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ON JUNGCK'S COMMON FIXED POINT THEOREM

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Abstract. Imdad and Javid proved an interesting generalization of Jungck's common fixed point theorem for two commuting self-mappings of a complete metric space. However, their result require that the range of one of the mappings is a complete subspace of the metric space. In this paper, we use the *(CLRg)* property to obtain one which does not require the completeness of the range of the mappings involved therein. Our main result generalizes, in particular, single-valued versions of the classical common fixed point results of Kaneko and Sessa [Internat. J. Math. & Math. Sci. 12 (2) (1989), 257 – 262] and Pathak [Acta Math. Hungar 67 (1995), 69 – 78]. Also, we provide an example to distinguish our result from previously known results.

Keywords: common fixed points and implicit functions; *(CLRg)* property; weak compatibility.

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1. Introduction

Jungck [1] introduced and discussed the notion of commuting mappings and proved a generalization of celebrated Banach contraction principle for two commuting self-mappings of a complete metric space. Sessa [3] and Jungck [2] introduced the notions of weakly commuting

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mappings and compatible mappings, respectively in common fixed point considerations. Consequently, the existing literature contains several common fixed point results established under weak commutativity conditions.

On the other hand, Popa [7] defined an implicit relation and proved some common fixed point theorems for compatible mappings satisfying the implicit relation. In [5], Imdad and Javid deduced several contractive conditions from the Popa's implicit relation. They further established a generalization of the Jungck's common fixed point theorem in [1] which satisfy the implicit relation under the (E.A) *property* due to Aamri and El Moutawakil [4].

In this paper, we state and prove a general common fixed point theorem for two self-mappings of a metric space under *(CLRg)* property satisfying an implicit relation.

2. Preliminaries

The following definitions and facts will be frequently used in the sequel.

Let X be a non-empty set, and $f, g : X \to X$ be mappings. A point $t \in X$ is called a common fixed point of the self-mappings f and g if $t = ft = gt$. If a point $b \in X$ is such that $fb = gb$, then such b is called a coincidence point of the mappings.

Definition 2.1. [2] Let (X,d) be a metric space. The mappings $f, g : X \to X$ are said to be compatible if and only if $d(fgx_n, gfx_n)$ approaches 0 whenever $\{x_n\}$ is a sequence in X such that $\{fx_n\}$ approaches t, $\{gx_n\}$ approaches t for some point $t \in X$.

In 1976, Jungck [1] proved the following common fixed point theorem:

Theorem 2.1. *Let f be a continuous mappings of a complete metric space (X,d) into itself. Then f has a fixed point in X if there exist $\alpha \in (0,1)$ and a mapping $g : X \to X$ which commutes with f and satisfies $g(X) \subset f(X)$ and $d(gx, gy) \leq \alpha d(fx, fy)$, for all $x, y \in X$.*

Definition 2.2. [8] Mappings $g : X \to X$, and $f : X \to X$ are said to be weakly compatible if $gfx = fgx$ whenever $gx = fx$.

Definition 2.3. [4] Let (X,d) be a metric space. Mappings $g, f : X \to X$ are said to satisfy *property* (E.A) if there exists a sequence $\{x_n\} \subset X$ such that both $\{gx_n\}$ and $\{fx_n\}$ converge to t for some $t \in X$.

Clearly, the class of mappings satisfying *property* $(E.A)$ contains both compatible and non-compatible mappings.

Definition 2.4. [9] Let (X,d) be a metric space and $f,g : X \to X$ be mappings. The mappings f and g are said to satisfy the *common limit in the range of* g property (*(CLRg)* property for short) if $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$ for some $x \in X$.

Example 2.1. Let $X = [-1,1]$ equipped with the usual metric and $f,g : X \to X$ be mappings defined as follows:

$$fx = \begin{cases} \frac{1}{3}, & \text{if } x = -1 \\ \frac{x}{4}, & \text{if } -1 < x < 1 \\ \frac{3}{5}, & \text{if } x = 1 \end{cases}$$

and

$$gx = \begin{cases} \frac{1}{3}, & \text{if } x = -1 \\ \frac{x}{2}, & \text{if } -1 < x < 1 \\ \frac{4}{5}, & \text{if } x = 1 \end{cases}$$

For a sequence $\{x_n\} = \{\frac{1}{n}\}$, we have $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = g0$. Thus, the mappings f and g satisfy the *(CLRg)* property.

