P} = (c - int\mathscr{P}) \cap (int\mathscr{P} - c)$.*

Definition 2.4. *([17]) Let $\mathscr{P}$ be a solid cone in a Banach algebra $\mathscr{A}$. A sequence $\{u_n\} \subseteq \mathscr{P}$ is a $c-$sequence if for each $c \in \mathscr{P}$, there exists $m \in \mathbb{N}$ such that $u_n \ll c$ for all $n \geqslant m$.*

Proposition 2.2. *([17])*

(i) *If $\{x_n\}$ and $\{y_n\}$ are $c-$sequences and $a,b > 0$, then $\{ax_n + by_n\}$ is also a $c-$sequence.*
(ii) *If $\{u_n\}$ is a $c-$sequence in $\mathscr{P}$ and $k \in \mathscr{P}$ is an arbitrarily given vector, then $\{ku_n\}$ is also a $c-$sequence.*
(iii) *If $\{x_n\}$ is a sequence in X converging to $x \in X$, then $\{d(x_n,x)\}$ and $\{d(x_n,x_{n+p})\}$ for any $p \in \mathbb{N}$ are also $c-$sequences.*
(iv) *If $\theta \preceq x \preceq y$ and $\theta \preceq u$ then $\theta \preceq ux \preceq uy$.*
(v) *If $\{x_n\},\{y_n\}$ are sequences in $\mathscr{A}$ with $x_n \longrightarrow x$ and $y_n \longrightarrow y(n \longrightarrow \infty)$ where $x,y \in \mathscr{A}$ then we have $x_ny_n \longrightarrow xy(n \longrightarrow \infty)$.*

Lemma 2.1. *([17]) Let $\mathscr{A}$ be a Banach algebra and $\mathscr{P}$ be the underlying solid cone.*

(i) *If $\lambda \in \mathscr{P}$ with $r(\lambda) < 1$, then $(e-\lambda)^{-1} \in \mathscr{P}$.*
(ii) *If $x,y \in \mathscr{A}$ and x,y commute, then the following holds:*
   (a) *$r(xy) \leqslant r(x)r(y)$*
   (b) *$r(x+y) \leqslant r(x) + r(y)$ and*
   (c) *$\mid r(x) - r(y) \mid \leqslant r(x-y)$.*
(iii) *If $k \in \mathscr{A}$ such that $0 \leqslant r(k) < 1$, then we have $r((e-k)^{-1}) \leqslant (1-r(k))^{-1}$.*

Let (X,d) be a cone 2-metric space over Banach algebra. The following properties are often used in proving the results, particularly when dealing with non-normal cones.

(p_1) If $x \preceq y$ and $y \ll z$, then $x \ll z$.

(p_2) If $\theta \preceq x \ll c$ for each $c \in int\mathscr{P}$, then $x = \theta$.

(p_3) If $x \preceq y + c$ for each $c \in int\mathscr{P}$, then $x \preceq y$.

(p_4) If $\theta \preceq x \preceq y$ and $u \geqslant 0$ then $\theta \preceq ux \preceq uy$.

(p_5) If $\theta \preceq x_n \preceq y_n$ for each $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} x_n = x$, $\lim\limits_{n\to\infty} y_n = y$ then $\theta \preceq x \preceq y$.

(p_6) If $\theta \preceq d(x_n, x) \preceq b_n$ and $b_n \longrightarrow \theta$, then $x_n \longrightarrow x$.

(p_7) If $u \preceq \lambda u$, where $u \in \mathscr{P}$ and $0 \leqslant \lambda < 1$, then $u = \theta$.

(p_8) If $c \in int\mathscr{P}, \theta \preceq u_n$ and $\|u_n\| \longrightarrow 0$, then there exists an $m \in \mathbb{N}$ such that $u_n \ll c$ for all $n \geqslant m$.

