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COINCIDENCE THEOREM FOR RECIPROCALLY CONTINUOUS SYSTEMS OF MULTI-VALUED AND SINGLE-VALUED MAPS

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Abstract. In this paper we eliminate completely the requirement of continuity from the main results of Baillon-Singh [1], Gairola et al. [9] and Gairola-Jangwan [7] and prove a coincidence theorem for systems of single-valued and multi-valued maps on finite product of metric spaces using the concept of coordinatewise reciprocal continuity.

Keywords: fixed point; coordinatewise commuting maps; weakly commuting maps; R-weakly commuting maps; asymptotically commuting maps; reciprocal continuous maps; hybrid contraction; product space.

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1. Introduction

Hybrid fixed point theory for nonlinear single-valued and multi-valued maps is a new development within the ambit of multi-valued fixed point theory. The study of hybrid maps was initiated during 1980-83 by Hadzic [11], Singh-Kulshrestha [28], Bhaskaran-Subrahmanyam [2], Rhoades et al. [23], Kaneko [14], Naimpally et al. [19] etc. For a history concerning the

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development of hybrid maps, one may refer to Kaneko [15], Kaneko-Sessa [16], Singh-Mishra [26], Beg-Azam [3], Jungck-Rhoades [13] and Pathak et al. [21]. Hybrid fixed point theory has potential application in Functional Inclusion, Optimization Theory, Fractal Graphics and Discrete Dynamics for set valued operator (see [4], [29]). In recent formulation, Corley [4] has shown that certain optimization problem are equivalent to a hybrid fixed point theorems. Such theorems also appear to be new tools, concerning problems of treatment of images in computer graphics.

In view to generalizing the celebrated Banach contraction principal, Matkowski [17] extend the concept of Banach contraction principal for system of n maps on finite product of metric spaces. Czerwik [5] generalized the result of Matkowski [op. cit.] (see also Nadler [18] and Reich [22]) and obtain a fixed point theorem for system of multi-valued maps. Further, Baillon-Singh [1] proved a hybrid fixed point theorem for systems of single-valued and multi-valued maps. Recently, Gairola et al. [9] and Gairola-Jangwan [7], motivated by the work of Baillon-Singh [op. cit.], proved some coincidence theorems for systems of single-valued and multi-valued maps by introducing a new class of maps- coordinatewise asymptotically commuting and R-weakly commuting maps.

In this paper we introduced the concept of coordinatewise reciprocal continuity for systems of single-valued and multi-valued maps on finite product of metric spaces (cf. Definition 2.5) and gave a coincidence theorem (cf. Theorem 3.1) for systems of single-valued and multi-valued maps. We showed that the continuity of any system of maps is not necessary for the existence of a coincidence point for systems of single-valued and multi-valued maps. Our result extends and generalizes numerous coincidence and hybrid fixed point results of Czerwik [op. cit.], Kaneko [14], Kaneko-Sessa [16], Baillon-Singh [op. cit.], Gairola et al. [9], Gairola-Jangwan [7] and others.

2. Notations and Definitions

Throughout the paper we shall follow the following notations and definitions.

Let (Y,d) be a metric space and $CL(Y)$ denotes the set of non-empty closed subsets of Y. Then for $A,B \in CL(Y)$,

$D(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$,

$D(x,A) = \inf\{d(x,y) : y \in A\}$,

$H(A,B) = \max\left\{\sup_{x\in A} D(x,B), \sup_{y\in B} D(y,A)\right\}$,

where H is called the generalized Hausdorff metric induced by metric d and $(CL(Y),H)$ is called generalized Hausdorff metric space.

Let a_{ik} be non-negative numbers $i,k = 1,\dots,n$ and $c_{ik}^{(t)}$ square matrix defined in Matkowski [op. cit.](see also [1], [5]).

$$c_{ik}^{(0)} = \begin{cases} a_{ik}, & i \neq k \\ 1 - a_{ik}, & i = k \end{cases} \quad i,k = 1,\dots,n \tag{2.1}$$

$$c_{ik}^{(t+1)} = \begin{cases} c_{11}^{(t)} c_{i+1,k+1}^{(t)} + c_{i+1,1}^{(t)} c_{1,k+1}^{(t)}, & i \neq k \\ c_{11}^{(t)} c_{i+1,k+1}^{(t)} - c_{i+1,1}^{(t)} c_{1,k+1}^{(t)}, & i = k \end{cases} \tag{2.2}$$

