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BEST PROXIMITY POINTS IN PARTIALLY ORDERED METRIC SPACES

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Abstract. The existence of best proximity point is an important aspect of optimization theory. We define the concept of proximally monotone Lipschitzian mappings on a partially ordered metric space. Then we obtain sufficient conditions for the existence and uniqueness of best proximity points for these mappings in partially ordered CAT(0) spaces. This work is a continuation of the work of Ran and Reurings [Proc. Amer. Math. Soc. **132** (2004), 1435–1443] and Nieto and Rodríguez-López [Order, **22** (2005), 223–239] for the new class of mappings introduced herein.

Keywords: partially ordered set; CAT(0) space; fixed point; best proximity point; monotone mapping.

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1. Introduction

The theory of fixed points is one of the most powerful tools of modern mathematics. It has given a new impetus to modern fixed point theory via nonlinear functional analysis. For example, the existence of solutions of elliptic partial differential equations, or the existence of closed

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periodic orbits in dynamical systems, and the existence of answer sets in logic programming are usually translated into a fixed point problem.

Recently, a new direction has been discovered for the extension of Banach Contraction Principle [2] to metric spaces endowed with a partial order. Ran and Reurings [23] have successfully carried out the first attempt; in particular, they showed, how this extension is useful when dealing with some special matrix equations. Another similar approach was given by Nieto and Rodríguez-López [21] and they used it in solving some differential equations. Recently, Bachar and Khamsi [1] studied existence of fixed points of monotone nonexpansive mappings defined on partially ordered Banach spaces. Buthinah and Khamsi [7] have given an analogue of the fixed point theorem of Browder and Göhde for monotone nonexpansive mappings on a very general nonlinear domain.

If the fixed point equation $Tx = x$ of a given mapping T does not have a solution, then it is of interest to find an approximate solution for it. In other words, we are in search of an element in the domain of the mapping, whose image is as close to it as possible. This situation motivated to develop the notion of ”best proximity point” (see, [9, 15, 18]). The best proximity point theorems can be viewed as a generalization of fixed point theorems, since most fixed point theorems can be derived as corollaries of best proximity point theorems.

In this paper, we obtain an extension of Banach Contraction Principle for best proximity points in partially ordered CAT(0) spaces. The case of nonexpansive mappings is also discussed. To the best of our knowledge, investigations on these lines have not been carried out so far.

2. Preliminaries

The extension of Banach Contraction Principle in metric spaces endowed with a partial order was initiated by Ran and Reurings [23]. Let a metric space (X,d) be endowed with a partial order $\preceq$. We will say that $x,y \in X$ are comparable whenever $x \preceq y$ or $y \preceq x$. Recall that an order interval is a subset of the form $[a,b] = \{x \in X; a \preceq x \preceq b\}$, $[a,\rightarrow) = \{x \in X; a \preceq x\}$, $(\leftarrow,a] = \{x \in X; x \preceq a\}$, for any $a,b \in X$.

The definition of proximally monotone mappings has roots in [3].

Let us define the concept of proximally monotone Lipschitzian mappings on a partially ordered metric space.

Definition 2.1. Let A,B be nonempty subsets of X. Define

$$d(A,B) = \inf\{d(a,b);\ a \in A,\ b \in B\}.$$

Let $T : A \to B$ be a mapping.

(1) T is said to be proximally monotone if it satisfies the condition:

$$x \preceq y,\ d(u,Tx) = d(A,B) \text{ and } d(v,Ty) = d(A,B) \text{ imply } u \preceq v$$

for all $x,y,u,v \in A$.

(2) T is said to be proximally monotone Lipschitzian mapping if T is proximally monotone and there exists $k \geq 0$ such that

$$d(Tx,Ty) \leq k\, d(x,y),$$

where $x,y \in A$ and x and y are comparable.

If $k < 1$ ($k = 1$), then we say that T is a monotone contraction (nonexpansive) mapping.

(3) A point $x \in A$ is said to be a best proximity point of T if

$$d(x,Tx) = d(A,B).$$

If $A = B$, then the above definition coincides with the definition of monotone Lipschitzian mappings [1] and the best proximity point x reduces to a fixed point of T.