Lemma 2.2 ([10]). *Let $\mathscr{P}$ be a cone and $\{x_n\}$ be a sequence in $\mathscr{A}$. If $c \in int\mathscr{P}$ and $\theta \preceq x_n \longrightarrow \theta$ as $n \longrightarrow \infty$, then there exists N such that for all $n > N$, we have $x_n \ll c$.*

3. Main Results

3.1. Cone convex contraction mappings.

Definition 3.1. *A continuous mapping $T : X \to X$ defined on a cone 2-metric space over Banach Algebra (X,d) is said to be cone convex contraction mapping of order 2 if there exists $\lambda_1, \lambda_2 \in \mathscr{P}$ with $\lambda_1\lambda_2 = \lambda_2\lambda_1$ and $r(\lambda_1) + r(\lambda_2) < 1$ such that to each $a \in X$, we have*

$$d(T^2x, T^2y, a) \preceq \lambda_1 d(Tx, Ty, a) + \lambda_2 d(x, y, a) \tag{3.1}$$

for all $x, y \in X$.

Theorem 3.1. *Let (X,d) be a complete cone 2-metric space over Banach Algebra and $\mathscr{P}$ be a solid cone. Let $T : X \to X$ be a cone convex contraction mapping of order 2. Then T has a unique fixed point in X. Furthermore, for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.*

Proof. Let $x_0 \in X$ be arbitrary. Define the sequence $\{x_n\}$ as $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}_0$. Let $\lambda = \lambda_1 + \lambda_2$.

Set to each $a \in X$,

$$v_a = d(x_2, x_1, a) + d(x_1, x_0, a)$$

Using (3.1), it is easy to see that, to each $a \in X$

$d(x_3, x_2, a) \preceq \lambda v_a, \qquad d(x_4, x_3, a) \preceq \lambda v_a,$

$d(x_5, x_4, a) \preceq \lambda^2 v_a, \qquad d(x_6, x_5, a) \preceq \lambda^2 v_a$

$d(x_7, x_6, a) \preceq \lambda^3 v_a, \qquad d(x_8, x_7, a) \preceq \lambda^3 v_a,$

... ...

An Induction argument shows that, to each $a \in X$

$$d(x_{m+1}, x_m, a) \preceq \lambda^l v_a \tag{3.2}$$

when $m = 2l$ or $m = 2l + 1$.

First we prove that, for all $i, j, k \in \mathbb{N}$

$$d(x_i, x_j, x_k) = \theta \tag{3.3}$$

Since $r(\lambda_1) < 1$ and $r(\lambda_2) < 1$, applying (3.1) repeatedly with $a = x_k$, and $i = j + 1$, we have

$$\begin{aligned}
d(x_{j+1}, x_j, x_k) &\preceq \lambda_1 d(x_j, x_{j-1}, x_k) + \lambda_2 d(x_{j-1}, x_{j-2}, x_k) \\
&\preceq d(x_j, x_{j-1}, x_k) + d(x_{j-1}, x_{j-2}, x_k) \\
&\preceq d(x_{j-1}, x_{j-2}, x_k) + d(x_{j-2}, x_{j-3}, x_k) \\
&\qquad + d(x_{j-2}, x_{j-3}, x_k) + d(x_{j-3}, x_{j-4}, x_k) \\
&= d(x_{j-1}, x_{j-2}, x_k) + 2d(x_{j-2}, x_{j-3}, x_k) + d(x_{j-3}, x_{j-4}, x_k) \\
&\preceq d(x_{j-2}, x_{j-3}, x_k) + 3d(x_{j-3}, x_{j-4}, x_k) \\
&\qquad + 3d(x_{j-4}, x_{j-5}, x_k) + d(x_{j-5}, x_{j-6}, x_k)
\end{aligned}$$

It can be easily seen that, for any k with $1 < k < j + 2$, we have

$$d(x_{j+1}, x_j, x_k) = \theta \tag{3.4}$$

Suppose $k \geqslant j+2$. Using the pyramidal inequality and (3.4), we have

$$\begin{aligned}
d(x_{j+1},x_j,x_k) &\preceq d(x_{j+1},x_j,x_{k-1}) + d(x_{j+1},x_{k-1},x_k) + d(x_{k-1},x_j,x_k) \\
&= d(x_{j+1},x_j,x_{k-1}) \\
&\preceq d(x_{j+1},x_j,x_{k-2}) + d(x_{j+1},x_{k-2},x_k) + d(x_{k-2},x_j,x_k) \\
&= d(x_{j+1},x_j,x_{k-2}) \\
&\quad \ldots \quad \ldots \quad \ldots \\
&\preceq d(x_{j+1},x_j,x_{j+1}) \\
&= \theta
\end{aligned}$$