$$t = 1,\dots,n-1, i,k = 1,\dots,n-t.$$

$$c_{ii}^{(t)} > 0, t = 1,\dots,n, i = 1,\dots,n+1-t. \tag{2.3}$$

In this paper we shall assume that $(X_i,d_i), i = 1,...,n$, are metric spaces, $(CL(X_i),H_i)$ the generalized Hausdorff metric spaces induced by d_i. Further, let $X = X_1 \times \cdots \times X_n, x = (x_1,\dots,x_n)$ and $\{x^m\} = \{(x_1^m,\dots,x_n^m)\}, m \in \mathbb{N}$ (natural numbers) be a sequence in X. $P_i,Q_i : X \to CL(X_i), i = 1,\dots,n$, are multi-valued maps and $S_i,T_i : X \to X_i, i = 1,\dots,n$, are single-valued maps. For $M = (M_1,\dots,M_n) \subset X$, we use the notation $T(M) = (T_1M_1,\dots,T_nM_n)$ as in [1].

Definition 2.1. [1]. Two systems of maps $(T_1,\dots,T_n)$ and $(P_1,\dots,P_n)$ are coordinatewise commuting at a point $x \in X$ if and only if

$$T_i(P_1x,\dots,P_nx) \subseteq P_i(T_1x,\dots,T_nx), i = 1,\dots,n.$$

For $n = 1$ this definition is that of Itoh-Takahashi [12]. The two systems of maps are coordinatewise commuting on X if and only if they are coordinatewise commuting at every point of X.

Definition 2.2. [1]. Two systems of maps $(T_1,\dots,T_n)$ and $(P_1,\dots,P_n)$ are coordinatewise weakly commuting at a point $x \in X$ if and only if

$$H_i(T_i(P_1x,\dots,P_nx),P_i(T_1x,\dots,T_nx)) \le D_i(P_ix,T_ix), i=1,\dots,n.$$

Two systems are coordinatewise weakly commuting on X if and only if they are coordinatewise weakly commuting at every point of X.

For $n=1$ this definition is due Kaneko [15] (see, Singh et al. [25]). An equivalent formulation of Definition 2.2 for two systems of single-valued maps on X appears in [24].

In [1], Baillon-Singh has shown that coordinatewise weakly commuting systems of maps need not be coordinatewise commuting . However the Example 2.2 (below) shows that coordinatewise commutativity does not imply coordinatewise weak commutativity.

Definition 2.3. [7]. Two systems of maps $(T_1,\dots,T_n)$ and $(P_1,\dots,P_n)$ are coordinatewise R-weakly commuting at a point $x \in X$ if and only if

$$H_i(T_i(P_1x,\dots,P_nx),P_i(T_1x,\dots,T_nx)) \le RD_i(P_ix,T_ix), i=1,\dots,n, R \ge 0.$$

Two systems are coordinatewise R-weakly commuting on X if and only if they are coordinatewise R-weakly commuting at every point of X. An equivalent formulation of Definition 2.3 for two systems of single-valued maps on X appears in [6].

Remark 2.1. Notice that coordinatewise weakly commuting maps are coordinatewise R-weakly commuting. However R-weak commutativity implies coordinatewise weak commutativity only when $R \le 1$ (see [6-7]).

Definition 2.4. [9]. Two systems of maps $(T_1,\dots,T_n)$ and $(P_1,\dots,P_n)$ are coordinatewise asymptotically commuting (or simply asymptotically commuting) if and only if

$$H_i(T_i(P_1x^m,\dots,P_nx^m),P_i(T_1x^m,\dots,T_nx^m)) \to 0 \text{ as } m \to \infty,$$

whenever $\{x^m\}$ is a sequence in X such that $P_ix^m \to M_i \in CL(X_i)$ and $T_ix^m \to t_i \in M_i, i=1,\dots,n$. An equivalent formulation of Definition 2.4 for two systems of single-valued maps appears in [10].

If two systems of maps $(T_1,\dots,T_n)$ and $(P_1,\dots,P_n)$ are coordinatewise R-weakly commuting on X, then for all $x \in X$, there exist a $R \geq 0$, such that

$$H_i(T_i(P_1x,\dots,P_nx),P_i(T_1x,\dots,T_nx)) \leq RD_i(P_ix,T_ix), i=1,\dots,n. \tag{2.4}$$

Suppose there exist a sequence $\{x^m\} \in X$ such that $P_ix^m \to M_i \in CL(X_i)$ and $T_ix^m \to t_i \in M_i, i = 1,\dots,n$, as $m \to \infty$. Then by (2.4),

$$H_i(T_i(P_1x^m,\dots,P_nx^m),P_i(T_1x^m,\dots,T_nx^m)) \leq RD_i(T_ix^m,P_ix^m), i=1,\dots,n.$$

Making $m \to \infty$, we get $H_i(T_i(P_1x^m,\dots,P_nx^m),P_i(T_1x^m,\dots,T_nx^m)) \to 0$. Hence systems of maps $(T_1,\dots,T_n)$ and $(P_1,\dots,P_n)$ are coordinatewise asymptotically commuting on X.