Next, we introduce the concept of a hyperbolic space. Suppose that there exists a family F of metric segments such that any two points x,y in X are endpoints of a unique metric segment $[x,y] \in F$ ($[x,y]$ is an isometric image of the real line interval $[0,d(x,y)]$). We shall denote by $\beta x \oplus (1-\beta)y$ the unique point z of $[x,y]$ which satisfies

$$d(x,z) = (1-\beta)d(x,y), \text{ and } d(z,y) = \beta d(x,y),$$

where $\beta \in [0,1]$. Such metric spaces with a family F of metric segments are usually called *convex metric spaces* [20]. Moreover, if we have

$$d\Big(\alpha p \oplus (1-\alpha)x, \alpha q \oplus (1-\alpha)y\Big) \leq \alpha d(p,q) + (1-\alpha)d(x,y),$$

for all p,q,x,y in X, and $\alpha \in [0,1]$, then X is said to be a *hyperbolic space* (see [24]).

Obviously, normed linear spaces are hyperbolic spaces. As nonlinear examples, one can consider the Hadamard manifolds [6], the Hilbert open unit ball equipped with the hyperbolic metric [12], and the CAT(0) spaces [16, 17, 19]. We will say that a subset C of a hyperbolic space X is convex if $[x,y] \subset C$ whenever x,y are in C.

The definition of uniform convexity in Banach spaces finds its origin in [8]. The first attempt to generalize this concept to metric spaces was made in [13]. The reader may also consult [12] and [24].

Definition 2.2. [14] Let (X,d) be a hyperbolic space. We say that X is uniformly convex if for any $a \in X$, for every $r > 0$, and for each $\varepsilon > 0$

$$\delta(r,\varepsilon) = \inf\Big\{1 - \frac{1}{r}\, d\Big(\frac{1}{2}x \oplus \frac{1}{2}y, a\Big); d(x,a) \leq r, d(y,a) \leq r, d(x,y) \geq r\varepsilon\Big\} > 0.$$

Example 2.1. Let (X,d) be a metric space. A *geodesic* from x to y in X is a mapping c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y. The space (X,d) is said to be a *geodesic space* if any two points of X are joined by a geodesic and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x,y \in X$, which we denote by $[x,y]$, and call it the segment joining x to y.

A *geodesic triangle* $\Delta(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three points x_1,x_2,x_3 in X (the *vertices* of Δ) and a geodesic segment between each pair of vertices (the *edges* of Δ). A *comparison triangle* for geodesic triangle $\Delta(x_1,x_2,x_3)$ in (X,d) is a triangle $\overline{\Delta}(x_1,x_2,x_3) := \Delta(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}\left(\bar{x}_i,\bar{x}_j\right) = d\left(x_i,x_j\right)$ for $i,j \in \{1,2,3\}$. Such a triangle always exists (see [4]).

A geodesic metric space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following CAT(0) comparison axiom:

Let Δ be a geodesic triangle in X and let $\overline{\Delta} \subset \mathbb{R}^2$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) *inequality* if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \overline{\Delta}$, we have

$$d(x,y) \leq d(\bar{x},\bar{y}).$$

Complete CAT(0) spaces are often called *Hadamard spaces (see* [17]*)*. If x, y_1, y_2 are points of a CAT(0) space and y_0 is the midpoint of the segment $[y_1, y_2]$, which we denote by $\frac{y_1 \oplus y_2}{2}$, then the CAT(0) inequality implies:

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2}d^2(x,y_1) + \frac{1}{2}d^2(x,y_2) - \frac{1}{4}d^2(y_1,y_2).$$

This inequality is the (CN) inequality of Bruhat and Tits [5]. The (CN) inequality implies that CAT(0) spaces are uniformly convex with modulus of convexity [14]:

$$\delta(r,\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}.$$

We shall say that a subset C of a metric space (X,d) is a Chebyshev set if to each point p in X, there corresponds a unique point p_0 in C such that $d(p,p_0) = \inf\{d(p,c) : \ c \in C\}$. In this case, we define the nearest point projection $P_C : X \to C$ by assigning p to p_0.

Lemma 2.1. [4] Let C be a closed and convex subset of a complete CAT(0) space. Then the following hold:

(i) C is a Chebyshev set.

(ii) The nearest point projection P_C is a nonexpansive mapping.

We now introduce the terminology needed for the development of our results.