Thus, for all $j,k \in \mathbb{N}$, we have

$$d(x_{j+1},x_j,x_k) = \theta \tag{3.5}$$

Finally, let $i < j$. Using the pyramidal inequality and (3.5), we have

$$\begin{aligned}
d(x_i,x_j,x_k) &\preceq d(x_i,x_j,x_{j-1}) + d(x_i,x_{j-1},x_k) + d(x_{j-1},x_j,x_k) \\
&= d(x_i,x_{j-1},x_k) \\
&\preceq d(x_i,x_{j-2},x_k) \\
&\quad \ldots \quad \ldots \quad \ldots \\
&\preceq d(x_i,x_i,x_k) \\
&= \theta.
\end{aligned}$$

proving (3.3).

Now we show that $\{x_n\}$ is a Cauchy sequence in X. Without loss of generality, let $m < n$. To show this, first we see that

$$d(x_m,x_n,a) \preceq 2\lambda^l (e-\lambda)^{-1} v_a \tag{3.6}$$

when $m = 2l$ or $m = 2l+1$ where $l \geqslant 1$.

Case(i) When m is even and n is even:

Let $m = 2l$ and $n = 2p$ with $p \geqslant 2$ and $l \geqslant 1$. Using (3.2), (3.3) and pyramidal inequality repeatedly, we have

$$\begin{aligned} d(x_m, x_n, a) &\preceq d(x_m, x_n, x_{m+1}) + d(x_m, x_{m+1}, a) + d(x_{m+1}, x_n, a) \\ &\preceq \lambda^l v_a + d(x_{m+1}, x_n, a) \\ &\preceq \lambda^l v_a + d(x_{m+1}, x_n, x_{m+2}) + d(x_{m+1}, x_{m+2}, a) + d(x_{m+2}, x_n, a) \\ &\preceq \lambda^l v_a + \lambda^l v_a + d(x_{m+2}, x_n, a) \\ &\preceq \lambda^l v_a + \lambda^l v_a + d(x_{m+2}, x_n, x_{m+3}) \\ &\qquad\qquad + d(x_{m+2}, x_{m+3}, a) + d(x_{m+3}, x_n, a) \\ &\preceq \lambda^l v_a + \lambda^l v_a + \lambda^{l+1} v_a + d(x_{m+3}, x_n, a) \\ &\preceq 2\lambda^l v_a + 2\lambda^{l+1} v_a + \ldots + \lambda^{p-1} v_a + d(x_{2p-1}, x_{2p}, a) \\ &\preceq 2(\lambda^l v_a + \lambda^{l+1} v_a + \ldots + \lambda^{p-1} v_a) \\ &\preceq 2\lambda^l (e - \lambda)^{-1} v_a. \end{aligned}$$

Case(ii) When m is even and n is odd:

Let $m = 2l$ and $n = 2p+1$ with $p \geqslant 2$ and $l \geqslant 1$.

This case is similar to case (i).

Case(iii) When m is odd and n is even:

Let $m = 2l+1$ and $n = 2p$ with $p \geqslant 2$ and $l \geqslant 1$.

Using (3.2), (3.3) and pyramidal inequality repeatedly, we have

$$\begin{aligned} d(x_m, x_n, a) &\preceq d(x_m, x_n, x_{m+1}) + d(x_m, x_{m+1}, a) + d(x_{m+1}, x_n, a) \\ &\preceq \lambda^l v_a + d(x_{m+1}, x_n, a) \\ &\preceq \lambda^l v_a + d(x_{m+1}, x_n, x_{m+2}) + d(x_{m+1}, x_{m+2}, a) + d(x_{m+2}, x_n, a) \end{aligned}$$

$$\preceq \lambda^l v_a + \lambda^{l+1} v_a + \ldots + d(x_{2p-1}, x_{2p}, a)$$

$$\preceq \lambda^l v_a + \lambda^{l+1} v_a + \lambda^{l+1} v_a + \ldots + \lambda^p v_a$$

$$\preceq 2\lambda^l (e-\lambda)^{-1} v_a.$$

Case(iv) When m is odd and n is odd:

Let $m = 2l+1$ and $n = 2p+1$ with $p \geqslant 2$ and $l \geqslant 1$.

This case is similar to case(iii), proving (3.6) for all $m,n \in \mathbb{N}$.

