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TRIPLED BEST PROXIMITY POINT THEOREM FOR MIXED g-MONOTONE MAPPINGS IN PARTIALLY ORDERED METRIC SPACES

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Abstract. In this paper, we obtain the Tripled best proximity point theorems for mixed g-monotone mappings in partially ordered metric spaces. The main results of this paper are generalizations of the main results of Nantadilok and Chaipornjareansri (Adv. Fixed Point Theory 5 (2015), No. 2, 168-191).

Keywords: partially ordered set; tripled fixed point; tripled best proximity points; g-monotone property.

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1. Introduction

The existence and uniqueness of a fixed point of non-self mappings is one of the interesting subjects in fixed point theory. In fact, given nonempty closed subsets A and B of a complete metric space (X, d), a contraction non-self-mapping $T : A \to B$ does not necessarily yield a fixed

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point $Tx = x$. In this case, it is very natural to investigate whether there is an element x such that d(x, Tx) is minimum. A notion of best proximity point appears at this point.

Let (X, d) is a metric space, and A, B are subsets of X. A point x is called best proximity point of $T : A \to B$ if $d(x,Tx) = d(A,B)$, where $d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$.

A best proximity point represents an optimal approximate solution to the equation $Tx = x$ whenever a non-self-mapping T has no fixed point. It is clear that a fixed point coincides with a best proximity point if $d(A,B) = 0$. Since a best proximity point reduces to a fixed point if the underlying mapping is assumed to be self-mappings, the best proximity point theorems are natural generalizations of the Banachs contraction principle.

In 1969, Fan [1] introduced the notion of a best proximity and established a classical best approximation theorem. Subsequently, many researchers have studied the best proximity point results in many ways (see in [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]). The study of the existence of best proximity point in the setting of partially ordered metric spaces has been considered in [15, 16, 17, 18, 19, 20]. Bhaskar and Lakshmikanthan [21] proved the existence of a new fixed point theorem for a mixed monotone mapping in a metric space with the help of partial order, this new type of fixed point called as coupled fixed point. This concept is extended to tripled fixed point by Berinde and Borcut [22]. They obtained the existence and uniqueness theorems for contractive mappings in partially ordered complete metric spaces. In recent years many authors established various coupled and tripled fixed point theorems in partially ordered metric space (see [23, 24, 25, 26, 27, 28, 29, 30, 31, 32] and references there in).

2. **Preliminaries**

We recall the main concepts needed to present our results.

Let A and B be two nonempty subsets of a metric space (X,d). We denote by A_0 and B_0 the following sets:

$$A_0 = \{x \in A : d(x,y) = d(A,B) \quad for\ some\ y \in B\}$$

$$B_0 = \{y \in B : d(x,y) = d(A,B) \quad for\ some\ x \in A\}$$

where $d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$.
We refer to [2] for sufficient conditions that guarantee that A_0 and B_0 are nonempty.

Now, we endow the set X with a partial order. Let $(X, \leq)$ be a partially ordered set and let d be a metric on X such that (X,d) is a complete metric space. Consider on the product space X^3 the following partial order: for $(x,y,z),(u,v,w) \in X^3$,

$$(x,y,z) \leq (u,v,w) \Leftrightarrow x \leq u, y \geq v, z \leq w.$$

Definition 2.1. [22] *Let $(X, \leq)$ be a partially ordered set and $F : X^3 \to X$. We say that F has the mixed monotone property if F(x,y,z) is monotone nondecreasing in x and z, and is monotone nonincreasing in y, that is, for any $x,y,z \in X$*

$$x_1,x_2 \in X \quad x_1 \leq x_2 \Rightarrow F(x_1,y,z) \leq F(x_2,y,z),$$

$$y_1,y_2 \in X \quad y_1 \leq y_2 \Rightarrow F(x,y_1,z) \geq F(x,y_2,z),$$

$$z_1,z_2 \in X \quad z_1 \leq z_2 \Rightarrow F(x,y,z_1) \leq F(x,y,z_2).$$

Definition 2.2. [22] *Let X be a non-empty set. An element $(x,y,z) \in X^3$ is called a tripled fixed point of the mapping F if $F(x,y,z) = x$ and $F(y,x,y) = y$ and $F(z,y,x) = z$.*

Definition 2.3. [33] *A mapping $F : A^3 \to B$ is said to be the proximal mixed monotone property if F(x,y,z) is proximally nondecreasing in x and z, and is proximally nonincreasing in y, that is*

$$\begin{cases} x_1 \leq x_2 \leq x_3 \\ d(u_1, F(x_1,y,z)) = d(A,B) \\ d(u_2, F(x_2,y,z)) = d(A,B) \\ d(u_3, F(x_3,y,z)) = d(A,B) \end{cases} \Rightarrow u_1 \leq u_2 \leq u_3,$$

$$\begin{cases} y_1 \leq y_2 \leq y_3 \\ d(v_1, F(x,y_1,z)) = d(A,B) \\ d(v_2, F(x,y_2,z)) = d(A,B) \\ d(v_3, F(x,y_3,z)) = d(A,B) \end{cases} \Rightarrow v_3 \leq v_2 \leq v_1,$$

and

$$\begin{cases} z_1 \leq z_2 \leq z_3 \\ d(w_1, F(x,y,z_1)) = d(A,B) \\ d(w_2, F(x,y,z_2)) = d(A,B) \\ d(w_3, F(x,y,z_3)) = d(A,B) \end{cases} \Rightarrow w_1 \leq w_2 \leq w_3,$$

where $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3, u_1, u_2, u_3, v_1, v_2, v_3, w_1, w_2, w_3 \in A$.

If A = B in the above definition, the notion of proximal mixed monotone property reduces to that of mixed monotone property.

Definition 2.4. *Let* Φ *denote all functions* $\phi : [0,\infty) \to [0,\infty)$ *which satisfy*

(i) ϕ *is continuous and nondecreasing,*

(ii) $\phi(t) = 0$ *if and only if t = 0,*

(iii) $\phi(t+s) \leq \phi(t) + \phi(s), \forall t,s \in (0,\infty]$.

Definition 2.5. *Let* ψ *denote all functions* $\psi : [0,\infty) \to [0,\infty)$ *which satisfy* $\lim_{t\to r} \psi(t) > 0$ *for all* $r > 0$ *and* $\lim_{t\to 0^+} \psi(t) = 0$.

Luong and Thuan [24], obtained a more general result of coupled fixed point following.

Theorem 2.6. *Let* $(X, \leq)$ *be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let* $F : X \times X \to X$ *be mapping having the mixed monotone property on X such that*

$$\phi\big(d(F(x,y),F(u,v))\big) \leq \frac{1}{2}\phi\big(d(x,u)+d(y,v)\big) - \psi\Big(\frac{d(x,u)+d(y,v)}{2}\Big)$$

for all $x,y,u,v \in X$ *with* $x \geq u$ *and* $y \leq v$, *where* $\psi \in \Psi$ *and* $\phi \in \Phi$. *If there exist* $x_0, y_0 \in X$ *such that* $x_0 \leq F(x_0,y_0)$ *and* $y_0 \geq F(y_0,x_0)$. *Suppose either*

(a) F is continuous or

(b) X has the following property:

(i) if a non-decreasing sequence $\{x_n\} \to x$, *then* $x_n \leq x$ *for all n,*

(ii) if a non-increasing sequence $\{y_n\} \to y$, *then* $y \geq y_n$ *for all n.*

Then there exist $x, y \in X$ *such that* $F(x,y) = x$ *and* $F(y,x) = y$.

In [16], Kumam et al. generalized the results of Luong and Thuan [24]. Recently, Nantadilok and Chaipornjareansri [33], extended the main result of Kumam et al. [16]. The main result of in [33] is the following.

Theorem 2.7. *Let* $(X, d, \leq)$ *be a partially ordered complete metric space. Let A, B be nonempty closed subsets of the metric space (X, d) such that* $A_0 \neq \emptyset$. *Let* $F : A \times A \times A \to B$ *satisfy the following conditions:*

- *(a) F is continuous proximally tripled weak* (ψ, ϕ) *contraction on A having the proximal mixed monotone property on A such that* $F(A_0, A_0, A_0) \subseteq B_0$.
- *(b) there exist elements* (x_0, y_0, z_0) *and* $(x_1, y_1, z_1) \in A_0 \times A_0 \times A_0$ *such that*

$$d(x_1, F(x_0, y_0, z_0)) = d(A,B) \quad with \quad x_0 \leq x_1,$$

$$d(y_1, F(y_0, x_0, z_0)) = d(A,B) \quad with \quad y_0 \geq y_1, \text{ and}$$

$$d(z_1, F(z_0, y_0, x_0)) = d(A,B) \quad with \quad z_0 \leq z_1.$$

Then, there exists $(x,y,z) \in A \times A \times A$ *such that* $d(x, F(x,y,z)) = d(A,B)$, $d(y, F(y,x,z)) = d(A,B)$ *and* $d(z, F(z,y,x)) = d(A,B)$.

