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GENERALIZATIONS OF BANACH-KANNAN-CHATTERJEA TYPE FIXED POINT THEOREMS ON NON-NORMAL CONE METRIC SPACES WITH BANACH ALGEBRAS

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Abstract. In this paper, we obtain generalizations of Banach contraction principle, Kannan fixed point theorem and Chatterjea fixed point theorem for mappings satisfying Suzuki type contractive conditions on cone metric spaces over Banach algebras without the assumption of normality. The obtained results generalize and improve the corresponding conclusions in the literature.

Keywords: cone metric spaces with Banach algebras; fixed point; Suzuki type contraction.

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1. Introduction

In 2007, cone metric spaces were reviewed by Huang and Zhang, as a generalization of metric spaces (see [1]). The distance $d(x,y)$ of two elements x and y in a cone metric space X is defined to be a vector in an ordered Banach space E, quite different from that which is defined a non-negative real numbers in general metric space. In 2011, I. Beg, A. Azam and M. Arshad([2])

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introduced the concept of topological vector space-valued cone metric spaces, where the ordered Banach space in the definition of cone metric spaces is replaced by a topological vector space.

Recently, some authors investigated the problems of whether cone metric spaces are equivalent to metric spaces in terms of the existence of fixed points of the mappings and successfully established the equivalence between some fixed point results in metric spaces and in (topological vector space-valued) cone metric spaces, see [3-6]. Actually, they showed that any cone metric space (X,d) is equivalent to a usual metric space (X,d^*), where the real-metric function d^* is defined by a nonlinear scalarization function ξ_e(see [4]) or by a Minkowski function q_e(see[5]). After that, some other interesting generalizations were developed, see [7].

In 2013, Liu and Xu [8] introduced the concept of cone metric spaces over Banach algebras, replacing a Banach space E by a Banach algebra $\mathscr{A}$ as the underlying spaces of cone metric spaces. The authors in [8-10] discussed and obtained Banach fixed point theorem, Kannan type fixed point theorem, Chatterjea type fixed point theorem and ćirić type fixed point theorem in cone metric spaces over Banach algebras. Especially, the authors in [10] gave an example to show that fixed point results of mappings in this new space are indeed more different than the standard results of come metric spaces presented in literature.

In 1968, Kannan[11] obtained the generalization of Banach contractive principle, that is, Kannan fixed point theorem:

Theorem 1.1. Let X be a metric space and $f: X \to X$ a mapping. If there is a $\alpha \in [0, \frac{1}{2})$ such that for each $x, y \in X$,

$$d(fx, fy) \leq \alpha\,[d(x, fx) + d(y, fy)].$$

Then f has a unique fixed point.

In 2011, Shukla and Tiwari[12] obtained the variant result of Kannan fixed point theorem:

Theorem 1.2. Let X be a metric space and $f: X \to X$ a mapping. If there is a $\alpha \in [0, \frac{1}{3})$ such that for each $x, y \in X$,

$$d(fx, fy) \leq \alpha\,[d(x, fx) + d(y, fy) + d(x, y)].$$

Then f has a unique fixed point.

In 1972, Chatterjea[13] obtained the another generalization of Banach contractive principle, that is, Chatterjea fixed point theorem:

Theorem 1.3. Let X be a metric space, $f : X \to X$ a mapping. If there is a $\alpha \in [0, \frac{1}{2})$ such that for each $x, y \in X$,

$$d(fx, fy) \leq \alpha\,[d(x, fy) + d(y, fx)].$$

Then f has a unique fixed point.

2. Preliminaries

Let $\mathscr{A}$ always be a Banach algebra. That is, $\mathscr{A}$ is a real Banach space in which an operation of multiplication is defined, subject to the following properties(for all $x, y, z \in \mathscr{A}$, $\alpha \in \mathbb{R}$):

1. $(xy)z = x(yz)$;
2. $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;
3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
4. $\| xy \| \leq \| x \| \| y \|$.

In this paper, we shall assume that a Banach algebra has a unit (i.e., a multiplicative identity) e such that $ex = xe = x$ for all $x \in \mathscr{A}$. an element $x \in \mathscr{A}$ is said to be invertible if there is an inverse element $y \in A$ such that $xy = yx = e$. The inverse of x denoted by x^{-1}. For more detail, we refer to [14].

