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FIXED POINT THEOREMS AND STABILITY OF ITERATIONS IN CONE METRIC SPACES

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Abstract. In this paper, fixed point problems of weak contractions are investigated in cone metric spaces. Theorems of convergence and theorems of stability for fixed points of some weak contraction are established in cone metric spaces.

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1. Introduction

Fixed point theory as an important branch of nonlinear functional analysis theory has been applied in many disciplines, including economics, image recovery, mechanics, quantum physics, and control theory; see, for example, [1-3]. The theory itself is a beautiful mixture of analysis, topology, and geometry. During the four decades, many famous existence theorems for fixed points of nonlinear mappings were established in Banach spaces. However, from the standpoint of real world applications it is not only to

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know the existence of fixed points of nonlinear mappings, but also to be able to construct an iteration to approximate their fixed points.

By replacing the real numbers with an ordered Banach Space, Huang and Zhang [4] defined cone metric spaces and proved some fixed point theorems of contractions on cone metric spaces. Since then, several authors have studied the fixed point problem of nonlinear mappings in cone metric spaces; see, for example, [4]-[8]. The purpose of this paper is to establish a convergence theorem and a stability theorem for fixed points of some general contractions in cone metric spaces.

In this paper, we always assume that E is a real Banach space and P is a subset of E with $intP \neq \phi$, where $intP$ denotes the interior of P.

Recall the following definitions which are related to cone metric spaces from [4].

Definition 1.1. P is said to be a cone if and only if:

(a) P is closed, nonempty, and $P \neq \{0\}$;

(b) If $a, b \in R$, $a, b \geq 0$, $x, y \in P$, then $ax + by \in P$;

(c) if $x \in P$ and $x \in -P$, then $x = 0$.

We define a partial ordering $\leq$ with respect to P as: $x \leq y$ if and only if $y - x \in P$.

We shall write $x < y$ if and only if $x \leq y$ but $x \neq y$, and $x \ll y$ if and only if $y - x \in intP$. Note that it is clear if $a \leq b$,$c \leq d$, then $a + c \leq b + d$, and for $\lambda \in R$, $\lambda \geq 0$, $\lambda a \leq \lambda b$.

Definition 1.2. The cone P is said to be normal if there is a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$. The least positive number satisfying above is said to be the normal constant of P.

Definition 1.3. Let X be a nonempty set. E is a real Banach space and P is a subset of E with $intP \neq \phi$. $\leq$ is a partial ordering with respect to P. Suppose the mapping $d : X \times X \to E$ satisfies:

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all x, $y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X,d) is called a cone metric space.

Definition 1.4. Let (X,d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$ there is N such that for all $n > N$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent. If $\{x_n\}$ converges to x, then x is said to be the limit of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$, or $x_n \to x$, as $n \to \infty$. If for any $c \in E$ with $0 \ll c$, there is N such that for all $n, m > N$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is said to be a Cauchy sequence in X. If $\{x_n\}$ converges, then $\{x_n\}$ is a Cauchy sequence. If every Cauchy sequence is convergent in (X,d), then (X,d) is said to be a complete cone metric space.

Definition 1.5. Let (X,d) be a cone metric space, T a selfmap of X. Let $x_{n+1} = f(T, x_n)$ be some iteration procedure. Suppose that $F(T)$, the fixed point set of T, is nonempty and that x_n converges to a point $p \in F(T)$. Let $\{y_n\} \subset X$, and define $\epsilon_n = d(y_{n+1}, f(T, y_n))$. If $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} y_n = p$, then $x_{n+1} = f(T, x_n)$ is said to be stable with respect to T.

2. Theorems of convergence

Theorem 2.1. *Let X be a nonempty complete cone metric space. Let T be a self-map on X such that*

$$d(Tx, Ty) \leq hd(x,y) + Ld(y, Tx), \quad \forall x, y \in X, \tag{2.1}$$

where h is some real number in $[0,1)$, and L is some real number in $(0,\infty)$. Then T has a fixed point in X. Let $\{x_n\}$ be a sequence generated by the following Picard iteration

$$x_0 \in X, \quad x_n = Tx_{n-1}, \quad \forall n \geq 1. \tag{2.2}$$

Then $\{x_n\}$ converges to a fixed point of T.