Motivated by the above theorems, we first define the concept of proximal mixed g-monotone property and proximally tripled weak (ψ, ϕ) contraction on A. We also explore the existence and uniqueness of tripled best proximity points in the setting of partially ordered metric spaces. Further, we attempt to give the generalization of Theorem 2.7.

3. Tripled best proximity point theorems

Let X be a nonempty set. We recall that an element $(x,y,z) \in X \times X \times X$ is called a tripled coincidence point of two mappings $F : X \times X \times X \to X$ and $g : X \to X$ provided that $F(x,y,z) = g(x)$, $F(y,x,y) = g(y)$ and $F(z,y,x) = g(z)$ for all $x,y,z \in X$. Also, we say that F and g are commutative if $g(F(x,y,z)) = F(g(x), g(y), g(z))$ for all $x,y,z \in X$. We now present the following definitions.

Definition 3.1. *Let $(X,d,\leq)$ be a partially ordered metric space. Let A, B be nonempty subsets of X, and $F : A\times A\times A \to B$ and $g : A \to A$ be two given mappings. We say that F has the proximal mixed g-monotone property provided that for all $x,y,z \in A$, if*

$$\begin{cases} g(x_1) \leq g(x_2) \leq g(x_3), \\ d\big(g(u_1),F(g(x_1),g(y),g(z))\big) = d(A,B) \\ d\big(g(u_2),F(g(x_2),g(y),g(z))\big) = d(A,B) \\ d\big(g(u_3),F(g(x_3),g(y),g(z))\big) = d(A,B) \end{cases} \Longrightarrow g(u_1) \leq g(u_2) \leq g(u_3),$$

$$\begin{cases} g(y_1) \leq g(y_2) \leq g(y_3), \\ d\big(g(v_1),F(g(x),g(y_1),g(z))\big) = d(A,B) \\ d\big(g(v_2),F(g(x),g(y_2),g(z))\big) = d(A,B) \\ d\big(g(v_3),F(g(x),g(y_3),g(z))\big) = d(A,B) \end{cases} \Longrightarrow g(v_3) \leq g(v_2) \leq g(v_1),$$

and

$$\begin{cases} g(z_1) \leq g(z_2) \leq g(z_3), \\ d\big(g(w_1),F(g(x),g(y),g(z_1))\big) = d(A,B) \\ d\big(g(w_2),F(g(x),g(y),g(z_2))\big) = d(A,B) \\ d\big(g(w_3),F(g(x),g(y),g(z_3))\big) = d(A,B) \end{cases} \Longrightarrow g(w_1) \leq g(w_2) \leq g(w_3),$$

where $x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3,u_1,u_2,u_3,v_1,v_2,v_3,w_1,w_2,w_3 \in A$.

Definition 3.2. *Let $(X,d,\leq)$ be a partially ordered metric space and A, B are nonempty subsets of X. Let $F : A\times A\times A \to B$ and $g : A \to A$ be two given mappings. F is said to be proximally tripled weak (ψ,ϕ) contraction on A, whenever*

$$\begin{cases} g(x_1) \leq g(x_2),\ g(y_1) \geq g(y_2),\ g(z_1) \leq g(z_2) \\ d\big(g(u_1),F(g(x_1),g(y_1),g(z_1))\big) = d(A,B) \\ d\big(g(u_2),F(g(x_2),g(y_2),g(z_2))\big) = d(A,B) \end{cases}$$

$$\begin{aligned}\Longrightarrow \phi\big(d(g(u_1),g(u_2))\big) \;\leq\; & \frac{1}{3}\phi\Big(d(g(x_1),g(x_2))+d(g(y_1),g(y_2))+d(g(z_1),g(z_2))\Big)\\ & -\psi\Big(\frac{d(g(x_1),g(x_2))+d(g(y_1),g(y_2))+d(g(z_1),g(z_2))}{3}\Big),\end{aligned} \tag{3.1}$$

where $x_1,x_2,y_1,y_2,z_1,z_2,u_1,u_2 \in A$.

Lemma 3.3. *Let* $(X,d,\leq)$ *be a partially ordered metric space and A, B be nonempty subsets of X,* $A_0 \neq \emptyset$ *and* $F : A\times A\times A \to B$ *and* $g : A \to A$ *be two given mappings. If F has the proximal mixed g-monotone property, with* $g(A_0) = A_0$, $F(A_0,A_0,A_0) \subseteq B_0$.

$$\begin{cases} g(x_1) \leq g(x_2) \leq g(x_3),\ g(y_3) \leq g(y_2) \leq g(y_1),\\ g(z_1) \leq g(z_2) \leq g(z_3)\\ d\big(g(u_1),F(g(x_1),g(y_1),g(z_1))\big) = d(A,B)\\ d\big(g(u_2),F(g(x_2),g(y_2),g(z_2))\big) = d(A,B)\\ d\big(g(u_3),F(g(x_3),g(y_3),g(z_3))\big) = d(A,B)\end{cases} \Longrightarrow g(u_1) \leq g(u_2) \leq g(u_3) \tag{3.2}$$

where $x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3,u_1,u_2,u_3 \in A_0$.

Proof. Since $g(A_0) = A_0$, $F(A_0,A_0,A_0) \subseteq B_0$, it follows that $F(g(x_3),g(y_1),g(z_1)) \in B_0$. Hence there exists $g(u_1^*) \in A_0$ such that

$$d(g(u_1^*),F(g(x_3),g(y_1),g(z_1))) = d(A,B). \tag{3.3}$$

Using the fact that F has the proximal mixed g-monotone property, together with (3.2) and (3.3), we get

$$\begin{cases} g(x_1) \leq g(x_2) \leq g(x_3)\\ d\big(g(u_1),F(g(x_1),g(y_1),g(z_1))\big) = d(A,B)\\ d\big(g(u_2),F(g(x_2),g(y_2),g(z_2))\big) = d(A,B)\\ d\big(g(u_1^*),F(g(x_3),g(y_1),g(z_1))\big) = d(A,B)\end{cases} \Longrightarrow g(u_1) \leq g(u_2) \leq g(u_1^*). \tag{3.4}$$

Also, from the proximal mixed g-monotone property of F with (3.2) and (3.4), we get

$$\begin{cases} g(y_3) \leq g(y_2) \leq g(y_1) \\ d\big(g(u_3), F(g(x_3), g(y_3), g(z_3))\big) = d(A,B) \\ d\big(g(u_1^*), F(g(x_3), g(y_1), g(z_1))\big) = d(A,B) \end{cases} \Longrightarrow g(u_1^*) \leq g(u_3). \tag{3.5}$$

From (3.4) and (3.5) , one can conclude the $g(u_1) \leq g(u_2) \leq g(u_3)$. Hence the proof is complete.

□

Lemma 3.4. *Let $(X, d, \leq)$ be a partially ordered metric space and A, B be nonempty subsets of X, $A_0 \neq \emptyset$ and $F : A \times A \times A \to B$ and $g : A \to A$ be two given mappings. Let F have the proximal mixed g-monotone property, with $g(A_0) = A_0$, $F(A_0, A_0, A_0) \subseteq B_0$. If*

$$\begin{cases} g(x_1) \leq g(x_2) \leq g(x_3), \quad g(y_3) \leq g(y_2) \leq g(y_1), \\ g(z_1) \leq g(z_2) \leq g(z_3) \\ d\big(g(v_1), F(g(y_1), g(x_1), g(z_1))\big) = d(A,B) \\ d\big(g(v_2), F(g(y_2), g(x_2), g(z_2))\big) = d(A,B) \\ d\big(g(v_3), F(g(y_3), g(x_3), g(z_3))\big) = d(A,B) \end{cases} \Longrightarrow g(v_3) \leq g(v_2) \leq g(v_1), \tag{3.6}$$

where $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3, v_1, v_2, v_3 \in A_0$.