Remark 2.2. Coordinatewise R-weak commutativity of two systems of maps $(T_1,\dots,T_n)$ and $(P_1,\dots,P_n)$ on X implies their coordinatewise asymptotic commutativity, however converse need not be true. The following example shows the coordinatewise asymptotic commutativity of two systems of maps and illustrates that coordinatewise asymptotic commutativity of two systems of maps does not imply their coordinatewise R-weak commutativity on X.

Example 2.1. Let $X_1 = X_2 = [0,\infty)$ be usual metric spaces and $T_i : X_1 \times X_2 \to X_i, P_i : X_1 \times X_2 \to CL(X_i), i = 1,2$, be such that

$$\begin{aligned} &T_1(x_1,x_2) = x_1^3, && T_2(x_1,x_2) = x_2^3,\\ &P_1(x_1,x_2) = [2x_1^3,\infty), && P_2(x_1,x_2) = [2x_2^3,\infty), \text{ for all } (x_1,x_2) \in X_1 \times X_2. \end{aligned}$$

Then for an arbitrary $x \in X_1 \times X_2$, $P_i(T_1x,T_2x) = [2x_i^9,\infty), T_i(P_1x,P_2x) = [8x_i^9,\infty)$ and $H_i(T_i(P_1x, P_2x),P_i(T_1x,T_2x)) = 6x_i^9 \leq Rx_i^3 = RD_i(T_ix,P_ix), i = 1,2$, where $R \geq 6\max\{x_1^6,x_2^6\}$. Since R depends on $x_i \in X_i, i = 1,2$, therefore there does not exist a fixed $R \geq 0$, for all $x \in X_1 \times X_2$, such that $H_i(T_i(P_1x,P_2x),P_i(T_1x,T_2x)) \leq RD_i(T_ix,P_ix)$. Hence systems of maps $\{P_1,P_2\}$ and $\{T_1,T_2\}$ are not coordinatewise R-weakly commuting on $X_1 \times X_2$.

Now if we consider a sequence $\{x^m\} \in X_1 \times X_2$ such that $T_ix^m = \{x^{3m}\} \to t_i \in X_i, P_ix^m = [2x_i^{3m},\infty) \to [2t_i,\infty) = M_i \in CL(X_i)$ and $t_i \in [2t_i,\infty) = M_i, i = 1,2$, as $m \to \infty$. Then $t_i = 0$ and $H_i(T_i(P_1x^m,P_2x^m),P_i(T_1x^m,T_2x^m)) = 6x_i^{9m} \to 0, i = 1,2$, as $m \to \infty$. Hence systems of maps $\{P_1,P_2\}$ and $\{T_1,T_2\}$ are coordinatewise asymptotically commuting on $X_1 \times X_2$.

Remark 2.3. Coordinatewise weak commutativity, R-weak commutativity and asymptotic commutativity of two systems of maps $(P_1,\dots,P_n)$ and $(T_1,\dots,T_n)$ at a coincidence point z (that is, when $T_iz \in P_iz, i=1,\dots,n$) is equivalent to their coordinatewise commutativity, however coordinatewise commutativity of systems $(P_1,\dots,P_n)$ and $(T_1,\dots,T_n)$ is more general than their weak commutativity, R-weak commutativity and asymptotic commutativity at their coincidence point z. The following example illustrates the above statement.

Example 2.2. Let $X_1 = X_2 = [0,\infty)$, be usual metric spaces and $T_i : X_1\times X_2 \to X_i, P_i : X_1 \times X_2 \to CL(X_i), i=1,2$, be such that

$$\begin{aligned} &T_1(x_1,x_2) = 2x_1, && T_2(x_1,x_2) = 2x_2,\\ &P_1(x_1,x_2) = [1+x_1,\infty), && P_2(x_1,x_2) = [1+x_2,\infty), \text{ for all } (x_1,x_2)\in X. \end{aligned}$$