We say that the set $\{x_1, x_2, \cdots, x_n\} \subset \mathscr{A}$ commute if $x_i x_j = x_j x_i$ for all $i, j \in \{1, 2, \cdots, n\}$.

Proposition 2.1.[14] Let $\mathscr{A}$ be a Banach algebra with a unit e, and $x \in \mathscr{A}$. If the spectral radius $r(x)$ of x is less than 1, i.e.,

$$r(x) = \lim_{n\to\infty} \| x^n \|^{\frac{1}{n}} = \inf_{n\to\infty} \| x^n \|^{\frac{1}{n}} < 1.$$

Then $(e - x)$ is invertible. Actually,

$$(e-x)^{-1} = \sum_{i=0}^{+\infty} x^i.$$

Remark 2.1. 1) $r(x) \leq \| x \|$ for any $x \in \mathscr{A}$(see [14]).

2) In Proposition 2.1, if the condition $r(x) < 1$ is replaced by the condition $\| x \| < 1$, then the conclusion remains true.

A subset P of a Banach algebra $\mathscr{A}$ is called a cone if

1. P is nonempty closed and $\{0, e\} \subset P$;
2. $\alpha P + \beta P \subset P$ for all non-negative real numbers α, β;
3. $P^2 = PP \subset P$;
4. $P \cap (-P) = \{\mathbf{0}\}$.

Where $\mathbf{0}$ denotes the null of the Banach algebra $\mathscr{A}$.

For a given cone $P \subset \mathscr{A}$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. $x < y$ stand for $x \leq y$ and $x \neq y$. While $x \ll y$ sill stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. A cone P is called solid if $\operatorname{int} P \neq \emptyset$.

The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in \mathscr{A}$.

$$0 \leq x \leq y \implies \| x \| \leq M \| y \| .$$

The least positive number satisfying the above is called the normal constant of P.

Here, we always assume that P is a solid and $\leq$ is the partial ordering with respect to P.

Definition 2.1.[1, 9-10] Let X be a non-empty set. Suppose that the mapping $d : X \times X \to \mathscr{A}$ satisfies

1. $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space(over a Banach algebra $\mathscr{A}$).

Remark 2.2. The examples of cone metric spaces(over a Banach algebra $\mathscr{A}$) can be found in [8-10].

Definition 2.2.[1, 8] Let (X, d) be a cone metric space over a Banach algebra $\mathscr{A}$, $x \in X$ and $\{x_n\}$ a sequence in X. Then:

1. $\{x_n\}$ converges to x whenever for each $c \in \mathscr{A}$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$.

2. $\{x_n\}$ is Cauchy sequence whenever for each $c \in \mathscr{A}$ with $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

3. (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

Definition 2.3.[15-16] Let P is a solid cone in a Banach space $\mathscr{A}$. A sequence $\{u_n\} \subset P$ is a c-sequence if for each $c \gg 0$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for all $n \geq n_0$.

Proposition 2.2.[15] Let P is a solid cone in a Banach space $\mathscr{A}$ and let $\{x_n\}$ and $\{y_n\}$ be sequences in P. If $\{x_n\}$ and $\{y_n\}$ are c-sequences and $\alpha, \beta > 0$, then $\{\alpha x_n + \beta y_n\}$ is a c-sequence.

Proposition 2.3.[15] Let P is a solid cone in a Banach algebra $\mathscr{A}$ and $\{x_n\}$ a sequence in P. Then the following conditions are equivalent:

(1) $\{x_n\}$ is a c-sequence;

(2) for each $c \gg 0$there exists $n_0 \in \mathbb{N}$ such that $x_n < c$ for all $n \geq n_0$;

(3) for each $c \gg 0$there exists $n_1 \in \mathbb{N}$ such that $x_n \leq c$ for all $n \geq n_1$.

Proposition 2.4.[10] Let P is a solid cone in a Banach algebra $\mathscr{A}$ and $\{u_n\}$ a sequence in P. Suppose that $k \in P$ is an arbitrarily given vector and $\{u_n\}$ is a c-sequence in P. Then $\{ku_n\}$ is a c-sequence.