Proof. Since $g(A_0) = A_0$, $F(A_0, A_0, A_0) \subseteq B_0$, it follows that $F(g(y_3), g(x_1), g(z_1)) \in B_0$. Hence there exists $g(v_1^*) \in A_0$ such that

$$d(g(v_1^*), F(g(y_3), g(x_1), g(z_1))) = d(A,B). \tag{3.7}$$

Using the fact that F has the proximal mixed g-monotone property, together with (3.6) and (3.7), we get

$$\begin{cases} g(x_1) \leq g(x_2) \leq g(x_3),\ g(y_3) \leq g(y_2) \leq g(y_1), \\ d\big(g(v_1), F(g(y_1), g(x_1), g(z_1))\big) = d(A,B) \\ d\big(g(v_2), F(g(y_2), g(x_2), g(z_2))\big) = d(A,B) \\ d\big(g(v_1^*), F(g(y_3), g(x_1), g(z_1))\big) = d(A,B) \end{cases} \Longrightarrow g(v_1^*) \leq g(v_2) \leq g(v_1). \tag{3.8}$$

Also, from the proximal mixed g-monotone property of F with (3.6) and (3.8), we get

$$\begin{cases} g(x_1) \le g(x_2) \le g(x_3), \quad g(z_1) \le g(z_2) \le g(z_3) \\ d\big(g(v_3), F(g(y_3), g(x_3), g(z_3))\big) = d(A,B) \\ d\big(g(v_1^*), F(g(y_3), g(x_1), g(z_1))\big) = d(A,B) \end{cases} \Longrightarrow g(v_3) \le g(v_1^*). \tag{3.9}$$

From (3.8) and (3.9) , one can conclude the $g(v_3) \le g(v_2) \le g(v_1)$. Hence the proof is complete.

□

Lemma 3.5. *Let $(X,d,\le)$ be a partially ordered metric space and A, B be nonempty subsets of X, $A_0 \neq \emptyset$ and $F : A \times A \times A \to B$ and $g : A \to A$ be two given mappings. If F has the proximal mixed g-monotone property, with $g(A_0) = A_0$, $F(A_0,A_0,A_0) \subseteq B_0$*

$$\begin{cases} g(x_1) \le g(x_2) \le g(x_3),\ g(y_3) \le g(y_2) \le g(y_1), \\ g(z_1) \le g(z_2) \le g(z_3) \\ d\big(g(w_1), F(g(x_1), g(y_1), g(z_1))\big) = d(A,B) \\ d\big(g(w_2), F(g(x_2), g(y_2), g(z_2))\big) = d(A,B) \\ d\big(g(w_3), F(g(x_3), g(y_3), g(z_3))\big) = d(A,B) \end{cases} \Longrightarrow g(w_1) \le g(w_2) \le g(w_3), \tag{3.10}$$

where $x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3,w_1,w_2,w_3 \in A_0$.

Proof. The proof is similar to that of Lemma 3.3 and Lemma 3.4. □

The following main result is a tripled best proximity point theorem for non-self weak (ψ,ϕ) proximal contractions.

Theorem 3.6. *Let $(X,d,\le)$ be a partially ordered complete metric space. Let A, B be nonempty closed subsets of the metric space (X, d) such that $A_0 \neq \emptyset$. Let $F : A \times A \times A \to B$ and $g : A \to A$ be two given mappings satisfying the following conditions:*

- *(a) F and g are continuous;*
- *(b) F has the proximal mixed g-monotone property on A such that $g(A_0) = A_0$, $F(A_0,A_0,A_0) \subseteq B_0$;*
- *(c) F is a proximally tripled weak (ψ,ϕ) contraction on A;*

(d) there exist elements (x_0, y_0, z_0) *and* $(x_1, y_1, z_1) \in A_0 \times A_0 \times A_0$ *such that*

$$d\big(g(x_1), F(g(x_0), g(y_0), g(z_0))\big) = d(A,B) \quad with \quad g(x_0) \leq g(x_1),$$

$$d\big(g(y_1), F(g(y_0), g(x_0), g(z_0))\big) = d(A,B) \quad with \quad g(y_0) \geq g(y_1),\ and$$

$$d\big(g(z_1), F(g(z_0), g(y_0), g(x_0))\big) = d(A,B) \quad with \quad g(z_0) \leq g(z_1).$$

Then there exists $(x,y,z) \in A \times A \times A$ *such that*

$$d\big(g(x), F(g(x), g(y), g(z))\big) = d(A,B),\ d\big(g(y), F(g(y), g(x), g(z))\big) = d(A,B)$$

and $d\big(g(z), F(g(z), g(y), g(x))\big) = d(A,B)$.

Proof. Let $(x_0, y_0, z_0), (x_1, y_1, z_1) \in A_0 \times A_0 \times A_0$ be such that

$$\begin{aligned} d\big(g(x_1), F(g(x_0), g(y_0), g(z_0))\big) &= d(A,B) \quad with \quad g(x_0) \leq g(x_1), \\ d\big(g(y_1), F(g(y_0), g(x_0), g(z_0))\big) &= d(A,B) \quad with \quad g(y_0) \geq g(y_1),\ and \\ d\big(g(z_1), F(g(z_0), g(y_0), g(x_0))\big) &= d(A,B) \quad with \quad g(z_0) \leq g(z_1). \end{aligned}$$

Since $F(A_0, A_0, A_0) \subseteq B_0$ and $g(A_0) = A_0$, there exists an element $(x_2, y_2, z_2) \in A_0 \times A_0 \times A_0$ such that

$$\begin{aligned} d\big(g(x_2), F(g(x_1), g(y_1), g(z_1))\big) &= d(A,B), \\ d\big(g(y_2), F(g(y_1), g(x_1), g(z_1))\big) &= d(A,B),\ and \\ d\big(g(z_2), F(g(z_1), g(y_1), g(x_1))\big) &= d(A,B). \end{aligned}$$

And also, there exists an element $(x_3, y_3, z_3) \in A_0 \times A_0 \times A_0$ such that

$$\begin{aligned} d\big(g(x_3), F(g(x_2), g(y_2), g(z_2))\big) &= d(A,B), \\ d\big(g(y_3), F(g(y_2), g(x_2), g(z_2))\big) &= d(A,B),\ and \\ d\big(g(z_3), F(g(z_2), g(y_2), g(x_2))\big) &= d(A,B). \end{aligned}$$

Hence from Lemma 3.3, Lemma 3.4 and Lemma 3.5, we obtain $g(x_1) \leq g(x_2) \leq g(x_3)$, $g(y_1) \geq g(y_2) \geq g(y_3)$, and $g(z_1) \leq g(z_2) \leq g(z_3)$. Continuing this process, we can construct the sequences $\{x_n\}, \{y_n\}$ *and* $\{z_n\} \in A_0$ such that

$$d\big(g(x_{n+1}), F(g(x_n), g(y_n), g(z_n))\big) = d(A,B)\ for\ all\ n \geq 0,$$

with

$$g(x_0) \leq g(x_1) \leq g(x_2) \leq \cdots \leq g(x_n) \leq g(x_{n+1}) \leq \cdots, \tag{3.11}$$

$$d\big(g(y_{n+1}), F(g(y_n), g(x_n), g(z_n))\big) = d(A,B) \ for \ all \ n \geq 0,$$

with

$$g(y_0) \geq g(y_1) \geq g(y_2) \geq \cdots \geq g(y_n) \geq g(y_{n+1}) \geq \cdots, \tag{3.12}$$

and

$$d\big(g(z_{n+1}), F(g(z_n), g(y_n), g(x_n))\big) = d(A,B) \ for \ all \ n \geq 0,$$

with

$$g(z_0) \leq g(z_1) \leq g(z_2) \leq \cdots \leq g(z_n) \leq g(z_{n+1}) \leq \cdots. \tag{3.13}$$

Then

$$d\big(g(x_n), F(g(x_{n-1}), g(y_{n-1}), g(z_{n-1}))\big) = d(A,B) \ and$$

$$d\big(g(x_{n+1}), F(g(x_n), g(y_n), g(z_n))\big) = d(A,B),$$

and also we have $g(x_{n-1}) \leq g(x_n)$, $g(y_{n-1}) \geq g(y_n)$ and $g(z_{n-1}) \leq g(z_n)$. Now using the fact that F is a proximally tripled weak (ψ, ϕ) contraction on A, we get

$$\begin{aligned} &\phi\big(d(g(x_n), g(x_{n+1}))\big) \\ &\leq \frac{1}{3}\phi\big(d(g(x_{n-1}), g(x_n)) + d(g(y_{n-1}), g(y_n)) + d(g(z_{n-1}), g(z_n))\big) \\ &\quad -\psi\Big(\frac{d(g(x_{n-1}), g(x_n)) + d(g(y_{n-1}), g(y_n)) + d(g(z_{n-1}), g(z_n))}{3}\Big). \end{aligned} \tag{3.14}$$