Then at $x=(2,2), T_i(2,2) = 4 \in [3,\infty) = P_i(2,2)$ and $T_i(P_1(2,2),P_2(2,2)) = [6,\infty) \subset [5,\infty) = P_i(T_1(2,2),T_2(2,2))$ but $H_i\left(T_i(P_1(2,2),P_2(2,2)),P_i(T_1(2,2),T_2(2,2))\right) = 1 \nleq 0 = D_i(T_i(1,1), P_i(1,1)), i=1,2$. Hence systems of maps $\{P_1,P_2\}$ and $\{T_1,T_2\}$ are coordinatewise commuting but neither coordinatewise weakly commuting nor R-weakly commuting at $x=(2,2)$. If we consider $\{x^m\} = \{(1,1)\}, m\in\mathbb{N}$ then $T_ix^m \to 2$ and $P_ix^m \to [2,\infty)$ for $i=1,2$, as $m\to\infty$. Since $2\in[2,\infty)$ and $H_i(T_i(P_1x^m,P_2x^m),P_i(T_1x^m,T_2x^m)) \to 1, i=1,2$, as $m\to\infty$. Hence systems of maps $\{P_1,P_2\}$ and $\{T_1,T_2\}$ are not coordinatewise asymptotically commuting at $x=(2,2)$.

Definition 2.5. Two systems of maps $(T_1,\dots,T_n)$ and $(P_1,\dots,P_n)$ are coordinatewise reciprocally continuous on X (resp. at $t\in X$) if and only if $T_i(P_1x,\dots,P_nx) \in CL(X_i)$ for each $x\in X$ (resp., $T_i(P_1t,\dots,P_nt)\in CL(X_i), i=1,\dots,n$) and

$$\lim_{m\to\infty} T_i(P_1x^m,\dots,P_nx^m) = T_iM,\ \lim_{m\to\infty} P_i(T_1x^m,\dots,T_nx^m) = P_it$$

whenever $\{x^m\}$ is a sequence in X_i such that $\lim_{m\to\infty} P_ix^m = M_i \in CL(X_i),\ \lim_{m\to\infty} T_ix^m = t_i \in M_i, i = 1,\dots,n$.

An equivalent formulation of Definition 2.5 for two systems of single-valued maps on X appears in [8]. As a special case of the above definition for $n=1$, we have the following definition introduced in [27].

Definition 2.6. The mapping $T_1 : X_1 \to X_1$ and $P_1 : X \to CL(X_1)$ are reciprocally continuous on X_1 (resp. at $t\in X_1$) if and only if $T_1P_1x \in CL(X_1)$ for each $x\in X_1$ (resp., $T_1P_1 \in CL(X_1)$) and

$$\lim_{n\to\infty} T_1P_1x_n = T_1M_1,\ \lim_{n\to\infty} P_1T_1x_n = P_1t$$

whenever $\{x_n\}$ is a sequence in X_1 such that $\lim_{n\to\infty} P_1x_n = M_1 \in CL(X_1)$, $\lim_{n\to\infty} T_1x_n = t \in M_1$.
If the map P_1 in Definition 2.6 is single-valued then M_1 has just a single element t, and we get the definition of reciprocally continuous single-valued maps introduced by Pant [20].

If two systems $(T_1,\dots,T_n)$ and $(P_1,\dots,P_n)$ both are continuous then they are obviously coordinatewise reciprocally continuous but converse need not be true. The following example shows the coordinatewise reciprocal continuity of two systems of maps and illustrates that the coordinatewise reciprocal continuity of two systems of maps does not imply continuity of any system of maps (see also [8], [27]).

Example 2.3. Let $X_1 = X_2 = [0,\infty)$ be usual metric spaces and $P_i : X_1\times X_2 \to CL(X_i), T_i : X_1\times X_2 \to X_i, i = 1,2$, be such that

$$P_1(x_1,x_2) = \begin{cases} [\frac{1}{2}, x_1+1] & \text{if } x_1 > 0 \\ \{0\} & \text{if } x_1 = 0 \end{cases},\quad P_2(x_1,x_2) = \begin{cases} [\frac{1}{2}, x_2+1] & \text{if } x_2 > 0 \\ \{0\} & \text{if } x_2 = 0 \end{cases},$$

$$T_1(x_1,x_2) = \begin{cases} 1 & \text{if } x_1 > 0 \\ 0 & \text{if } x_1 = 0 \end{cases},\qquad T_2(x_1,x_2) = \begin{cases} 1 & \text{if } x_2 > 0 \\ 0 & \text{if } x_2 = 0 \end{cases}.$$