Proposition 2.5.[10]. Let $\mathscr{A}$ be a Banach algebra with a unit e, P a cone in $\mathscr{A}$ and $\leq$ be the semi-order generated by the cone P. The following assertions hold true:

(i) For any $x, y \in \mathscr{A}$, $a \in P$ with $x \leq y$, $ax \leq ay$;

(ii) For any sequences $\{x_n\}, \{y_n\} \subset \mathscr{A}$ with $x_n \to x$ and $y_n \to y$ as $n \to \infty$, where $x, y \in \mathscr{A}$, we have $x_n y_n \to xy$ as $n \to \infty$.

Proposition 2.6.[10] Let $\mathscr{A}$ be a Banach algebra with a unit e, P a cone in $\mathscr{A}$ and $\leq$ be the semi-order generated by the cone P. Let $\lambda \in P$. If the spectral radius $r(\lambda)$ of λ is less than 1, then the following assertions hold true:

(i) Suppose that x is invertible and that $x^{-1} > 0$ implies $x > 0$, then for any integer $n \geq 1$, we have $\lambda^n \leq \lambda \leq e$.

(ii) For any $u > 0$, we have $u \nleq \lambda u$, i.e., $\lambda u - u \notin P$.

(iii) If $\lambda \geq 0$, then $(e - \lambda)^{-1} \geq 0$.

Proposition 2.7.[10] Let (X,d) be a complete cone metric space over a Banach algebra $\mathscr{A}$ and P a solid cone in Banach algebra A. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to $x \in X$, then we have

(i) $\{d(x_n,x)\}$ is a c-sequence.

(ii) For any $p \in \mathbb{N}$, $\{d(x_n,x_{n+p})\}$ is a c-sequence.

Lemma 2.1.[17] If E is a real Banach space with a cone P and if $a \leq \lambda a$ with $a \in P$ and $0 \leq \lambda < 1$, then $a = 0$.

Lemma 2.2.[18] If E is a real Banach space with a cone P and if $0 \leq u \ll c$ for all $0 \ll c$, then $u = 0$.

Lemma 2.3.[18] If E is a real Banach space with a solid cone P and if $\| x_n \| \to 0$ as $n \to \infty$, then for any $0 \ll c$, there exists $N \in \mathbb{N}$ such that, for any $n > N$, we have $x_n \ll c$.

Lemma 2.4.[10] If $\mathscr{A}$ is a Banach algebra and $k \in \mathscr{A}$ with $r(k) < 1$, then $\| k^n \| \to 0$ as $n \to \infty$.

Lemma 2.5.[10] Let A be a Banach algebra and $x, y \in \mathscr{A}$. If x and y commute, then the following hold:

(i) $r(xy) \leq r(x)r(y)$;

(ii) $r(x+y) \leq r(x) + r(y)$;

(iii) $| r(x) - r(y) | \leq r(x-y)$.

Lemma 2.6.[10] Let A be a Banach algebra and $\{x_n\}$ a sequence in $\mathscr{A}$. Suppose that $\{x_n\}$ converge to $x \in \mathscr{A}$ and that x_n and x commute for all n, then $r(x_n) \to r(x)$ as $n \to \infty$.

Lemma 2.7.[19-20] Let $\mathscr{A}$ be a Banach algebra and $\{\alpha,\beta,\gamma\} \subset \mathscr{A}$ with $r(\gamma) < 1$. If $\{\alpha,\beta,\gamma\}$ commute, then

$$r\Big((e-\gamma)^{-1}(\alpha+\beta)\Big) \leq \frac{r(\alpha+\beta)}{1-r(\gamma)} \leq \frac{r(\alpha)+r(\beta)}{1-r(\gamma)}.$$

Lemma 2.8.[19-20] (Cauchy Principle) Let (X,d) be a cone metric space over a Banach algebra $\mathscr{A}$, $k \in P$ with $r(k) < 1$. If a sequence $\{x_n\} \subset X$ satisfies that

$$d(x_{n+1},x_{n+2}) \leq kd(x_n,x_{n+1}), \forall n = 0,1,2,\cdots.$$

Then $\{x_n\}$ is a Cauchy sequence.