Similarly,

$$\begin{aligned} &\phi\big(d(g(y_n), g(y_{n+1}))\big) \\ &\leq \frac{1}{3}\phi\big(d(g(y_{n-1}), g(y_n)) + d(g(x_{n-1}), g(x_n)) + d(g(z_{n-1}), g(z_n))\big) \\ &\quad -\psi\Big(\frac{d(g(y_{n-1}), g(y_n)) + d(g(x_{n-1}), g(x_n)) + d(g(z_{n-1}), g(z_n))}{3}\Big). \end{aligned} \tag{3.15}$$

$$
\begin{aligned}
&\phi\big(d(g(z_n),g(z_{n+1}))\big)\\
&\le \frac{1}{3}\phi\big(d(g(z_{n-1}),g(z_n))+d(g(y_{n-1}),g(y_n))+d(g(x_{n-1}),g(x_n))\big)\\
&\quad -\psi\Big(\frac{d(g(z_{n-1}),g(z_n))+d(g(y_{n-1}),g(y_n))+d(g(x_{n-1}),g(x_n))}{3}\Big).
\end{aligned}
\tag{3.16}
$$

Adding (3.14), (3.15) and (3.16), we get

$$
\begin{aligned}
&\phi\big(d(g(x_n),g(x_{n+1}))\big)+\phi\big(d(g(y_n),g(y_{n+1}))\big)+\phi\big(d(g(z_n),g(z_{n+1}))\big)\\
&\le \phi\big(d(g(x_{n-1}),g(x_n))+d(g(y_{n-1}),g(y_n))+d(g(z_{n-1}),g(z_n))\big)\\
&\quad -3\psi\Big(\frac{d(g(x_{n-1}),g(x_n))+d(g(y_{n-1}),g(y_n))+d(g(z_{n-1}),g(z_n))}{3}\Big).
\end{aligned}
\tag{3.17}
$$

By the definition of ϕ, we have

$$
\begin{aligned}
&\phi\big(d(g(x_n),g(x_{n+1}))+d(g(y_n),g(y_{n+1}))+d(g(z_n),g(z_{n+1}))\big)\\
&\le \phi\big(d(g(x_n),g(x_{n+1}))\big)+\phi\big(d(g(y_n),g(y_{n+1}))\big)+\phi\big(d(g(z_n),g(z_{n+1}))\big).
\end{aligned}
\tag{3.18}
$$

From (3.17) and (3.18), we get

$$
\begin{aligned}
&\phi\big(d(g(x_n),g(x_{n+1}))+d(g(y_n),g(y_{n+1}))+d(g(z_n),g(z_{n+1}))\big)\\
&\le \phi\big(d(g(x_{n-1}),g(x_n))+d(g(y_{n-1}),g(y_n))+d(g(z_{n-1}),g(z_n))\big)\\
&\quad -3\psi\Big(\frac{d(g(x_{n-1}),g(x_n))+d(g(y_{n-1}),g(y_n))+d(g(z_{n-1}),g(z_n))}{3}\Big).
\end{aligned}
\tag{3.19}
$$

Since ϕ is nondecreasing, we get

$$
\begin{aligned}
&d(g(x_n),g(x_{n+1}))+d(g(y_n),g(y_{n+1}))+d(g(z_n),g(z_{n+1}))\\
&\le d(g(x_{n-1}),g(x_n))+d(g(y_{n-1}),g(y_n))+d(g(z_{n-1}),g(z_n)).
\end{aligned}
\tag{3.20}
$$

Set $\delta_n = d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})) + d(g(z_n), g(z_{n+1}))$, then the sequence (δ_n) is decreasing. Therefore, there is some $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \left[d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})) + d(g(z_n), g(z_{n+1})) \right] = \delta. \tag{3.21}$$

We shall show that $\delta = 0$. Suppose, to the contrary, that $\delta > 0$. Then taking the limit as $n \to \infty$ both sides of (3.19) and having in mind that we assume $\lim_{t\to r} \psi(t) > 0$ for all $r > 0$ and ϕ is continuous, we have

$$\phi(\delta) = \lim_{n\to\infty} \phi(\delta_n) \leq \lim_{n\to\infty} \phi(\delta_{n-1}) - 3\psi\Big(\frac{\delta_{n-1}}{3}\Big) = \phi(\delta) - 3 \lim_{n\to\infty} \psi\Big(\frac{\delta_{n-1}}{3}\Big) < \phi(\delta), \tag{3.22}$$

a contradiction. Thus $\delta = 0$, that is,

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \left[d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})) + d(g(z_n), g(z_{n+1})) \right] = 0. \tag{3.23}$$

Now, we prove that $\{g(x_n)\}$, $\{g(y_n)\}$ and $\{g(z_n)\}$ are Cauchy sequences. Suppose that at least one of the sequences $\{g(x_n)\}$, $\{g(y_n)\}$ and $\{g(z_n)\}$ is not a Cauchy sequence. This implies that

$$\lim_{n,m\to\infty} d(g(x_n), g(x_m)) \nrightarrow 0,\ or\ \lim_{n,m\to\infty} d(g(y_n), g(y_m)) \nrightarrow 0,$$

$$or\ \lim_{n,m\to\infty} d(g(z_n), g(z_m)) \nrightarrow 0,$$

and, consequently

$$\lim_{n,m\to\infty} \left[d(g(x_n), g(x_m)) + d(g(y_n), g(y_m)) + d(g(z_n), g(z_m)) \right] \nrightarrow 0. \tag{3.24}$$

Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{g(x_{n(k)})\}$, $\{g(x_{m(k)})\}$ of $\{g(x_n)\}$, $\{g(y_{n(k)})\}$, $\{g(y_{m(k)})\}$ of$\{g(y_n)\}$ and $\{g(z_{n(k)})\}$, $\{g(z_{m(k)})\}$ of $\{g(z_n)\}$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$,

$$\left[d(g(x_{n(k)}), g(x_{m(k)})) + d(g(y_{n(k)}), g(y_{m(k)})) + d(g(z_{n(k)}), g(z_{m(k)})) \right] \geq \varepsilon. \tag{3.25}$$

This means that

$$\left[d(g(x_{n(k)-1}), g(x_{m(k)})) + d(g(y_{n(k)-1}), g(y_{m(k)})) + d(g(z_{n(k)-1}), g(z_{m(k)})) \right] < \varepsilon. \tag{3.26}$$

Therefore by using (3.25), (3.26) and the triangle inequality, we obtain

$$\begin{aligned}
\varepsilon &\leq r_k = d(g(x_{n(k)}),g(x_{m(k)})) + d(g(y_{n(k)}),g(y_{m(k)})) + d(g(z_{n(k)}),g(z_{m(k)})) \\
&\leq d(g(x_{n(k)}),g(x_{n(k)-1})) + d(g(x_{n(k)-1}),g(x_{m(k)})) + d(g(y_{n(k)}),g(y_{n(k)-1})) \\
&\quad + d(g(y_{n(k)-1}),g(y_{m(k)})) + d(g(z_{n(k)}),g(z_{n(k)-1})) + d(g(z_{n(k)-1}),g(z_{m(k)})) \\
&\leq d(g(x_{n(k)}),g(x_{n(k)-1})) + d(g(y_{n(k)}),g(y_{n(k)-1})) + d(g(z_{n(k)}),g(z_{n(k)-1})) + \varepsilon.
\end{aligned}$$

On taking the limit $k \to \infty$ and using (3.23), we obtain

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} \left[g(x_{n(k)}),g(x_{m(k)})) + d(g(y_{n(k)}),g(y_{m(k)})) + g(z_{n(k)}),g(z_{m(k)})) \right] = \varepsilon. \tag{3.27}$$

By the triangle inequality

$$\begin{aligned}
r_k &= d(g(x_{n(k)}),g(x_{m(k)})) + d(g(y_{n(k)}),g(y_{m(k)})) + d(g(z_{n(k)}),g(z_{m(k)})) \\
&\leq d(g(x_{n(k)}),g(x_{n(k)+1})) + d(g(x_{n(k)+1}),g(x_{m(k)+1})) + d(g(x_{m(k)+1}),g(x_{m(k)})) \\
&\quad + d(g(y_{n(k)}),g(y_{n(k)+1})) + d(g(y_{n(k)+1}),g(y_{m(k)+1})) + d(g(y_{m(k)+1}),g(y_{m(k)})) \\
&\quad + d(g(z_{n(k)}),g(z_{n(k)+1})) + d(g(z_{n(k)+1}),g(z_{m(k)+1})) + d(g(z_{m(k)+1}),g(z_{m(k)})) \\
&= \delta_{n(k)} + \delta_{m(k)} + d(g(x_{n(k)+1}),g(x_{m(k)+1})) + d(g(y_{n(k)+1},g(y_{m(k)+1})) \\
&\quad + d(g(z_{n(k)+1}),g(z_{m(k)+1})).
\end{aligned}$$