Since $\|2\lambda^l (e-\lambda)^{-1} v_a\| \to 0$ as $l \to \infty$, using Lemma 2.2 and Proposition 2.2, for any $c \in \mathscr{A}$ with $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that for all $m,n \geqslant n_0 \in \mathbb{N}$, we have

$$d(x_m, x_n, a) \ll c$$

showing that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that $x_n \longrightarrow z$ as $n \longrightarrow \infty$. Since T is continuous, $x_{n+1} = Tx_n \longrightarrow Tz$ as $n \longrightarrow \infty$. The uniqueness of the limit for any sequence in cone 2-metric space shows that $Tz = z$ or z is the fixed point of T. Suppose y be another fixed point of T. Then to each $a \in X$, we have

$$d(z,y,a) = d(T^2 z, T^2 y, a) \preceq \lambda_1 d(z,y,a) + \lambda_2 d(z,y,a) = \lambda d(z,y,a)$$

Since $r(\lambda) < 1$, we get $d(z,y,a) = \theta$ which implies $z = y$, proving the uniqueness of the fixed point. □

Example 3.1. *Let $\mathscr{A} = \mathbb{R}^2$. For each $(x_1,x_2) \in \mathscr{A}$, we define the norm as $\|(x_1,x_2)\| = |x_1| + |x_2|$. The multiplication is defined by $xy = (x_1,x_2)(y_1,y_2) = (x_1y_1, x_1y_2 + x_2y_1)$. Let $\mathscr{P} = \{(x,y) \in \mathbb{R}^2 | x,y \geqslant 0\}$ be the underlying normal cone. Take $X = \{(x,0) \in \mathbb{R}^2 | 0 \leqslant x \leqslant 1\} \cup \{(0,1)\}$. Define $d : X \times X \times X \to \mathscr{A}$ as follows: For all $A,B,C \in X$*

$$d(A,B,C) = \begin{cases} d(p(A,B,C)) & \text{where } p \text{ denotes permutation} \\ (\triangle, \gamma.\triangle) & \text{otherwise} \end{cases}$$

where $\gamma \in (0,1)$ and $\triangle =$ twice the area of triangle A,B,C. Then (X,d) is a complete cone 2-metric space over Banach algebra. Take the continuous map $T : X \to X$ as

$$T(x,0) = \left(\frac{x^2}{2}, 0\right) \qquad \text{and} \qquad T(0,1) = (0,0)$$

Now,

$$\begin{aligned}
d(T(\alpha,0),T(\beta,0),(0,1)) &= d\left(\left(\frac{\alpha^2}{2},0\right),\left(\frac{\beta^2}{2},0\right),(0,1)\right)\\
&= \left(\frac{|\alpha^2-\beta^2|}{2},\gamma\frac{|\alpha^2-\beta^2|}{2}\right)\\
&= \left(\frac{\alpha+\beta}{2},0\right)(|\alpha-\beta|,\gamma|\alpha-\beta|)\\
&= \left(\frac{\alpha+\beta}{2},0\right)d((\alpha,0),(\beta,0),(0,1))
\end{aligned}$$

i.e., $\quad d(T(\alpha,0),T(\beta,0),(0,1)) \preceq \lambda d((\alpha,0),(\beta,0),(0,1))$

which is not true for all $(\alpha,0),(\beta,0) \in X$ *and fixed* $\lambda \in \mathscr{P}$. *Otherwise, we would have* $\left(\frac{\alpha+\beta}{2},0\right) \preceq \lambda$, *which is impossible for fixed* $\lambda \in \mathscr{P}$ *satisfying* $r(\lambda) \in [0,1)$ *when* $\alpha,\beta \longrightarrow 1$. *Hence* T *is not a Banach cone contraction mapping on cone 2-metric space* X.

But we have,

$$\begin{aligned}
d(T^2(\alpha,0),T^2(\beta,0),(0,1)) &= d\left(\left(\frac{\alpha^4}{8},0\right),\left(\frac{\beta^4}{8},0\right),(0,1)\right)\\
&= \left(\frac{|\alpha^4-\beta^4|}{8},\gamma\frac{|\alpha^4-\beta^4|}{8}\right)\\
&= \left(\frac{\alpha^2+\beta^2}{4},0\right)\left(\frac{|\alpha^2-\beta^2|}{2},\gamma\frac{|\alpha^2-\beta^2|}{2}\right)\\
&\preceq \left(\frac{1}{2},0\right)d(T(\alpha,0),T(\beta,0),(0,1))
\end{aligned}$$

Thus T *is a cone convex contraction mapping of order 2 but not a Banach cone contraction in cone 2-metric space. Clearly* $(0,0)$ *is the unique fixed point of* X.