Lemma 2.9.[19-20] Let (X,d) be a cone metric space over a Banach algebra $\mathscr{A}$, $\{x_n\} \subset X$ a sequence. If $\{x_n\}$ is convergent, then the limits of $\{x_n\}$ is unique.

Xu and Radenović [10] obtained Banach, Kannan and Chatterjea type fixed point theorems on cone metric spaces over Banach algebras without the assumption of normality. These results generalize and improve Banach contraction principle, Theorem 1.1 and Theorem 1.3.

In this paper, we discuss the existence problems of fixed points for mappings satisfying Suzuki type contractive conditions on cone metric spaces over Banach algebras without the assumption of normality. Our results further generalize Theorem 1.1-1.3 and the conclusions in [8-10] and others.

3. Fixed point results

Theorem 3.1. Let (X,d) be a complete cone metric space over a Banach algebra $\mathscr{A}$, $f: X \to X$ a mapping. Suppose that there exist four commutable elements $\{\alpha,\beta,\gamma,\delta\} \subset P$ satisfying $e \geq \alpha$ and $r(\alpha) \leq \frac{1}{2}$ and $r(\beta)+r(\gamma)+r(\delta) < 1$ such that $\alpha d(x,fx) \leq d(x,y)$ for $x,y \in X$ with $x \neq y$ implies

$$d(fx,fy) \leq \beta d(x,fx) + \gamma d(y,fy) + \delta d(x,y). \tag{3.1}$$

If $d(X \times X)$ is a totally ordered subset of $\mathscr{A}$, then f has a unique fixed point u and $\lim_{n\to\infty} f^n x = u$ for all $x \in X$.

Proof. Take any element $x_0 \in X$ and let $x_1 = fx_0$. If $x_1 = x_0$, then x_0 is a fixed point of f, hence we assume that $x_1 \neq x_0$. Since $(e-\alpha)d(x_0,fx_0) \geq 0$, i.e., $\alpha d(x_0,fx_0)) \leq d(x_0,x_1)$, by (3.1),

$$d(fx_0,fx_1) \leq \beta d(x_0,fx_0) + \gamma d(x_1,fx_1) + \delta d(x_0,x_1).$$

Let $x_2 = fx_1$, then

$$d(x_1,x_2) \leq \beta d(x_0,x_1) + \gamma d(x_1,x_2) + \delta d(x_0,x_1).$$

Since $r(\gamma) < 1$ implies $(e-\gamma)$ is invertible and $(e-\gamma)^{-1} \geq 0$, we obtain

$$d(x_1,x_2) \leq (e-\gamma)^{-1}(\beta+\delta) d(x_0,x_1). \tag{3.2}$$

If $x_2 = x_1$, then x_1 is a fixed point of f, hence we assume that $x_2 \neq x_1$. Similarly, since $(e-\alpha)d(x_1, fx_1) \geq 0$, i.e,, $\alpha d(x_1, fx_1) \leq d(x_1, x_2)$, by (3.1),

$$d(fx_1, fx_2) \leq \beta d(x_1, fx_1) + \gamma d(x_2, fx_2) + \delta d(x_1, x_2). \tag{3.3}$$

Let $x_3 = fx_2$, then we obtain

$$d(x_2, x_3) \leq (e-\gamma)^{-1}(\beta+\delta)d(x_1, x_2).$$

If $x_3 = x_2$, then x_2 is a fixed point of f, hence we assume $x_3 \neq x_2$. Repeating this process, we obtain a sequence $\{x_n\}$ satisfying

$$x_{n+1} = fx_n, x_{n+1} \neq x_n, d(x_{n+1}, x_{n+2}) \leq (e-\gamma)^{-1}(\beta+\delta)d(x_n, x_{n+1}), \forall n = 0,1,2,\cdots. \tag{3.4}$$

Since $r((e-\gamma)^{-1}(\beta+\delta)) \leq \frac{r(\beta)+r(\delta)}{1-(\gamma)} < 1$ by Lemma 2.7, $\{x_n\}$ is Cauchy by Lemma 2.8 and (3.4). Hence there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$ by the completeness of X.