Let A, B be nonempty subsets of a metric space (X,d). Then the proximity pair associated with the pair (A,B), denoted by (A_0,B_0), is defined by

$$A_0 = \{x \in A : d(x,y) = d(A,B); \text{ for some } y \in B\},$$

and

$$B_0 = \{y \in B : d(x,y) = d(A,B); \text{ for some } x \in A\}.$$

A pair of subsets (A,B) is said to be proximinal if and only if $A = A_0$ and $B = B_0$. It is said to be sharp proximinal if and only if for any $(x,y) \in A \times B$, there exist a unique $x' \in B$ and $y' \in A$ such that

$$d(x,x') = d(y,y') = d(A,B).$$

It is clear that A_0 is nonempty if and only if B_0 is so. Espinola and Fernández-León [10] have given more on the structure of the proximity pair associated with the pair (A,B) as follows:

Proposition 2.1. Let A and B be nonempty closed and convex subsets of a complete CAT(0) space X. Then the pair (A_0,B_0) is nonempty, closed and convex (that is, both sets A_0 and B_0 are nonempty, closed and convex subsets) in X.

Recently, based on geometrical properties of Hilbert spaces, the so-called Pythagorean property is introduced.

Definition 2.3. [11] A sharp proximinal pair (A,B) in a metric space X is said to have the Pythagorean property if and only if, for each $(x,y) \in A \times B$,

$$d(x,y)^2 = d(x,y')^2 + d(y',y)^2 \text{ and } d(x,y)^2 = d(y,x')^2 + d(x',x)^2,$$

where x' and y' are the (unique) points in B and A, respectively, such that $d(x,x') = d(A,B)$ and $d(y',y) = d(A,B)$.

It is shown in [11] that the following facts hold in CAT(0) spaces.

Lemma 2.2. Let (A,B) be a nonempty, closed and convex pair in a complete CAT(0) space X. Then the pair (A_0,B_0) is a sharp proximinal pair.

Theorem 2.1. Let A and B be nonempty, closed, convex and proximinal subsets of a complete CAT(0) space. Then the pair (A,B) has the Pythagorean property.

We prove the following fact which will be needed in the next section.

Theorem 2.2. Let A and B be nonempty, closed and convex subsets of a complete CAT(0) space X. Then the pair (A_0,B_0) has the Pythagorean property.

Proof. By Lemma 2.2, (A_0,B_0) is a sharp proximinal pair. Proposition 2.1 implies that the pair (A_0,B_0) is nonempty, closed and convex in X. As (A_0,B_0) is proximinal pair, so Theorem 2.1

implies that the pair (A_0, B_0) has the Pythagorean property. To see this, let (A_{00}, B_{00}) be the proximity pair associated with the pair (A_0, B_0). Indeed, $A_{00} \subseteq A_0$. Conversely, let $x \in A_0$, then there exists $y \in B$ such that $d(x,y) = d(A,B)$. Hence, $y \in B_0$ and $d(x,y) = d(A,B) = d(A_0, B_0)$. i.e., $x \in A_{00}$. Therefore, $A_0 \subseteq A_{00}$. Hence, $A_0 = A_{00}$. In a similar way, we can show that $B_0 = B_{00}$. Therefor, the pair (A_0, B_0) is proximinal pair.

3. Main results

In this section, we obtain sufficient conditions for the existence and uniqueness of best proximity points for proximally monotone mappings in partially ordered CAT(0) spaces.

The following best proximity point theorem provides an extended version of Banach Contraction Principle for proximally monotone contraction mappings on partially ordered CAT(0) spaces.

Theorem 3.1. Let (A,B) be a pair of nonempty, bounded, closed, and convex subsets of a partially ordered CAT(0) space $(X,d,\preceq)$ such that order intervals are closed. Let $T : A \to B$ be a proximally monotone contraction mapping such that $T(A_0) \subseteq B_0$. If there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and $d(x_1, Tx_0) = d(A,B)$, then T has a best proximity point x in A. Moreover, if y in A is a best proximity point of T comparable to x, then $y = x$.

Proof. Since $Tx_1 \in T(A_0) \subseteq B_0$, therefore there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A,B)$. Applying the definition of proximally monotone mappings to $x = x_0, y = u = x_1, v = x_2$, we obtain $x_1 \preceq x_2$.

Continuing this process, we can find a sequence $\{x_n\}$ in A_0 such that, for all $n \in \mathbb{N}$, $x_{n-1} \preceq x_n$ and $d(x_n, Tx_{n-1}) = d(A,B)$.

By Theorem 2.2, the pair (A_0, B_0) has the Pythagorean property. Hence, we have

$$\left.\begin{array}{l} d(x_{n+1}, Tx_n)^2 + d(A,B)^2 = d(x_n, Tx_n)^2 \\ d(x_n, Tx_{n-1})^2 + d(A,B)^2 = d(x_n, Tx_n)^2 \end{array}\right\} \Longrightarrow d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}),$$

for any $n \geq 1$.