Using the property of ϕ, we obtain

$$\begin{aligned}
\phi(r_k) &= \phi\Big(\delta_{n(k)} + \delta_{m(k)} + d(g(x_{n(k)+1}),g(x_{m(k)+1})) + d(g(y_{n(k)+1},g(y_{m(k)+1})) \\
&\quad + d(g(z_{n(k)+1}),g(z_{m(k)+1}))\Big) \\
&\leq \phi\left(\delta_{n(k)}\right) + \phi\left(\delta_{m(k)}\right) + \phi\left(d(g(x_{n(k)+1}),g(x_{m(k)+1}))\right) \\
&\quad + \phi\left(d(g(y_{n(k)+1},g(y_{m(k)+1}))\right) + \phi\left(d(g(z_{n(k)+1}),g(z_{m(k)+1}))\right).
\end{aligned} \tag{3.28}$$

Since $g(x_{n(k))} \geq g(x_{m(k)})$, $g(y_{n(k)}) \leq g(y_{m(k)})$ and $g(z_{n(k))} \geq g(z_{m(k)})$, using the fact that F is a proximally tripled weak (ψ, ϕ) contraction on A, we get

$$\begin{aligned}
&\phi\big(d(g(x_{n(k)+1}), g(x_{m(k)+1}))\big)\\
&\leq \frac{1}{3}\phi\big(d(g(x_{n(k)}), g(x_{m(k)})) + d(g(y_{n(k)}), g(y_{m(k)})) + d(g(z_{n(k)}), g(z_{m(k)}))\big)\\
&\quad -\psi\Big(\frac{d(g(x_{n(k)}), g(x_{m(k)})) + d(g(y_{n(k)}), g(y_{m(k)})) + d(g(z_{n(k)}), g(z_{m(k)}))}{3}\Big)\\
&\leq \frac{1}{3}\phi(r_k) - \psi\Big(\frac{r_k}{3}\Big). \qquad (3.29)
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
&\phi\big(d(g(y_{n(k)+1}), g(y_{m(k)+1}))\big)\\
&\leq \frac{1}{3}\phi\big(d(g(y_{n(k)}), g(y_{m(k)})) + d(g(x_{n(k)}), g(x_{m(k)})) + d(g(z_{n(k)}), g(z_{m(k)}))\big)\\
&\quad -\psi\Big(\frac{d(g(y_{n(k)}), g(y_{m(k)})) + d(g(x_{n(k)}), g(x_{m(k)})) + d(g(z_{n(k)}), g(z_{m(k)}))}{3}\Big)\\
&\leq \frac{1}{3}\phi(r_k) - \psi\Big(\frac{r_k}{3}\Big). \qquad (3.30)
\end{aligned}$$

and

$$\begin{aligned}
&\phi\big(d(g(z_{n(k)+1}), g(z_{m(k)+1}))\big)\\
&\leq \frac{1}{3}\phi\big(d(g(z_{n(k)}), g(z_{m(k)})) + d(g(y_{n(k)}), g(y_{m(k)})) + d(g(x_{n(k)}), g(x_{m(k)}))\big)\\
&\quad -\psi\Big(\frac{d(g(z_{n(k)}), g(z_{m(k)})) + d(g(y_{n(k)}), g(y_{m(k)})) + d(g(x_{n(k)}), g(x_{m(k)}))}{3}\Big)\\
&\leq \frac{1}{3}\phi(r_k) - \psi\Big(\frac{r_k}{3}\Big). \qquad (3.31)
\end{aligned}$$

From (3.28),(3.29), (3.30) and (3.31), we obtain

$$\phi(r_k) \leq \phi\big(\delta_{n(k)} + \delta_{m(k)}\big) + \phi(r_k) - 3\psi\big(\frac{r_k}{3}\big). \qquad (3.32)$$

On taking the limit $k \to \infty$ using (3.23),(3.27) and (3.32), we have

$$\phi(\varepsilon) \leq \phi(0) + \phi(\varepsilon) - 3\lim_{k\to\infty}\psi\big(\frac{r_k}{3}\big) = \phi(\varepsilon) - 3\lim_{k\to\infty}\psi\big(\frac{r_k}{3}\big) < \phi(\varepsilon). \qquad (3.33)$$

Which is a contradiction. This shows that $\{g(x_n)\}$, $\{g(y_n)\}$ and $\{g(z_n)\}$ are Cauchy sequences. Since A is a closed subset of a complete metric space X, there exist $x', y', z' \in A$ such that $g(x_n) \to x'$, $g(y_n) \to y'$ and $g(z_n) \to z'$ as $n \to \infty$. Here $x_n, y_n, z_n \in A_0$, $g(A_0) = A_0$ so that $g(x_n), g(y_n), g(z_n) \in A_0$. Since A_0 is closed, we conclude that $x', y', z' \in A_0 \times A_0 \times A_0$, i.e., there exist $x, y, z \in A_0$ such that $g(x) = x'$, $g(y) = y'$ and $g(z) = z'$. Therefore

$$g(x_n) \to g(x), g(y_n) \to g(y) \textit{ and } g(z_n) \to g(z). \tag{3.34}$$

Since $\{g(x_n)\}$ is monotone increasing, $\{g(y_n)\}$ is monotone decreasing and $\{g(z_n)\}$ is monotone increasing, we have $g(x_n) \leq g(x)$, $g(y_n) \geq g(y)$ and $g(z_n) \leq g(z)$. From (3.11),(3.12) and (3.13), we have

$$d\big(g(x_{n+1}), F(g(x_n), g(y_n), g(z_n))\big) = d(A,B), \tag{3.35}$$

$$d\big(g(y_{n+1}), F(g(y_n), g(x_n), g(z_n))\big) = d(A,B), \tag{3.36}$$

and

$$d\big(g(z_{n+1}), F(g(z_n), g(y_n), g(x_n))\big) = d(A,B). \tag{3.37}$$

Since F is continuous, we have, from (3.34),

$$F(g(x_n), g(y_n), g(z_n)) \to F(g(x), g(y), g(z)),$$

$$F(g(y_n), g(x_n), g(z_n)) \to F(g(y), g(x), g(z))$$

and

$$F(g(z_n), g(y_n), g(x_n)) \to F(g(z), g(y), g(x)).$$

Thus, the continuity of the metric d implies that

$$d\big(g(x_{n+1}), F(g(x_n), g(y_n), g(z_n))\big) \to d\big(g(x), F(g(x), g(y), g(z))\big), \tag{3.38}$$

$$d\big(g(y_{n+1}), F(g(y_n), g(x_n), g(z_n))\big) \to d\big(g(y), F(g(y), g(x), g(z))\big) \tag{3.39}$$

and

$$d\big(g(z_{n+1}), F(g(z_n), g(y_n), g(x_n))\big) \to d\big(g(z), F(g(z), g(y), g(x))\big). \tag{3.40}$$

Therefore from (3.35), (3.36), (3.38),(3.41) and (3.40)

$$d\big(g(x), F(g(x), g(y), g(z))\big) = d(A,B) \qquad d\big(g(y), F(g(y), g(x), g(z))\big) = d(A,B),$$

$$d\big(g(z), F(g(z), g(y), g(x))\big) = d(A, B).$$

□

If g is assumed to be the identity mappings in Theorem 3.6.

Corollary 3.7. *Let $(X, d, \leq)$ be a partially ordered complete metric space. Let A, B be nonempty closed subsets of the metric space (X, d) such that $A_0 \neq \emptyset$. Let $F : A \times A \times A \to B$ given mappings satisfying the following conditions:*

(a) F be continuous;

(b) F has the proximal mixed monotone property on A such that $F(A_0, A_0, A_0) \subseteq B_0$;

(c) F is a proximally tripled weak (ψ, ϕ) contraction on A;

(d) there exist elements (x_0, y_0, z_0) and $(x_1, y_1, z_1) \in A_0 \times A_0 \times A_0$ such that

$$d\big(x_1, F(x_0, y_0, z_0)\big) = d(A, B) \text{ with } x_0 \leq x_1,\ d\big(y_1, F(y_0, x_0, z_0)\big) = d(A, B) \text{ with } y_0 \geq y_1,$$

$$\text{and } d\big(z_1, F(z_0, y_0, x_0)\big) = d(A, B) \text{ with } z_0 \leq z_1.$$

Then there exists $(x, y, z) \in A \times A \times A$ such that

$$d\big(x, F(x, y, z)\big) = d(A, B), \quad d\big(y, F(y, x, z)\big) = d(A, B) \text{ and } d\big(z, F(z, y, x)\big) = d(A, B).$$