We claim that for any $n = 0,1,2,\cdots$, one of the following relations holds:

$$\alpha d(x_n, fx_n) \leq d(x_n, u), \alpha d(x_{n+1}, fx_{n+1}) \leq d(x_{n+1}, u). \tag{3.5}$$

Otherwise, by the property of $d(X \times X)$, there exists n such that $\alpha d(x_n, fx_n) > d(x_n, u)$ and $\alpha d(x_{n+1}, fx_{n+1}) > d(x_{n+1}, u)$, hence using (3.4), we obtain

$$\begin{aligned} & d(x_n, x_{n+1}) \\ \leq & d(x_n, u) + d(x_{n+1}, u) \\ < & \alpha d(x_n, fx_n) + \alpha d(x_{n+1}, fx_{n+1}) \leq \alpha\big(e + (e-\gamma)^{-1}(\beta+\delta)\big)d(x_n, x_{n+1}), \end{aligned}$$

that is,

$$\Big[e - \Big(\alpha\big(e + (e-\gamma)^{-1}(\beta+\delta)\big)\Big)\Big]d(x_n, x_{n+1}) < 0. \tag{3.6}$$

Since $\{\alpha, \beta, \gamma, \delta\}$ commute, by Lemma 2.5 and Lemma 2.7,

$$r\Big(\alpha\big(e + (e-\gamma)^{-1}(\beta+\delta)\big)\Big) = r(\alpha + \alpha(e-\gamma)^{-1}(\beta+\delta)) \leq r(\alpha) + \frac{r(\alpha)(r(\beta)+r(\delta))}{1-r(\gamma)} < 1,$$

hence $\Big[e - \Big(\alpha\big(e + (e-\gamma)^{-1}(\beta+\delta)\big)\Big)\Big]$ is invertible and $\Big[e - \Big(\alpha\big(e + (e-\gamma)^{-1}(\beta+\delta)\big)\Big)\Big]^{-1} \geq 0$ by Proposition 2.1 and Proposition 2.6, therefore $d(x_n, x_{n+1}) < 0$ by Proposition 2.5 and (3.6). This is a contradiction.

By (3.5), there exists an infinite sub-sequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\alpha d(x_{n_i}, fx_{n_i}) \le d(x_{n_i}, u)$ for all $i \in \mathbb{N}$, hence by (3.1), for each $i \in \mathbb{N}$,

$$\begin{aligned} & d(x_{n_i+1}, fu) \\ =& d(fx_{n_i}, fu) \\ \le& \beta d(x_{n_i}, fx_{n_i}) + \gamma d(u, fu) + \delta d(x_{n_i}, u) \\ =& \beta d(x_{n_i}, x_{n_i+1}) + \gamma d(u, fu) + \delta d(x_{n_i}, u), \end{aligned}$$

which implies that

$$d(x_{n_i+1}, fu) \le \beta d(x_{n_i}, x_{n_i+1}) + \gamma [d(u, x_{n_i+1}) + d(x_{n_i+1}, fu)] + \delta d(x_{n_i}, u), \forall i \in \mathbb{N},$$

hence

$$d(x_{n_i+1}, fu) \le (e-\gamma)^{-1} \beta d(x_{n_i}, x_{n_i+1}) + (e-\gamma)^{-1} \gamma d(u, x_{n_i+1}) + (e-\gamma)^{-1} \delta d(x_{n_i}, u), \forall i \in \mathbb{N}. \tag{3.7}$$

Since $\{x_n\}$ converges to u, the right-hand side of (3.7) is a c-sequence by Proposition 2.2, 2.4 and 2.7, hence $\{d(x_{n_i+1}, fu)\}$ is also a c-sequence. Therefore, it is easy to prove that $\{x_{n_i+1}\}$ converges to fu by Definition 2.3, so $fu = u$, i.e., u is the fixed point of f.