As T is a proximally monotone contraction, so there exists $k < 1$ such that

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq k d(x_n, x_{n-1}),$$

for any $n \geq 1$, which implies that

$$d(x_{n+1}, x_n) \leq k^n d(x_1, x_0),$$

for any $n \geq 1$. Hence $\sum_{n \in \mathbb{N}} d(x_{n+1}, x_n)$ is convergent which implies that $\{x_n\}$ is Cauchy. Since X is complete, therefore there exists $x \in X$ such that $\{x_n\}$ converges to x. Since $\{x_n\} \subset A_0$ and by Proposition 2.1 A_0 is closed, we conclude that $x \in A_0$. As the order intervals are closed, so we conclude that $x_n \preceq x$, for any $n \in \mathbb{N}$. Furthermore, $d(Tx_n, Tx) \leq kd(x_n, x) \to 0$ as $n \to \infty$. So $\{Tx_n\}$ converges to $Tx \in B_0$. By $d(x_{n+1}, Tx_n) = d(A, B)$, we get $d(x, Tx) = d(A, B)$, i.e., $x \in A_0$ and $Tx \in B_0$. Clearly, x is a best proximity point of T in A.

Next, we prove that if y in A is a best proximity point of T comparable to x, then $y = x$. Without loss of any generality, assume that $x \preceq y$. Since T is a proximally monotone contraction mapping, therefore by the Pythagorean property, we have

$$\left. \begin{array}{l} d(x, Tx) = d(A, B) \\ d(y, Ty) = d(A, B) \end{array} \right\} \Longrightarrow d(x, y) = d(Tx, Ty).$$

As T is a contraction mapping, so we get

$$d(x, y) = d(Tx, Ty) < d(x, y),$$

which implies $d(x, y) = 0$, i.e., $y = x$.

Remark 3.1. If $A = B$ in Theorem 3.1, then $d(A, B) = 0$ i.e., $x_1 = Tx_0$. Therefore, $x_0 \preceq Tx_0$. Hence, our result extends the work of Ran and Reurings [23] and Nieto and Rodríguez-López [21] for best proximity points.

One may wonder what happens to the conclusion of Theorem 3.1 if T is proximally monotone nonexpansive.

To answer this quastion, we recall the following result.

Theorem 3.2. [7] Let $(X, d, \preceq)$ be a partially ordered hyperbolic space such that order intervals are closed and convex. Assume that (X, d) is uniformly convex. Let C be a nonempty, convex, closed and bounded subset of X not reducible to one point. Let $T : C \to C$ be a monotone nonexpansive mapping. Assume there exists $x_0 \in C$ such that x_0 and Tx_0 are comparable. Then T has a fixed point.

Armed with Theorem 3.2, we extend the conclusion of Theorem 3.1 for proximally monotone nonexpansive mappings as follows:

Theorem 3.3. Let (A,B) be a pair of nonempty, bounded, closed, and convex subsets of a partially ordered CAT(0) space $(X,d,\preceq)$ such that order intervals are closed and convex with A_0 not reducible to one point. Let $T : A \to B$ be a proximally monotone nonexpansive mapping such that $T(A_0) \subseteq B_0$. If there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and $d(x_1, Tx_0) = d(A,B)$, then T has a best proximity point x in A.

Proof. Note that, for any $y_0 \in B_0$, there exists unique $x_0 \in A_0$ such that $d(x_0,y_0) = d(A,B)$. Hence

$$d(A,B) \leq d(A,y_0) \leq d(A_0,y_0) \leq d(x_0,y_0) = d(A,B).$$

That is, $d(A,y) = d(A_0,y) = d(A,B)$, for all $y \in B_0$.

Consider the mapping $P_{A_0} \circ T : A_0 \to A_0$. By Lemma 2.1 (ii), the nearest point projection P_{A_0} is nonexpansive. Furthermore, our assumption on the mapping T implies that $P_{A_0} \circ T$ is monotone nonexpansive.

Moreover, let $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and $d(x_1, Tx_0) = d(A,B)$. By Lemma 2.1 (i), A_0 is a Chebyshev subset, so $P_{A_0}(T(x_0)) = x_1$. Hence, $x_0 \preceq P_{A_0}(T(x_0))$.