Corollary 3.8. *Let $(X, d, \leq)$ be a partially ordered complete metric space. Let A be a nonempty closed subsets of the metric space (X, d). Let $F : A \times A \times A \to A$ and $g : A \to A$ be two given mappings satisfying the following conditions:*

(a) F and g are continuous;

(b) F has the mixed g-monotone property on A such that $g(A) = A$, $F(A, A, A) \subseteq A$;

(c) F is a tripled weak (ψ, ϕ) contraction on A;

(d) there exist elements (x_0, y_0, z_0) and $(x_1, y_1, z_1) \in A \times A \times A$ such that

$$g(x_1) = F(g(x_0), g(y_0), g(z_0)) \quad \text{with} \quad g(x_0) \leq g(x_1),$$

$$g(y_1) = F(g(y_0), g(x_0), g(z_0)) \quad \text{with} \quad g(y_0) \geq g(y_1), \text{ and}$$

$$g(z_1) = F(g(z_0), g(y_0), g(x_0)) \quad \text{with} \quad g(z_0) \leq g(z_1).$$

Then there exists $(x, y, z) \in A \times A \times A$ such that

$$d\big(g(x), F(g(x), g(y), g(z))\big) = 0, \quad d\big(g(y), F(g(y), g(x), g(z))\big) = 0,$$

$$d\big(g(z),F(g(z),g(y),g(x))\big)=0.$$

Theorem 3.9. *Let $(X,d,\leq)$ be a partially ordered complete metric space. Let A, B be nonempty closed subsets of the metric space (X, d) such that $A_0 \neq \emptyset$. Let $F: A\times A\times A \to B$ and $g: A\to A$ be two given mappings satisfying the following conditions:*

(a) g is continuous;

(b) F has the proximal mixed g-monotone property on A such that $g(A_0)=A_0$, $F(A_0,A_0,A_0)\subseteq B_0$;

(c) F is a proximally tripled weak (ψ,ϕ) contraction on A;

(d) there exist elements (x_0,y_0,z_0) and $(x_1,y_1,z_1)\in A_0\times A_0\times A_0$ such that

$$d\big(g(x_1),F(g(x_0),g(y_0),g(z_0))\big)=d(A,B) \quad with \quad g(x_0)\leq g(x_1),$$

$$d\big(g(y_1),F(g(y_0),g(x_0),g(z_0))\big)=d(A,B) \quad with \quad g(y_0)\geq g(y_1) \text{ and}$$

$$d\big(g(z_1),F(g(z_0),g(y_0),g(x_0))\big)=d(A,B) \quad with \quad g(z_0)\leq g(z_1).$$

(e) if $\{x_n\}$ is a nondecreasing sequence in A such that $x_n\to x$, then $x_n\leq x$ and if $\{y_n\}$ is a nonincreasing sequence in A such that $y_n\to y$, then $y_n\geq y$ and if $\{z_n\}$ is a nondecreasing sequence in A such that $z_n\to z$, then $z_n\leq z$.

Then there exists $(x,y,z)\in A\times A\times A$ such that

$$d\big(g(x),F(g(x),g(y),g(z))\big)=d(A,B), \quad d\big(g(y),F(g(y),g(x),g(z))\big)=d(A,B)$$

$$\text{and } d\big(g(z),F(g(z),g(y),g(x))\big)=d(A,B).$$

Proof. As in the proof of Theorem 3.6, there exist sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in A_0 such that

$$d\big(g(x_{n+1}),F\big(g(x_n),g(y_n),g(z_n)\big)\big)=d(A,B) \text{ with } g(x_n)\leq g(x_{n+1}) \text{ for all } n\geq 0, \tag{3.41}$$

$$d\big(g(y_{n+1}),F\big(g(y_n),g(x_n),g(z_n)\big)\big)=d(A,B) \text{ with } g(y_n)\geq g(y_{n+1}) \text{ for all } n\geq 0. \tag{3.42}$$

and

$$d\big(g(z_{n+1}),F\big(g(z_n),g(y_n),g(x_n)\big)\big)=d(A,B) \text{ with } g(z_n)\leq g(z_{n+1}) \text{ for all } n\geq 0. \tag{3.43}$$

Also, $g(x_n) \to g(x)$, $g(y_n) \to g(y)$ and $g(z_n) \to g(z)$. From (e), we get $g(x_n) \leq g(x)$, $g(y_n) \geq g(y)$ and $g(z_n) \leq g(z)$. Since $F(A_0, A_0, A_0) \subseteq B_0$, it follows that $F(g(x), g(y), g(z))$, $F(g(y), g(x), g(z))$ and $F(g(z), g(y), g(x))$ are in B_0. Therefore, there exists $(x_1^*, y_1^*, z_1^*) \in A_0 \times A_0 \times A_0$ such that

$$\begin{aligned} d\big(x_1^*, F(g(x), g(y), g(z))\big) &= d(A,B), \\ d\big(y_1^*, F(g(y), g(x), g(z))\big) &= d(A,B), \\ d\big(z_1^*, F(g(z), g(y), g(x))\big) &= d(A,B). \end{aligned}$$

Since $g(A_0) = A_0$, there exist $x^*, y^*, z^* \in A_0$ such that $g(x^*) = x_1^*$, $g(y^*) = y_1^*$ and $g(z^*) = z_1^*$. Hence,

$$d\big(g(x^*), F(g(x), g(y), g(z))\big) = d(A,B), \tag{3.44}$$

$$d\big(g(y^*), F(g(y), g(x), g(z))\big) = d(A,B) \text{ and} \tag{3.45}$$

$$d\big(g(z^*), F(g(z), g(y), g(x))\big) = d(A,B). \tag{3.46}$$

Since $g(x_n) \leq g(x)$, $g(y_n) \geq g(y)$ and $g(z_n) \leq g(z)$ and F is a proximally tripled weak (ψ, ϕ) contraction on A for (3.41),(3.42),(3.43), (3.44), (3.45) and (3.46) we get

$$\begin{aligned} \phi\big(d\big(g(x_{n+1}), g(x^*)\big)\big) \leq\ & \frac{1}{3}\phi\big(d(g(x_n), g(x)) + d(g(y_n), g(y)) + d(g(z_n), g(z))\big) \\ & -\psi\Big(\frac{d(g(x_n), g(x)) + d(g(y_n), g(y)) + d(g(z_n), g(z))}{3}\Big), \end{aligned}$$

$$\begin{aligned} \phi\big(d\big(g(y_{n+1}), g(y^*)\big)\big) \leq\ & \frac{1}{3}\phi\big(d(g(y_n), g(y)) + d(g(x_n), g(x)) + d(g(z_n), g(z))\big) \\ & -\psi\Big(\frac{d(g(y_n), g(y)) + d(g(x_n), g(x)) + d(g(z_n), g(z))}{3}\Big), \end{aligned}$$

and

$$\begin{aligned} \phi\big(d\big(g(z_{n+1}), g(z^*)\big)\big) \leq\ & \frac{1}{3}\phi\big(d(g(z_n), g(z)) + d(g(y_n), g(y)) + d(g(x_n), g(x))\big) \\ & -\psi\Big(\frac{d(g(z_n), g(z)) + d(g(y_n), g(y)) + d(g(x_n), g(x))}{3}\Big), \end{aligned}$$

By taking the limit of the above inequalities, we get $g(x) = g(x^*)$, $g(y) = g(y^*)$ and $g(z) = g(z^*)$. Hence, from (3.44), (3.45), (3.46), we get

$$d\big(g(x), F(g(x), g(y), g(z))\big) = d(A,B), \quad d(g(y), F\big(g(y), g(x), g(z))\big) = d(A,B),$$

and $d\big(g(z),F(g(z),g(y),g(x))\big)=d(A,B)$. □

Remark 3.10. *Corollary 3.7 holds true if we replace the continuity of F by the condition (e) of Theorem 3.9.*

One can prove that the tripled best proximity point is in fact unique, provided that the product space $A\times A$ endowed with the partial order mentioned earlier has the following property:

$$Every\ pair\ of\ elements\ has\ either\ a\ lower\ bound\ or\ an\ upper\ bound.$$

It is known that this condition is equivalent to the following. For every pair of $(x,y,z),(x^*,y^*,z^*)\in A\times A\times A$, there exists $(u,v,w)\in A\times A\times A$ that is comparable to (x,y,z) and (x^*,y^*,z^*).