Suppose $\{x^m\}$ be a sequence in $X_1\times X_2$ such that $P_ix^m \to M_i \in CL(X_i)$ and $T_ix^m \to t_i$, for some $t_i \in M_i, i = 1,2$, as $m\to\infty$. Then for $t = (0,0)$ and $\{x^m\} = \{(0,0)\} \in X, m \in \mathbb{N}$, we have $P_ix^m \to \{0\} = M_i, T_ix^m \to 0 = t_i \in M_i$ and $P_i(T_1x^m, T_2x^m) \to P_i(0,0) = P_it, T_i(P_1x^m, P_2x^m) \to T_i(0,0) = T_i(M_1,M_2), i = 1,2$, as $m\to\infty$. Hence the systems of maps $\{P_1,P_2\}$ and $\{T_1,T_2\}$ are coordinatewise reciprocally continuous at $t = (0,0)$. However it is easy to see that each system of maps $\{P_1,P_2\}$ and $\{T_1,T_2\}$ is discontinuous at $t = (0,0)$.

Remark 2.4. The coordinatewise reciprocal continuity of two systems of maps $(P_1,\dots,P_n)$ and $(T_1,\dots,T_n)$ at a point $t \in X$ may be verified by considering all sequences $\{x^m\} \in X$ such that $P_ix^m = M_i \in CL(X_i)$ and $T_ix^m = t_i \in M_i, i = 1,\dots,n$. If there does not exist such a sequence then the definition of coordinatewise reciprocal continuity holds vacuously. The same observation applies for coordinatewise asymptotically commuting maps.The following example illustrates this point.

Example 2.4. Let $X_1 = X_2 = [2,\infty)$, be usual metric spaces and $P_i : X_1\times X_2 \to CL(X_i), T_i : X_1\times X_2 \to X_i, i = 1,2$, be such that

$$P_1(x_1,x_2)=\{1+x_1\},\quad P_2(x_1,x_2)=\{1+x_2\},$$
$$T_1(x_1,x_2)=2x_1+1,\qquad T_2(x_1,x_2)=2x_2+1,\ \text{for all } (x_1,x_2)\in X_1\times X_2.$$

We see that for an arbitrary $t\in X_1\times X_2$ there does not exist any sequence $\{x^m\}\in X_1\times X_2$ such that $P_ix^m\to M_i\in CL(X_i)$ and $T_ix^m\to t_i\in M_i$ for $i=1,2$ as $m\to\infty$. Thus the requirement of coordinatewise reciprocal continuity and asymptotic commutativity are vacuously satisfied.

3. Coincidence Theorem

Now we state our main result.

Theorem 3.1. *Let $(X_i,d_i), i=1,\ldots,n$, be complete metric spaces and $P_i,Q_i : X\to CL(X_i), S_i,T_i : X\to X_i$, be such that*

$$P_i(X)\subset T_i(X) \text{ and } Q_i(X)\subset S_i(X), i=1,\ldots,n. \tag{3.1}$$

The system $(P_1,\ldots,P_n)$ commutes coordinatewise asymptotically with the system $(S_1,\ldots,S_n)$ and the system $(Q_1,\ldots,Q_n)$ commutes coordinatewise asymptotically with the system $(T_1,\ldots,T_n)$.

(3.2)

The systems $(P_1,\ldots,P_n)$ and $(S_1,\ldots,S_n)$ are coordinatewise reciprocally continuous or the systems $(Q_1,\ldots,Q_n)$ and $(T_1,\ldots,T_n)$ are coordinatewise reciprocally continuous.

(3.3)

If there exist non-negative numbers $b<1$ and $a_{ik}, i,k=1,\ldots,n$, defined in (2.1) and (2.2) such that (2.3) and the following hold:

$$H_i(P_ix,Q_iy)\le\max_i\left\{\sum_{k=1}^{n}a_{ik}d_k(S_kx,T_ky),\, b\max\left\{\begin{array}{l}D_i(S_ix,P_ix),D_i(T_iy,Q_iy),\\ \frac{D_i(S_ix,Q_iy)+D_i(T_iy,P_ix)}{2}\end{array}\right\}\right\} \tag{3.4}$$

for all $x,y\in X$, then there exists a point $z\in X$ such that

$$S_iz\in P_iz \text{ and } T_iz\in Q_iz, i=1,\ldots,n. \tag{3.5}$$

Proof. First we note that the system (2.3) and

$$\sum_{k=1}^{n}a_{ik}r_k<r_i, i=1,\ldots,n,$$

are equivalent for some positive numbers $r_i, i=1,\ldots,n$. Further if we put

$$h = \max_i \left\{ r_i^{-1} \sum_{k=1}^{n} a_{ik} r_k \right\}$$

then $h \in (0,1)$ and we may choose positive numbers $r_1, \ldots, r_n$ such that