Since A_0 is a nonempty, closed and convex subset of A, therefore A_0 is bounded. As CAT(0) spaces are uniformly convex hyperbolic spaces, so Theorem 3.2 implies that $P_{A_0} \circ T$ has a fixed point $x \in A_0$, i.e., $x = P_{A_0}(Tx)$. Then, we have $x \in A_0$ and $Tx \in B_0$. Hence

$$d(x,Tx) = d(P_{A_0}(Tx), Tx) = d(Tx, A_0) = d(A,B).$$

Therefore, x is a best proximity point of T in A.

As nonexpansive mappings may fail to have a fixed point, so there is no reason to seek uniqueness of the best proximity point result obtained here.

Remark 3.2. Theorem 3.3 generalizes the corresponding results in [1] and [7] for the best proximity points.

We now set out to give an example to illustrate Theorems 3.1 and 3.3.

Definition 3.1. [22] A pair (A,B) of nonempty subsets of a normed linear space X is said to have the d-property if and only if

$$\left.\begin{array}{l} ||x-x'|| = d(A,B) \\ ||y'-y|| = d(A,B) \end{array}\right\} \Longrightarrow ||x-y'|| = ||x'-y||,$$

whenever $x, y' \in A$ and $x', y \in B$.

Definition 3.2. [22] A normed linear space X is said to have the d-property if and only if every pair (A,B) of nonempty, closed and convex subsets of X has the d-property.

Raj and Eldred have established a simple characterization of strictly convex normed linear spaces as follows:

Theorem 3.4. [22] Let X be a normed linear space, then X is strictly convex if and only if it has the d-property.

Let X be the Euclidean vector space $\mathbb{R}^2$. Let $A = \{(x,0) : 0 \leq x \leq 1\}$ and $B = \{(x,1) : 0 \leq x \leq 1\}$ be nonempty subsets of X. Clearly, the pair (A,B) is nonempty, bounded, closed, and convex in X. Hence, by Theorem 3.4, the pair (A,B) has the d-property. Moreover, $A_0 = A, B_0 = B$ and $d(A,B) = 1$.

We claim that the pair (A,B) has the Pythagorean property. We note that for any $((x,0),(y,1)) \in A \times B$, there exists unique $(x,1) \in B$ and $(y,0) \in A$ such that

$$||(x,0)-(x,1)|| = ||(y,0)-(y,1)|| = d(A,B).$$

Hence, the pair (A,B) is sharp proximinal.

Moreover, armed with the d-property, we have

$$||(x,0)-(y,1)||^2 = ||(x,0)-(y,0)||^2 + ||(y,0)-(y,1)||^2,$$

$$||(x,0)-(y,1)||^2 = ||(y,1)-(x,1)||^2 + ||(x,1)-(x,0)||^2.$$

Therefore, the sharp proximinal pair (A,B) has the Pythagorean property. Obviously, the $CAT(0)$ inequality holds in X and so it is a $CAT(0)$ space. The Pythagorean property and the d-property coincide on X.

Consider the product order $\preceq$ on X, i.e. $(a,b) \preceq (c,d)$ iff $a \leq c$ and $b \leq d$.

Clearly, order intervals are closed and convex.

Define a mapping $T : A \to B$ by

$$T((x,0)) = (kx,1),$$

for $k \in [0,1]$.

Now, $T(A_0) \subseteq B_0$. Let $x_0 = x_1 = (0,0)$. Then $x_0 \preceq x_1$ and $||x_1 - Tx_0|| = ||(0,0) - T((0,0))|| = ||(0,0) - (0,1)|| = 1$.

We now show that T is a proximally monotone Lipschitzian mapping. For $(x,0),(y,0),(u,0),(v,0) \in A$ with $(x,0) \preceq (y,0)$, $||(u,0) - T((x,0))|| = 1$ and $||(v,0) - T((y,0))|| = 1$, we have $(u,0) = (kx,0)$ and $(v,0) = (ky,0)$. Hence, $(u,0) \preceq (v,0)$.
Moreover, $||T((x,0)) - T((y,0))|| = ||(kx,1) - (ky,1)|| = k||(x,0) - (y,0)||$.

If $k < 1$, then T is a proximally monotone contraction mapping and $(0,0) \in A$ is the best proximity point of T.

Finally, If $k = 1$, then T is a proximally monotone nonexpansive mapping and any $x \in A$ is a best proximity point of T.

Conflict of Interests

The authors declare that there is no conflict of interests.

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