Theorem 3.11. *Suppose that all the hypotheses of Theorem 3.6 hold and further, for all (x,y,z) and $(x^*,y^*,z^*)\in A_0\times A_0\times A_0$, there exists $(u,v,w)\in A_0\times A_0\times A_0$ such that (u,v,w) is comparable to (x,y,z), (x^*,y^*,z^*) (with respect to the ordering in $A\times A\times A$). Then there exists a unique $(x,y,z)\in A\times A\times A$ such that $d\big(g(x),F\big(g(x),g(y),g(z)\big)\big)=d(A,B)$, $d\big(g(y),F\big(g(y),g(x),g(z)\big)\big)=d(A,B)$ and $d\big(g(z),F\big(g(z),g(y),g(x)\big)\big)=d(A,B)$.*

Proof. In Theorem3.6, there exists an element $(x,y,z)\in A\times A\times A$ such that

$$d\big(g(x),F\big(g(x),g(y),g(z)\big)\big)=d(A,B), \tag{3.47}$$

$$d\big(g(y),F\big(g(y),g(x),g(z)\big)\big)=d(A,B), \tag{3.48}$$

and

$$d\big(g(z),F\big(g(z),g(y),g(x)\big)\big)=d(A,B). \tag{3.49}$$

Now, suppose that there exists an element $x^*,y^*,z^*\in A\times A\times A$ such that

$$d\big(g(x^*),F(g(x^*),g(y^*),g(z^*))\big)=d(A,B), \tag{3.50}$$

$$d\big(g(y^*),F(g(y^*),g(x^*),g(z^*))\big)=d(A,B) \tag{3.51}$$

and

$$d\big(g(z^*),F(g(z^*),g(y^*),g(x^*))\big)=d(A,B). \tag{3.52}$$

First, let $(g(x),g(y),g(z))$ be comparable to $(g(x^*),g(y^*),g(z^*))$ with respect to the ordering in $A\times A\times A$.

Since $d\big(g(x),F(g(x),g(y),g(z))\big)=d(A,B)$ and $d\big(g(x^*),F(g(x^*),g(y^*),g(z^*))\big)=d(A,B)$ it follows from the fact that F is a proximally tripled weak (ψ,ϕ) contraction on A, we get

$$\begin{aligned}\phi\left(d\big(g(x),g(x^*)\big)\right) &\leq \frac{1}{3}\phi\left(d(g(x),g(x^*))+d(g(y),g(y^*))+d(g(z),g(z^*))\right)\\ &\quad -\psi\Big(\frac{d(g(x),g(x^*))+d(g(y),g(y^*))+d(g(z),g(z^*))}{3}\Big),\end{aligned} \tag{3.53}$$

$$\begin{aligned}\phi\left(d\big(g(y),g(y^*)\big)\right) &\leq \frac{1}{3}\phi\left(d(g(y),g(y^*))+d(g(x),g(x^*))+d(g(z),g(z^*))\right)\\ &\quad -\psi\Big(\frac{d(g(y),g(y^*))+d(g(x),g(x^*))+d(g(z),g(z^*))}{3}\Big),\end{aligned} \tag{3.54}$$

$$\begin{aligned}\phi\left(d\big(g(z),g(z^*)\big)\right) &\leq \frac{1}{3}\phi\left(d(g(z),g(z^*))+d(g(y),g(y^*))+d(g(x),g(x^*))\right)\\ &\quad -\psi\Big(\frac{d(g(z),g(z^*))+d(g(y),g(y^*))+d(g(x),g(x^*))}{3}\Big).\end{aligned} \tag{3.55}$$

Adding (3.53), (3.54), (3.55), we get

$$\begin{aligned}&\phi\left(d\big(g(x),g(x^*)\big)\right)+\phi\left(d\big(g(y),g(y^*)\big)\right)+\phi\left(d\big(g(z),g(z^*)\big)\right)\\ &\leq \phi\left(d(g(x),g(x^*))+d(g(y),g(y^*))+d(g(z),g(z^*))\right)\\ &\quad -3\psi\Big(\frac{d(g(x),g(x^*))+d(g(y),g(y^*))+d(g(z),g(z^*))}{3}\Big).\end{aligned} \tag{3.56}$$

By the definition of ϕ, we have

$$\begin{aligned}&\phi\left(d(g(x),g(x^*))+d(g(y),g(y^*))+d(g(z),g(z^*))\right)\\ &\leq \phi\left(d\big(g(x),g(x^*)\big)\right)+\phi\left(d\big(g(y),g(y^*)\big)\right)+\phi\left(d\big(g(z),g(z^*)\big)\right).\end{aligned} \tag{3.57}$$

From (3.56) and (3.57), we have

$$\begin{aligned}
&\phi\big(d(g(x),g(x^*))+d(g(y),g(y^*))+d(g(z),g(z^*))\big)\\
&\leq \ \phi\big(d(g(x),g(x^*))+d(g(y),g(y^*))+d(g(z),g(z^*))\big)\\
&\quad -3\psi\Big(\frac{d(g(x),g(x^*))+d(g(y),g(y^*))+d(g(z),g(z^*))}{3}\Big).
\end{aligned} \tag{3.58}$$

this implies that $3\psi\Big(\frac{d(g(x),g(x^*))+d(g(y),g(y^*))+d(g(z),g(z^*))}{3}\Big) \leq 0$ and using the property of ψ, we get $d(g(x),g(x^*))+d(g(y),g(y^*))+d(g(z),g(z^*))=0$, hence $gx=gx^*$, $gy=gy^*$ and $gz=gz^*$.

Second, let $(g(x),g(y),g(z))$ is not comparable to $(g(x^*),g(y^*),g(z^*))$, then there exists $(g(u_1),g(v_1),g(w_1)) \in A_0 \times A_0 \times A_0$ which is comparable to $(g(x),g(y),g(z))$ and $(g(x^*),g(y^*),g(z^*))$. Since $F(A_0,A_0,A_0) \subseteq B_0$ and $g(A_0)=A_0$, there exists $(g(u_2),g(v_2),g(w_2)) \in A_0 \times A_0 \times A_0$ such that $d\big(g(u_2),F(g(u_1),g(v_1),g(w_1))\big)=d(A,B)$, $d\big(g(v_2),F(g(v_1),g(u_1),g(w_1))\big)=d(A,B)$ and $d\big(g(w_2),F(g(w_1),g(v_1),g(u_1))\big)=d(A,B)$.

We assume, without loss of generality, that $(g(u_1),g(v_1),g(w_1)) \leq (g(x),g(y),g(z))$, i.e., $g(u_1) \leq g(x)$ and $g(v_1) \geq g(y)$ and $g(w_1) \leq g(z)$. Therefore $(g(y),g(x),g(z)) \leq (g(v_1),g(u_1),g(w_1))$ and $(g(w_1),g(v_1),g(u_1)) \leq (g(z),g(y),g(x))$. From Lemma 3.3 and Lemma 3.4, we get

$$\begin{cases}
g(u_1) \leq g(x), \quad g(v_1) \geq g(y), \quad g(w_1) \leq g(z),\\
d\big(g(u_2),F(g(u_1),g(v_1),g(w_1))\big)=d(A,B) \\
d\big(g(x),F(g(x),g(y),g(z))\big)=d(A,B)
\end{cases} \Longrightarrow g(u_2) \leq g(x),$$

$$\begin{cases}
g(u_1) \leq g(x) \quad g(v_1) \geq g(y), \quad g(w_1) \leq g(z),\\
d\big(g(v_2),F(g(v_1),g(u_1),g(w_1))\big)=d(A,B) \\
d\big(g(y),F(g(y),g(x),g(z))\big)=d(A,B)
\end{cases} \Longrightarrow g(v_2) \geq g(y),$$

$$\begin{cases}
g(u_1) \leq g(x), \quad g(v_1) \geq g(y), \quad g(w_1) \leq g(z),\\
d\big(g(w_2),F(g(w_1),g(v_1),g(u_1))\big)=d(A,B) \\
d\big(g(z),F(g(z),g(y),g(x))\big)=d(A,B)
\end{cases} \Longrightarrow g(w_2) \leq g(z).$$

On continuing this process, we construct sequences $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ such that

$$\begin{aligned} d\big(g(u_{n+1}),F(g(u_n),g(v_n),g(w_n))\big) &= d(A,B)\\ d\big(g(v_{n+1}),F(g(v_n),g(u_n),g(w_n))\big) &= d(A,B)\\ d\big(g(w_{n+1}),F(g(w_n),g(v_n),g(x_n))\big) &= d(A,B) \end{aligned}$$

with $(g(u_n),g(v_n),g(w_n)) \leq (g(x),g(y),g(z))$. By using the fact that F is a proximally tripled weak (ψ,ϕ) contraction on A, we get

$$\begin{cases} g(u_n) \leq g(x) \quad g(v_n) \geq g(y), \quad g(w_n) \leq g(z)\\ d\big(g(u_{n+1}),F(g(u_n),g(v_n),g(w_n))\big) = d(A,B)\\ d\big(g(x),F(g(x),g(y),g(z))\big) = d(A,B) \end{cases}$$

$$\begin{aligned} \Longrightarrow \phi\big(d\big(g(u_{n+1}),g(x)\big)\big) \;\; &\leq \;\; \frac{1}{3}\phi\big(d(g(u_n),g(x))+d(g(v_n),g(y))+d(g(w_n),g(z))\big)\\ &\quad -\psi\Big(\frac{d(g(u_n),g(x))+d(g(v_n),g(y))+d(g(w_n),g(z))}{3}\Big). \end{aligned} \tag{3.59}$$