Example 3.2. *Let* $\mathscr{A} = C^2_{\mathbb{R}}[0,1]$. *Define the norm as* $\|x\| = \|x\|_\infty + \|x'\|_\infty$ *and the multiplication as pointwise. Take* $\mathscr{P} = \{\varphi \in \mathscr{A} : \varphi \geqslant 0\}$ *as the underlying non-normal cone. Take* $X = \{(x,0) \in \mathbb{R}^2 | 0 \leqslant x \leqslant 1\} \cup \{(0,1)\}$. *Define* $d : X \times X \times X \to \mathscr{A}$ *as follows:*

$$d(A,B,C)(t) = \begin{cases} d(p(A,B,C))(t) & \text{where } p \text{ denotes permutation}\\ \triangle.\varphi(t) & \text{otherwise}\end{cases}$$

for all $A,B,C \in X$, where $\varphi : [0,1] \to \mathbb{R}$ is such that $\varphi(t) = e^t$ and $\triangle=$ twice the area of triangle A,B,C. Then (X,d) is a complete cone 2-metric space over Banach algebra. Take the continuous map $T : X \to X$ as

$$T(x,0) = \left(\frac{x^2}{2},0\right) \qquad \textit{and} \qquad T(0,1) = (0,0)$$

For the same reason as in the above example, T is a cone convex contraction mapping of order 2 but not a Banach cone contraction in cone 2-metric space.

Definition 3.2. *A continuous mapping $T : X \to X$ on a cone 2-metric space over a Banach Algebra is said to be cone convex contraction mapping of order n if there exists $\lambda_0, \lambda_1, \lambda_2, ...\lambda_{n-1} \in \mathscr{P}$ such that $\lambda_i's$ commute, $\sum_{i=1}^{n-1} r(\lambda_i) < 1$ and to each $a \in X$, we have*

$$d(T^n x, T^n y, a) \preceq \lambda_0 d(x,y,a) + \lambda_1 d(Tx,Ty,a) + \cdots + \lambda_{n-1} d(T^{n-1}x, T^{n-1}y, a) \tag{3.7}$$

for all $x,y \in X$.

The proof of the next theorem is similar to Theorem 3.1. So we just present the statement.

Theorem 3.2. *Let (X,d) be a complete cone 2-metric space over Banach algebra and $\mathscr{P}$ be a solid cone. Let $T : X \to X$ be a cone convex contraction mapping of order n. Then T has a unique fixed point in X.*

3.2. **Two sided cone convex contraction mappings.**

Definition 3.3. *A continuous mapping $T : X \to X$ on cone 2-metric space over a Banach Algebra (X,d) is said to be two sided cone convex contraction of order 2 if there exists $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathscr{P}$ such that $\lambda_i's (i = 1,2,3,4)$ commute, $\sum_{i=1}^{4} r(\lambda_i) < 1$ and to each $a \in X$, we have*

$$\begin{aligned} d(T^2x, T^2y, a) \preceq{} & \lambda_1 d(x,Tx,a) + \lambda_2 d(Tx,T^2x,a) \\ & + \lambda_3 d(y,Ty,a) + \lambda_4 d(Ty,T^2y,a) \end{aligned} \tag{3.8}$$

for all $x,y \in X$.

Theorem 3.3. *Let (X,d) be a complete cone 2-metric space over Banach Algebra and $\mathscr{P}$ be the underlying solid cone. Let $T : X \to X$ be a two sided cone convex contraction mapping of order 2. Then T has a unique fixed point in X.*

Proof. Let $x_0 \in X$ be arbitrary. Define the sequence $\{x_n\}$ as $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}_0$. Set to each $a \in X$,

$$v_a = d(x_2, x_1, a) + d(x_1, x_0, a)$$

Write $\lambda' = (\lambda_1 + \lambda_3 + \lambda_4)(e - \lambda_2)^{-1}$ and $\lambda'' = (\lambda_1 + \lambda_2 + \lambda_3)(e - \lambda_4)^{-1}$. Take

$$\lambda = \begin{cases} \lambda' & \text{if} \quad r(\lambda') < 1 \\ \lambda'' & \text{if} \quad r(\lambda'') < 1 \\ \lambda' \quad \text{or} \quad \lambda'' & \text{if} \quad r(\lambda') < 1 \quad \text{and} \quad r(\lambda'') < 1 \end{cases}$$