$$\sum_{k=1}^{n} a_{ik} r_k \leq h r_i, i = 1, \ldots, n \text{ (see, [17]).}$$

Pick x_i^0 in $X_i, i = 1, \ldots, n$. Since (3.1) holds, we can find a point $x^1 \in X$ such that $y_i^1 = T_i x^1 \in P_i x^0, i = 1, \ldots, n$. Since $Q_i(X) \subset S_i(X)$, for a suitable $x^2 \in X$ we can have a point $y_i^2 = S_i x^2 \in Q_i x^1, i = 1, \ldots, n$, such that $d_i(T_i x^1, S_i x^2) \leq c^{-1/2} H_i(P_i x^0, Q_i x^1), i = 1, \ldots, n$, where $c = \max\{h, b\}$. In general, we choose a sequence $\{x^m\}$ in X and $\{y_i^m\}$ in X_i such that

$$y_i^{2m+1} = T_i x^{2m+1} \in P_i x^{2m} \text{ and } y_i^{2m+2} = S_i x^{2m+2} \in Q_i x^{2m+1},$$

$i = 1, \ldots, n, m = 0, 1, \ldots$.

If at any stage $T_i x^{2m+1} = S_i x^{2m+2}$ then $T_i x^{2m+1} \in Q_i x^{2m+1}$ that is, T_i and Q_i have a coincidence point at x^{2m+1} and if $T_i x^{2m+3} = S_i x^{2m+2}$ then $S_i x^{2m+2} \in P_i x^{2m+2}$ that is, S_i and P_i have a coincidence point at x^{2m+2}. We may assume that

$$d_i(y_i^1, y_i^2) \leq r_i, i = 1, \ldots, n.$$

Then by (3.4), we have

$$\begin{aligned}
d_i(y_i^2, y_i^3) &\leq c^{-1/2} H_i(P_i x^2, Q_i x^1) \\
&\leq c^{-1/2} \max \left\{ \sum_{k=1}^{n} a_{ik} d_k(S_k x^2, T_k x^1), b \max \left\{ \begin{array}{c} D_i(S_i x^2, P_i x^2), D_i(T_i x^1, Q_i x^1), \\ \frac{D_i(S_i x^2, Q_i x^1) + D_i(T_i x^1, P_i x^2)}{2} \end{array} \right\} \right\} \\
&\leq c^{-1/2} \max \left\{ \sum_{k=1}^{n} a_{ik} d_k(y_i^2, y_i^1), b \max \left\{ \begin{array}{c} d_i(y_i^2, y_i^3), d_i(y_i^1, y_i^2), \\ \frac{d_i(y_i^2, y_i^2) + d_i(y_i^1, y_i^3)}{2} \end{array} \right\} \right\} \\
&\leq c^{-1/2} \max \left\{ \sum_{k=1}^{n} a_{ik} r_k, b r_i \right\} \leq c^{-1/2} \max\{h r_i, b r_i,\} \leq c^{-1/2} c r_i = c^{1/2} r_i.
\end{aligned}$$

Similarly

$$d_i(y_i^2, y_i^3) \leq c^{2/2} r_i, i = 1, \ldots, n.$$

Inductively

$$d_i(y_i^{m+1}, y_i^{m+2}) \leq c^{m/2} r_i, i = 1, \ldots, n, m = 1, 2, \ldots.$$

So each $\{y_i^m\}$ is a Cauchy sequence in $X_i, i = 1, \ldots, n$, and X_i is a complete metric space. Therefore there exist a point u_i (say) in X_i such that the sequence $\{y_i^m\}$ converges to u_i and their subsequences $\{T_i x^{2m+1}\}$ and $\{S_i x^{2m+2}\}$ also converges to u_i.

Since $T_i x^{2m+1} \in P_i x^{2m}$ and $S_i x^{2m+2} \in Q_i x^{2m+1}, i = 1, \ldots, n$. It follows that $\{P_i x^{2m}\}$ and $\{Q_i x^{2m+1}\}$ are also Cauchy sequences in $CL(X_i), i = 1, \ldots, n$. So there exists M_i in $CL(X_i)$ such that $\{P_i x^{2m}\}$ and $\{Q_i x^{2m+1}\}$ converges to M_i for each $i = 1, \ldots, n$. Thus

$$\begin{aligned} D_i(u_i, M_i) &\leq d_i(u_i, T_i x^{2m+1}) + D_i(T_i x^{2m+1}, M_i) \\ &< d_i(u_i, T_i x^{2m+1}) + H_i(P_i x^{2m}, M_i) \to 0, \end{aligned}$$

as $m \to \infty$. This gives $u_i \in M_i, i = 1, \ldots, n$.