Suppose that v is another fixed point of f, then $u \ne v$. Since $\alpha d(u, fu) = 0 \le d(u, v)$, by (3.1),

$$d(u, v) = d(fu, fv) \le \beta d(u, fu) + \gamma d(v, fv) + \delta d(u, v) = \delta d(u, v),$$

that is

$$(e - \delta) d(u, v) \le 0.$$

Hence $u = v$ since $r(\delta) < 1$, therefore, u is the unique fixed point of f.

If $\beta = \gamma = 0$, then we obtain the following generalization of famous Banach contraction principle for Suzuki type contractive mappings in the setting of cone metric spaces over a Banach algebra without the assumption of normality of the underlying solid cone.

Theorem 3.2. Let (X, d) be a complete cone metric space over a Banach algebra $\mathscr{A}$, $f : X \to X$ a mapping. Suppose that there exist two commutable elements $\{\alpha, \beta\} \subset P$ satisfying $e \ge \alpha$ and

$r(\alpha) \leq \frac{1}{2}$ and $r(\beta) < 1$ such that $\alpha d(x, fx) \leq d(x,y)$ for $x, y \in X$ with $x \neq y$ implies

$$d(fx, fy) \leq \beta d(x,y). \tag{3.8}$$

If $d(X \times X)$ is a totally ordered subset of $\mathscr{A}$, then f has a unique fixed point u and $\lim_{n\to\infty} f^n x = u$ for all $x \in X$.

Remark 3.1. (1) if $\beta = \gamma$ and $\delta = 0$, then Theorem 3.1 is a new generalization of Theorem 1.1 and the corresponding results in [8,10]; if $\beta = \gamma = \delta$, then Theorem 3.2 is a new generalization of Theorem 1.2.

(2) If (3.1) and (3.8) holds for all $x, y \in X$ instead of all $x, y \in X$ with $\alpha d(x, dx) \leq d(x,y)$ and $x \neq y$, then Theorem 3.1 and Theorem 3.2 reduce to Theorem 1.1-1.2 and Banach type fixed point theorem respectively.

Example 3.1. Let $\mathscr{A} = C^1_{\mathbb{R}}[0,1]$ and define a norm on $\mathscr{A}$ by $\| x \| = \| x \|_\infty + \| x' \|_\infty$ for $x \in \mathscr{A}$. Define multiplication in $\mathscr{A}$ as just pointwise multiplication. Then $\mathscr{A}$ is a real Banach algebra with unit e, i.e., $e(t) = 1$ for all $t \in [0,1]$. The set $P = \{x \in \mathscr{A} : x \geq 0\}$ is not normal(see[10, 21]).

Let $X = \{a, b, c\}$ and define $d : X \times X \to \mathscr{A}$ as follows: for each $t \in [0,1]$ and $x \in X$,

$$d(a,b)(t) = d(b,a)(t) = 1.9e^t, d(a,c)(t) = d(c,a)(t) = 2.1e^t,$$

$$d(b,c)(t) = d(c,b)(t) = 0.2e^t, d(x,x)(t) = 0.$$

Then (X,d) is a complete cone metric space over a Banach algebra $\mathscr{A}$ without normality and $d(X \times X) = \{1.9e^t, 2.1e^t, 0.2e^t, 0\}$ is a totally ordered subset of $\mathscr{A}$.

Define a mapping $f : X \to X$ by $fa = a, fb = a, fc = b$. Let $\alpha, \beta \in P$ be $\alpha(t) = \frac{2}{5} + \frac{t}{15}, \beta(t) = \frac{98}{100} + \frac{t}{100}$ for all $t \in [0,1]$. It is easy to prove that $e(t) = 1 \geq \alpha(t)$ for all $t \in [0,1]$ and $r(\alpha) = \frac{7}{15} < \frac{1}{2}$, $r(\beta) = \frac{99}{100} < 1$.