Using (3.8) and because of symmetry of cone 2-metric, we have

$d(x_3, x_2, a) \preceq (\lambda_1 + \lambda_3 + \lambda_4)(e - \lambda_2)^{-1} v_a$ and

$d(x_3, x_2, a) \preceq (\lambda_1 + \lambda_2 + \lambda_3)(e - \lambda_4)^{-1} v_a$

Therefore, $d(x_3, x_2, a) \preceq \lambda v_a$. Similarly we can have $d(x_4, x_3, a) \preceq \lambda v_a$.

Continuing, to each $a \in X$, we have

$d(x_3, x_2, a) \preceq \lambda v_a, \qquad d(x_4, x_3, a) \preceq \lambda v_a,$

$d(x_5, x_4, a) \preceq \lambda^2 v_a, \qquad d(x_6, x_5, a) \preceq \lambda^2 v_a$

$d(x_7, x_6, a) \preceq \lambda^3 v_a, \qquad d(x_8, x_7, a) \preceq \lambda^3 v_a,$

... ...

An Induction argument shows that, to each $a \in X$

$$d(x_{m+1}, x_m, a) \preceq \lambda^l v_a \tag{3.9}$$

when $m = 2l$ or $m = 2l + 1$.

Now, applying (3.8) with $a = x_k$ and $i = j + 1$, we have

$$\begin{aligned} d(x_{j+1}, x_j, x_k) &\preceq \lambda_0 d(x_{j-1}, x_{j-2}, x_k) + \lambda_1 d(x_j, x_{j-1}, x_k) + \lambda_2 d(x_{j-1}, x_j, x_k) \\ &\quad + \lambda_3 d(x_j, x_{j+1}, x_k) + \lambda_4 d(x_{j-2}, x_{j-1}, x_k) + \lambda_5 d(x_{j-1}, x_j, x_k) \\ &= (\lambda_0 + \lambda_4) d(x_{j-1}, x_{j-2}, x_k) + (\lambda_1 + \lambda_2 + \lambda_5) d(x_j, x_{j-1}, x_k) \\ &\quad + \lambda_3 d(x_j, x_{j+1}, x_k) \end{aligned}$$

Taking $k = j-1$, we have $d(x_{j+1}, x_j, x_k) = \theta$. Applying (3.8) repeatedly, we get for any k with $1 < k < j+2$,

$$d(x_{j+1}, x_j, x_k) = \theta \tag{3.10}$$

As in Theorem 3.1, we can get $d(x_i, x_j, x_k) = \theta$ for all $k \geqslant j+2$. Clearly, the rest of the proof is essentially the same as in the Theorem (3.1). □

Definition 3.4. *A continuous mapping $T : X \to X$ on cone 2-metric space over a Banach Algebra (X, d) is said to be cone convex contraction mapping of type 2 if there exists $\lambda_0, \lambda_1, \ldots, \lambda_5 \in \mathscr{P}$ such that $\lambda_i's$ $(i = 0, 1, \ldots 5)$ commute, $\sum_{i=0}^{5} r(\lambda_i) < 1$ and to each $a \in X$, we have*

$$\begin{aligned} d(T^2x, T^2y, a) \preceq\ & \lambda_0 d(x, y, a) + \lambda_1 d(Tx, Ty, a) + \lambda_2 d(x, Tx, a) \\ & + \lambda_3 d(Tx, T^2x, a) + \lambda_4 d(y, Ty, a) + \lambda_5 d(Ty, T^2y, a) \end{aligned} \tag{3.11}$$

for all $x, y \in X$.

Corresponding to this new class of cone convex contraction mappings of type 2, we can have the following theorem.

Theorem 3.4. *Let (X, d) be a complete cone 2-metric space over a Banach Algebra with the underlying solid cone $\mathscr{P}$ and $T : X \to X$ a cone convex contraction mapping of type 2. Then T has a unique fixed point in X.*

Conflict of Interests

The authors declare that there is no conflict of interests.

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