If systems $(P_1, \ldots, P_n)$ and $(S_1, \ldots, S_n)$ are coordinatewise reciprocally continuous then $S_i(P_1 x, \ldots, P_n x) \in CL(X_i)$, for each $x \in X$ and

$$\lim_{m\to\infty} S_i(P_1 x^{2m}, \ldots, P_n x^{2m}) = S_i M, \lim_{m\to\infty} P_i(S_1 x^{2m}, \ldots, S_n x^{2m}) = P_i u, i = 1, \ldots, n.$$

Now coordinatewise asymptotic commutativity of systems of maps $(P_1, \ldots, P_n)$ and $(S_1, \ldots, S_n)$ gives

$$H_i(P_i(S_1 x^{2m}, \ldots, S_n x^{2m}), S_i(P_1 x^{2m}, \ldots, P_n x^{2m})) \to 0, \text{ as } m \to \infty.$$

Making $m \to \infty$,

$$S_i M = P_i u, i = 1, \ldots, n,$$

and as $u_i \in M_i, i = 1, \ldots, n$, we obtain

$$S_i u \in P_i u, i = 1, \ldots, n.$$

Thus u is a coincidence point of systems of maps $(P_1, \ldots, P_n)$ and $(S_1, \ldots, S_n)$. Since $P_i(X) \subset T_i(X)$, therefore there exist a point $v \in X$ such that

$$S_i u = T_i v = z_i \text{ (say) }, i = 1, \ldots, n.$$

By (3.4),

$$\begin{aligned} D_i(T_iv, Q_iv) &\leq H_i(P_iu, Q_iv) \\ &\leq \max\left\{ \sum_{k=1}^{n} a_{ik} d_k(S_ku, T_kv), b\max\left\{ \begin{array}{c} D_i(S_iu, P_iu), D_i(T_iv, Q_iv), \\ \frac{D_i(S_iu, Q_iv) + D_i(T_iv, P_iu)}{2} \end{array} \right\} \right\} \\ &\leq bD_i(T_iv, Q_iv). \end{aligned}$$

This gives $D_i(T_iv, Q_iv) = 0$. Since Q_iv is closed therefore

$$z_i = T_iv \in Q_iv, i = 1, \ldots, n.$$

Thus v is a coincidence point of systems of maps $(Q_1, \ldots, Q_n)$ and $(T_1, \ldots, T_n)$. Since coord-natewise asymptotic commutativity of two systems of maps $(P_1, \ldots, P_n)$ and $(S_1, \ldots, S_n)$ at their coincidence point $u \in X$ gives

$$S_i(P_1u, \ldots, P_nu) \subseteq P_i(S_1u, \ldots, S_nu), i = 1, \ldots, n.$$

Therefore

$$S_iz = S_i(S_1u, \ldots, S_nu) \in S_i(P_1u, \ldots, P_nu) \subseteq P_i(S_1u, \ldots, S_nu) = P_iz$$

or

$$S_iz \in P_iz, i = 1, \ldots, n.$$

Similarly coordnatewise asymptotic commutativity of systems $(Q_1, \ldots, Q_n)$ and $(T_1, \ldots, T_n)$ at their coincidence point $v \in X$ gives

$$T_iz \in Q_iz.$$

Thus the system of inclusion (3.5) has a common solution. In case, when systems of maps $(Q_1, \ldots, Q_n)$ and $(T_1, \ldots, T_n)$ are coordinatewise reciprocally continuous, the proof may be accomplished in an analogous manner. This completes the proof.

If we take $P_ix = Q_ix, i = 1, \ldots, n$, for all $x \in X$, in Theorem 3.1, then we get the following result as a corollary.

Corollary 3.1. *Let $(X_i, d_i), i = 1, \ldots, n$, be complete metric spaces and $P_i : X \to CL(X_i), S_i, T_i : X \to X_i$, be such that*

$$P_i(X) \subset S_i(X) \cap T_i(X), i = 1, \ldots, n.$$

The system $(P_1,\ldots,P_n)$ commutes coordinatewise asymptotically with systems $(S_1,\ldots,S_n)$ and $(Q_1,\ldots,Q_n)$.

The system $(P_1,\ldots,P_n)$ is coordinatewise reciprocally continuous with the system $(S_1,\ldots,S_n)$ or $(T_1,\ldots,T_n)$.