Similarly, we have

$$\begin{cases} g(u_n) \leq g(x) \quad g(v_n) \geq g(y), \quad g(w_n) \leq g(z)\\ d\big(g(v_{n+1}),F(g(v_n),g(u_n),g(w_n))\big) = d(A,B)\\ d\big(g(y),F(g(y),g(x),g(z))\big) = d(A,B) \end{cases}$$

$$\begin{aligned} \Longrightarrow \phi\big(d\big(g(v_{n+1}),g(y)\big)\big) \;\; &\leq \;\; \frac{1}{3}\phi\big(d(g(v_n),g(y))+d(g(u_n),g(x))+d(g(w_n),g(z))\big)\\ &\quad -\psi\Big(\frac{d(g(v_n),g(y))+d(g(u_n),g(x))+d(g(w_n),g(z))}{3}\Big), \end{aligned} \tag{3.60}$$

$$\begin{cases} g(u_n) \leq g(x) \quad g(v_n) \geq g(y), \quad g(w_n) \leq g(z)\\ d\big(g(w_{n+1}),F(g(w_n),g(v_n),g(u_n))\big) = d(A,B)\\ d\big(g(z),F(g(z),g(y),g(x))\big) = d(A,B) \end{cases}$$

$$\Longrightarrow \phi\big(d\big(g(w_{n+1}),g(z)\big)\big) \;\le\; \frac{1}{3}\phi\big(d(g(w_n),g(z))+d(g(v_n),g(y))+d(g(u_n),g(x))\big) -\psi\Big(\frac{d(g(w_n),g(z))+d(g(v_n),g(y))+d(g(u_n),g(x))}{3}\Big). \tag{3.61}$$

Adding (3.59), (3.60) and (3.61), we obtain

$$\begin{aligned}
&\phi\big(d\big(g(u_{n+1}),g(x)\big)\big)+\phi\big(d\big(g(v_{n+1}),g(y)\big)\big)+\phi\big(d\big(g(w_{n+1}),g(z)\big)\big)\\
&\le \phi\big(d(g(u_n),g(x))+d(g(v_n),g(y))+d(g(w_n),g(z))\big)\\
&\quad -3\psi\Big(\frac{d(g(u_n),g(x))+d(g(v_n),g(y))+d(g(w_n),g(z))}{3}\Big).
\end{aligned}$$

But

$$\begin{aligned}
&\phi\big(d\big(g(u_{n+1}),g(x)\big)+d\big(g(v_{n+1}),g(y)\big)+d\big(g(w_{n+1}),g(z)\big)\big)\\
&\le \phi\big(d\big(g(u_{n+1}),g(x)\big)\big)+\phi\big(d\big(g(v_{n+1}),g(y)\big)\big)+\phi\big(d\big(g(w_{n+1}),g(z)\big)\big),
\end{aligned}$$

hence

$$\begin{aligned}
&\phi\big(d\big(g(u_{n+1}),g(x)\big)+d\big(g(v_{n+1}),g(y)\big)+d\big(g(w_{n+1}),g(z)\big)\big)\\
&\le \phi\big(d(g(u_n),g(x))+d(g(v_n),g(y))+d(g(w_n),g(z))\big)\\
&\quad -3\psi\Big(\frac{d(g(u_n),g(x))+d(g(v_n),g(y))+d(g(w_n),g(z))}{3}\Big).
\end{aligned} \tag{3.62}$$

Using the fact that ϕ is nondecreasing, we get

$$\begin{aligned}
&d\big(g(u_{n+1}),g(x)\big)+d\big(g(v_{n+1}),g(y)\big)+d\big(g(w_{n+1}),g(z)\big)\\
&\le d\big(g(u_n),g(x)\big)+d\big(g(v_n),g(y)\big)+d\big(g(w_n),g(z)\big).
\end{aligned}$$

Therefore $d\big(g(u_n),g(x)\big)+d\big(g(v_n),g(y)\big)+d\big(g(w_n),g(z)\big)$ is a decreasing sequence. Hence there exists $r \ge 0$ such that

$$\lim_{n\to\infty}\big[d\big(g(u_n),g(x)\big)+d\big(g(v_n),g(y)\big)+d\big(g(w_n),g(z)\big)\big]=r.$$

We shall show that $r = 0$. Suppose, to the contrary, that $r > 0$. On taking the limit as $n \to \infty$ in (3.62), we have

$$\phi(r) \le \phi(r) - 3 \lim_{n\to\infty} \psi\Big(\frac{d(g(u_n), g(x)) + d(g(v_n), g(y))}{3}\Big) < \phi(r),$$

which is a contradiction. Hence, $r = 0$, that is,

$$\lim_{n\to\infty} \big[d\big(g(u_n), g(x)\big) + d\big(g(v_n), g(y)\big) + d\big(g(w_n), g(z)\big)\big] = 0,$$

so that $g(u_n) \to g(x)$, $g(v_n) \to g(y)$ and $g(w_n) \to g(z)$. Analogously, one can prove that $g(u_n) \to g(x^*)$, $g(v_n) \to g(y^*)$ and $g(w_n) \to g(z^*)$. Therefore, $g(x) = g(x^*)$, $g(y) = g(y^*)$ and $g(z) = g(z^*)$. Hence the proof is complete. □

Considering g is assumed to be the identity mappings in Theorem 3.11 then we obtained the following result.

Corollary 3.12. *Suppose that all the hypotheses of Corollary 3.7 hold and further, for all $(x,y,z), (x^*,y^*,z^*) \in A_0 \times A_0 \times A_0$, there exists $(u,v,w) \in A_0 \times A_0 \times A_0$ such that (u,v,w) is comparable to (x,y,z), (x^*,y^*,z^*) (with respect to the ordering in $A \times A \times A$). Then there exists a unique $(x,y,z) \in A \times A \times A$ such that $d\big(x, F(x,y,z)\big) = d(A,B)$ and $d\big(y, F(y,x,z)\big) = d(A,B)$ and $d\big(z, F(z,y,x)\big) = d(A,B)$.*

If A = B in Theorem 3.11, we obtained the result of tripled fixed point.

Corollary 3.13. *Suppose that all the hypotheses of Corollary 3.8 hold and further, for all $(x,y,z), (x^*,y^*,z^*) \in A \times A \times A$, there exists $(u,v,w) \in A \times A \times A$ such that (u,v,w) is comparable to (x,y,z), (x^*,y^*,z^*) (with respect to the ordering in $A \times A \times A$). Then there exists a unique $(x,y,z) \in A \times A \times A$ such that*

$$d\big(g(x), F\big(g(x), g(y), g(z)\big)\big) = 0,\ d\big(g(y), F\big(g(y), g(x), g(z)\big)\big) = 0$$

$$\textit{and } d\big(g(z), F\big(g(z), g(y), g(x)\big)\big) = 0.$$

We shall illustrate our results by the following example.

Example 3.14. *Let* $X = \mathbb{R}$ *and* $d(x,y) = \mid x-y \mid$ *be the usual metric on* X *and let the usual ordering* $(x,y,z) \leq (u,v,w) \Leftrightarrow x \leq u, y \geq v, z \leq w$. *Assume that* $A = [1,\infty)$ *and* $B = (\infty,-1]$ *and* A, B *are nonempty closed subsets of* X. *We also have* $A_0 = \{1\}$ *and* $B_0 = \{-1\}$ *and d(A,B) = 2. Let* $F : A \times A \times A \to B$ *and* $g : A \to A$ *be two mappings such that* $F(x,y,z) = -\frac{x+y+z}{3}$ *and* $g(x) = x^2$. *Then F and g are continuous and F(1,1,1) = -1 and g(1) = 1, i.e.,* $F(A_0,A_0,A_0) \subseteq B_0$ *and* $g(A_0) = A_0$. *Notice that the all the hypotheses of Theorem 3.11 are satisfied, then there exists a unique point* $(1,1,1) \in A \times A \times A$ *such that* $d\big(g(1),F(g(1),g(1),g(1))\big) = 2 = d(A,B)$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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