Case (i) For $x = a, y = b$,

$$\alpha d(a, fa)(t) = \alpha(t)\, d(a,a)(t) = 0 \leq 1.9e^t = d(a,b)(t)$$

and

$$d(fa, fb)(t) = d(a,a)(t) = 0 \leq (\frac{98}{100} + \frac{t}{100}) \times 1.9e^t = \beta d(a,b)(t)$$

for all $t \in [0,1]$, hence (3.8) holds;

Case (ii) For $x=a, y=c$,

$$\alpha d(a,fa)(t)=0\leq 2.1e^t=d(a,c)(t)$$

and

$$d(fa,fc)(t)=d(a,b)(t)=1.9e^t\leq (\frac{98}{100}+\frac{t}{100})\times 2.1e^t=\beta d(a,c)(t)$$

for all $t\in[0,1]$, hence (3.8) holds;

Case (iii) For $x=b, y=c$,

$$\alpha d(b,fb)(t)=\alpha d(b,a)(t)=(\frac{2}{5}+\frac{t}{15})1.9e^t>0.2e^t=d(b,c)(t)$$

and

$$d(fb,fc)(t)=d(a,b)(t)=1.9e^t>(\frac{98}{100}+\frac{t}{100})\times 0.2e^t=\beta d(b,c)(t)$$

for all $t\in[0,1]$, hence (3.8) holds.

Hence f, α, β satisfy all of the conditions in Theorem 3.2, so f has a unique fixed point a.

On the other hand, the second inequality in case (iii) shows that f does not satisfy the contractive condition in Banach contraction principle. Hence Theorem 3.2 really generalizes the Banach fixed point theorem and other corresponding conclusion.

Theorem 3.3. Let (X,d) be a complete cone metric space over a Banach algebra $\mathscr{A}$, $f:X\to X$ a mapping. Suppose that there exist commutable elements $\{\alpha,\beta,\gamma,\delta\}\subset P$ satisfying $e\geq\alpha$ and $r(\alpha)\leq\frac{1}{2}$ and $2\max\{r(\beta),r(\gamma)\}+r(\delta)<1$ such that $\alpha d(x,fx)\leq d(x,y)$ for $x,y\in X$ with $x\neq y$ implies

$$d(fx,fy)\leq\beta d(x,fy)+\gamma d(y,fx)+\delta d(x,y). \tag{3.9}$$

If $d(X\times X)$ is a totally ordered subset of $\mathscr{A}$, then f has a unique fixed point u and $\lim_{n\to\infty}f^nx=u$ for all $x\in X$.

Proof. Take any element $x_0\in X$ and let $x_1=fx_0$. If $x_1=x_0$, then x_0 is a fixed point of f, hence we assume $x_1\neq x_0$. Since $(e-\alpha)d(x_0,fx_0)\geq 0$, i.e., $\alpha d(x_0,fx_0))\leq d(x_0,x_1)$, by (3.9),

$$d(fx_0,fx_1)\leq\beta d(x_0,fx_1)+\gamma d(x_1,fx_0)+\delta d(x_0,x_1)=\beta d(x_0,fx_1)+\delta d(x_0,x_1).$$

Let $x_2 = fx_1$, then we have

$$\begin{aligned} & d(x_1,x_2) \\ \leq & \beta d(x_0,x_2) + \delta d(x_0,x_1) \\ \leq & \beta [d(x_0,x_1) + d(x_1,x_2)] + \delta d(x_0,x_1) \\ = & (\beta + \delta) d(x_0,x_1) + \beta d(x_1,x_2), \end{aligned}$$

hence

$$(e - \beta) d(x_1,x_2) \leq (\beta + \delta) d(x_0,x_1). \tag{3.10}$$

Since $r(\beta) < 1$, $(e-\beta)$ is invertible and $(e-\beta)^{-1} \geq 0$, hence by (3.10), we obtain

$$d(x_1,x_2) \leq (e-\beta)^{-1}(\beta + \delta) d(x_0,x_1).$$

If $x_2 = x_1$, then x_1 is a fixed point of f, hence we assume $x_2 \neq x_1$. Similarly, since $(e - \alpha) d(x_1, fx_1) \geq 0$, i.e., $\alpha d(x_1, fx_1) \leq d(x_1,x_2)$, by (3.9),

$$d(fx_1, fx_2) \leq \beta d(x_1, fx_2) + \gamma d(x_2, fx_1) + \delta d(x_1,x_2).$$