If there exist non-negative numbers $b<1$ and $a_{ik},i,k=1,\ldots,n$, defined in (2.1)and (2.2)such that (2.3) and the following hold:

$$H_i(P_ix,P_iy)\le\max_i\left\{\sum_{k=1}^n a_{ik}d_k(S_kx,T_ky),b\max\left\{\begin{array}{c}D_i(S_ix,P_ix),D_i(T_iy,P_iy),\\ \frac{D_i(S_ix,P_iy)+D_i(T_iy,P_ix)}{2}\end{array}\right\}\right\}$$

for all $x,y\in X$, then there exists a point $z\in X$ such that

$$T_iz\in P_iz \text{ and } S_iz\in P_iz, i=1,\ldots,n.$$

Remark 3.1. Corollary 3.1 generalizes the coincidence and fixed point results of Kaneko [14], Kaneko-Sessa [16], Gairola et al. [9] and others.

Putting $S_ix=T_ix,i=1,\ldots,n$, for all $x\in X$, in Theorem 3.1, we have the following result as a corollary.

Corollary 3.2. *Let $(X_i,d_i),i=1,\ldots,n$, be complete metric spaces and $P_i,Q_i:X\to CL(X_i),T_i:X\to X_i$, be such that*

$$P_i(X)\cup Q_i(X)\subset T_i(X), i=1,\ldots,n.$$

The system $(T_1,\ldots,T_n)$ is coordinatewise asymptotically commuting with systems $(S_1,\ldots,S_n)$ and $(Q_1,\ldots,Q_n)$.

The system $(T_1,\ldots,T_n)$ is coordinatewise reciprocally continuous with the system $(P_1,\ldots,P_n)$ or $(Q_1,\ldots,Q_n)$.

If there exist non-negative numbers $b<1$ and $a_{ik},i,k=1,\ldots,n$, defined in (2.1) and (2.2) such that (2.3) and the following hold:

$$H_i(P_ix,Q_iy)\le\max_i\left\{\sum_{k=1}^n a_{ik}d_k(T_kx,T_ky),b\max\left\{\begin{array}{c}D_i(T_ix,P_ix),D_i(T_iy,Q_iy),\\ \frac{D_i(T_ix,Q_iy)+D_i(T_iy,P_ix)}{2}\end{array}\right\}\right\}$$

for all $x,y\in X$, then there exists a point $u\in X$ such that

$$T_iu\in P_iu\cap Q_iu, i=1,\ldots,n.$$

Remark 3.2. Corollary 3.2 extend and generalize the results of Czerwik [5], Kaneko-Sessa [16], Baillon-Singh [1], Gairola-Jangwan [7] and others.

Remark 3.3. If we take $S_ix = T_ix = x_i, i = 1, \dots, n$, in Theorem 3.1, and $T_ix = x_i, i = 1, \dots, n$, in Corollary 3.2, then the condition of reciprocal continuity is not needed. So the following fixed point theorem is an immediate consequence from Theorem 3.1 and Corollary 3.2.

Corollary 3.3. *Let $(X_i, d_i), i = 1, \dots, n$, be complete metric spaces and $P_i, Q_i : X \to CL(X_i)$, be such that*

$$H_i(P_ix, Q_iy) \leq \max_i \left\{ \sum_{k=1}^{n} a_{ik} d_k(x_k, y_k), b \max \left\{ \begin{array}{c} D_i(x_i, P_ix), D_i(y_i, Q_iy), \\ \frac{D_i(x_i, Q_iy) + D_i(y_i, P_ix)}{2} \end{array} \right\} \right\}$$

for all $x, y \in X$, where $b \in [0, 1), a_{ik} \geq 0, i, k = 1, \dots, n$. If $c_{ik}^{(t)}$ defined in (2.1) and (2.2) satisfying (2.3) then the system of multivalued maps $(P_1, \dots, P_n)$ and $(Q_1, \dots, Q_n)$ has a common fixed point.

The following result may be obtained as a special case from Corollary 3.3, taking $(Y, d) = (X_i, d_i), P = P_i = Q_i, i = 1, \dots, n, n = 1$ and $k = \max\{a_{11}, b\}$ in Corollary 3.3.

Corollary 3.4. *Let Y be a complete metric space and take a mapping $P : Y \to CL(Y)$. If there exist a constant $k, 0 \leq k < 1$, such that for all $x, y \in Y$.*

$$H(Px, Py) \leq k \max \{d(x, y), D(x, Px), D(y, Py), [D(x, Py) + D(y, Px)]/2\},$$

then P has a fixed point in Y.

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Conflict of Interests

The authors declare that there is no conflict of interests.

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