Proof. Taking $x = x_{n-1}$ and $y = x_n$ in (2.1), we obtain that

$$d(Tx_{n-1}, Tx_n) \leq hd(x_{n-1}, x_n),$$

which implies that

$$d(x_n, x_{n+1}) \leq hd(x_{n-1}, x_n),$$

and

$$d(x_{n+1}, x_n) \leq hd(x_n, x_{n-1}) \leq h^2 d(x_{n-1}, x_{n-2}) \leq ... \leq h^n d(x_1, x_0).$$

For $n > m$, we therefore have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + ... + d(x_{m+1}, x_m) \\ &\leq (h^{n-1} + h^{n-2} + ... + h^m) d(x_1, x_0) \\ &= \frac{h^m}{1-h} d(x_1, x_0). \end{aligned}$$

For a given $c \in E$ with $0 \ll c$, that is, $c \in intP$, there exists $B(0, r)$ such that $c + B(0, r) \subseteq P$, where $B(0, r) = \{x \in E, \|x\| \leq r\}$. But there exits a positive number N such that $\frac{h^m}{1-h} d(x_1, x_0) \in B(0, r)$ for all $m > N$. therefore, we have $\frac{h^m}{1-h} d(x_1, x_0) \ll c$, for all $m > N$. It follows that

$$d(x_n, x_m) \leq \frac{h^m}{1-h} d(x_1, x_0) \ll c.$$

This implies that $\{x_n\}$ is a Cauchy sequence and is convergent because of the completeness of X. We denote $p = \lim_{n\to\infty} x_n$. Notice that

$$\begin{aligned} d(p, Tp) &\leq d(p, x_{n+1}) + d(x_{n+1}, Tp) \\ &= d(p, x_{n+1}) + d(Tx_n, Tp) \\ &\leq d(p, x_{n+1}) + hd(x_n, p) + Ld(p, Tx_n) \\ &= (1 + L)d(p, x_{n+1}) + hd(x_n, p) \to 0. \end{aligned}$$

This implies that $\lim_{n\to\infty} d(p, Tp) = 0$. This finds that $Tp = p$. This completes the proof. □

Remark 2.2. In Theorem 2.1, the fixed point set may not be singleton, e.g., $Tx = x$.

3. Theorems of Stability

Lemma 3.1. *Let (X,d) be a nonempty cone metric space with respect to a normal cone P of E. Let $\{a_n\}$ and $\{\epsilon_n\}$ be sequences in E satisfying, $0 \leq a_n \leq ha_{n-1} + \epsilon_n$, where $h \in [0,1)$, and $\lim_{n\to\infty} \epsilon_n = 0$, then $\lim_{n\to\infty} a_n = 0$.*

Proof. Notice that

$$\begin{aligned} 0 &\leq a_n \\ &\leq ha_{n-1} + \epsilon_n \\ &\leq h^2 a_{n-2} + h\epsilon_{n-1} + \epsilon_n \\ &\leq \cdots \\ &\leq h^n a_0 + h^{n-1}\epsilon_1 + h^{n-2}\epsilon_2 + ... + h\epsilon_{n-1} + \epsilon_n. \end{aligned}$$

It follows that

$$\begin{aligned} \|a_n\| &\leq K\|h^n a_0 + h^{n-1}\epsilon_1 + h^{n-2}\epsilon_2 + ... + h\epsilon_{n-1} + \epsilon_n\| \\ &\leq K(h^n\|a_0\| + h^{n-1}\|\epsilon_1\| + h^{n-2}\|\epsilon_2\| + ... + h\|\epsilon_{n-1}\| + \|\epsilon_n\|) \\ &\leq Kh^n\|a_0\| + K(h^{n-1} + h^{n-2} + ... + h + 1)\sup\{\|\epsilon_n\|\} \\ &\leq Kh^n\|a_0\| + K\frac{1}{1-h}\sup\{\|\epsilon_n\|\} \to 0. \end{aligned}$$

This implies that $\lim_{n\to\infty} a_n = 0$. This completes the proof. □

Theorem 3.2. *Let X be a nonempty complete cone metric space with respect to a normal cone P. Let T be a self-map on X such that*

$$d(Tx,Ty) \leq hd(x,y) + Ld(x,Tx), \quad \forall x,y \in X.$$

where $k \in [0,1)$, and $L \geq 0$. Assume that T enjoys a fixed point. Let $\{x_n\}$ be a sequence generated by Picard iteration (2.2) Then Picard iteration (2.2) is stable with respect to T.

Proof. Let $\{y_n\}$ be an arbitrary sequence, $\epsilon_n = d(y_{n+1}, Ty_n)$, and $\lim_{n\to\infty} \epsilon_n = 0$.

$$\begin{aligned} d(y_{n+1}, p) &\leq d(y_{n+1}, Ty_n) + d(Ty_n, p) \\ &= d(y_{n+1}, Ty_n) + d(Ty_n, Tp) \\ &\leq hd(y_n, p) + Ld(p, Tp) + \epsilon_n \\ &= hd(y_n, p) + \epsilon_n \end{aligned}$$

In view of Lemma 3.1, we can obtain $\lim\limits_{n\to\infty} d(y_n, p) = 0$ i.e., $\lim\limits_{n\to\infty} y_n = p$. $\square$

Remark 3.3. In Theorem 3.2, it is interesting whether the condition P is normal can be removed.

Remark 3.4. Our weak contraction condition is quite a general one. And if (X, d) is a usual metric space, the generalized contraction in our theorems contains many special cases. The theorems presented in this paper extend many results on the mappings and mapping pairs defined in Rhoades [8] from metric spaces to cone metric spaces.

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