Let $x_3 = fx_2$, then

$$d(x_2,x_3) \leq \beta d(x_1,x_3) + \delta d(x_1,x_2) \leq \beta [d(x_1,x_2) + d(x_2,x_3)] + \delta d(x_1,x_2). \tag{3.11}$$

Hence we obtain

$$d(x_2,x_3) \leq (e-\beta)^{-1}(\beta + \delta) d(x_1,x_2).$$

Repeating this process, we obtain a sequence $\{x_n\}$ satisfying

$$x_{n+1} = fx_n,\, x_{n+1} \neq x_n,\, d(x_{n+1},x_{n+2}) \leq (e-\beta)^{-1}(\beta + \delta) d(x_n,x_{n+1}), \forall n = 0,1,2,\cdots. \tag{3.12}$$

Since $r((e-\beta)^{-1}(\beta+\delta)) \leq \frac{r(\beta)+r(\delta)}{1-(\beta)} < 1$ by Lemma 2.7, $\{x_n\}$ is Cauchy by Lemma 2.8 and (3.12). Hence there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$ by the completeness of X.

Using the similar proof in Theorem 3.1, we can sure that (3.5) also holds. Hence there exists an infinite sub-sequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\alpha d(x_{n_i}, fx_{n_i}) \leq d(x_{n_i}, u)$ for all $i \in \mathbb{N}$ by (3.5).

In view of (3.9), for each $i \in \mathbb{N}$,

$$\begin{aligned} & d(x_{n_i+1}, fu) \\ =& d(fx_{n_i}, fu) \\ \leq& \beta d(x_{n_i}, fu) + \gamma d(u, fx_{n_i}) + \delta d(x_{n_i}, u) \\ =& \beta d(x_{n_i}, fu) + \gamma d(u, x_{n_i+1}) + \delta d(x_{n_i}, u), \end{aligned}$$

which implies that

$$d(x_{n_i+1}, fu) \leq \beta[d(x_{n_i}, x_{n_i+1}) + d(x_{n_i+1}, fu)] + \gamma d(u, x_{n_i+1}) + \delta d(x_{n_i}, u), \forall i \in \mathbb{N},$$

hence

$$d(x_{n_i+1}, fu) \leq (e-\beta)^{-1}\beta d(x_{n_i}, x_{n_i+1}) + (e-\beta)^{-1}\gamma d(u, x_{n_i+1}) + (e-\beta)^{-1}\delta d(x_{n_i}, u), \forall i \in \mathbb{N}. \tag{3.13}$$

Since $\{x_n\}$ converges to u, the right-hand side of (3.13) is a c-sequence by Proposition 2.2, 2.4 and 2.7, hence $\{d(x_{n_i+1}, fu)\}$ is also a c-sequence. Therefore, it is easy to prove that $\{x_{n_i+1}\}$ converges to fu by Definition 2.3, so $fu = u$, i.e., u is the fixed point of f.

Suppose that v is another fixed point of f, then $u \neq v$. Since $\alpha d(u, fu) = 0 \leq d(u, v)$, by (3.9),

$$d(u,v) = d(fu, fv) \leq \beta d(u, fv) + \gamma d(v, fu) + \delta d(u,v) = (\beta + \gamma + \delta) d(u,v),$$

that is,

$$[e - (\beta + \gamma + \delta)] d(u,v) \leq 0. \tag{3.14}$$

But $r(\beta + \gamma + \delta) \leq r(\beta) + r(\gamma) + r(\delta) < 1$, hence $[e - (\beta + \gamma + \delta)]$ is invertible and $[e - (\beta + \gamma + \delta)]^{-1} \geq 0$, therefore by Proposition 2.5 and (3.14),

$$d(u,v) = 0,$$

that is, $u = v$. Hence u is the unique fixed point of f.

Remark 3.2. If $\beta = \gamma = 0$, then Theorem 3.3 is the Banach type fixed point theorem; if $\beta = \gamma$ and $\delta = 0$, then Theorem 3.3 is a generalization of Theorem 1.3 and the corresponding results in [8,10]; if $\beta = \gamma = \delta$, then Theorem 3.3 is a new version and generalization of Theorem 1.3.

Conflict of Interests

The authors declare that there is no conflict of interests.

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