Notice that neither the range of f nor the range of g contains the other.

In 1999, Popa [7] introduced the following implicit relation and proved some fixed point theorems for compatible mappings satisfying the relation. To describe the implicit relation, let Ψ be the family of real lower semi-continuous functions $F(t_1,t_2,...,t_6) : [0,\infty)^6 \to \mathbb{R}$ satisfying the following conditions:

(ψ_1) F is non-increasing in the variables t_5 and t_6,

(ψ_2) there exists $h \in (0,1)$ such that for every $u,v \geq 0$ with

(ψ_{21}) $F(u,v,v,u,u+v,0) \leq 0$ or

(ψ_{22}) $F(u,v,u,v,0,u+v) \leq 0$ we have $u \leq hv$, and

(ψ_3) $F(u,u,0,0,u,u) > 0$, $\forall u > 0$.

The following examples of such functions appear in [5, 7].

Example 2.2. Define $F(t_1,t_2,...,t_6):[0,\infty)^6 \to \mathbb{R}$
as $F(t_1,t_2,...,t_6)=t_1-k\max\{t_2,t_3,t_4,\frac{1}{2}(t_5+t_6)\}$, where $k\in(0,1)$

Example 2.3. Define $F(t_1,t_2,...,t_6):[0,\infty)^6 \to \mathbb{R}$
as $F(t_1,t_2,...,t_6)=t_1-h\max\left\{t_2,\dfrac{t_3+t_4}{2},\dfrac{t_5+t_6}{2}\right\}$, where $h\in(0,1)$.

Example 2.4. Define $F(t_1,t_2,...,t_6):[0,\infty)^6 \to \mathbb{R}$
as $F(t_1,t_2,...,t_6)=t_1^2-at_2^2-\dfrac{bt_5t_6}{1+t_3^2+t_4^2}$, where $a>0$, $b\geq 0$ and $a+b<1$.

Example 2.5. Define $F(t_1,t_2,...,t_6):[0,\infty)^6 \to \mathbb{R}$
as $F(t_1,t_2,...,t_6)=t_1^2-t_1(at_2+bt_3+ct_4)-dt_5t_6$, where $a>0$, $b,c,d\geq 0$, $a+b+c<1$, and $a+d<1$.

Imdad and Javid [5] proved the following interesting generalization of Theorem 2.1. satisfying the implicit relation described just above.

Theorem 2.2. *Let f and g be self-mappings of a metric space (X,d) such that :*

(i) *f and g satisfy* property $(E.A)$,

(ii) $\forall x,y\in X$ *and* $F\in\Psi$,

$$F(d(fx,fy),d(gx,gy),d(gx,fx),d(gy,fy),d(gx,fy),d(gy,fx))\leq 0,$$

(iii) *$g(X)$ is a complete subspace X,*

Then

(a) *the pair (f,g) has a point of coincidence,*

(b) *the pair (f,g) has a common fixed point provided it is weakly compatible.*

We notice that Theorem 2.2. require that $g(X)$ is a complete subspace of X, which may not always be the case. Therefore, we cannot apply Theorem 2.2. in the event that $g(X)$ is not complete.

The purpose of this work is to prove a generalization of Theorem 2.1. that relaxes the requirement on completeness of the range of g.

3. Main results

Theorem 3.1. *Let (X,d) be a metric space and $f,g : X \to X$ be mappings such that :*

(i) *f and g satisfy the* (CLRg) *property,*

(ii) *$\forall x,y \in X$ and $F \in \Psi$,*

$$F(d(fx,fy),d(gx,gy),d(gx,fx),d(gy,fy),$$

$$d(gx,fy),d(gy,fx)) \leq 0, \tag{1}$$

Then

(a) *the pair (f,g) has a point of coincidence,*

(b) *the pair (f,g) has a common fixed point provided it is weakly compatible.*

Proof. Since f and g satisfy *(CLRg)* property, then there exists a sequence $\{x_n\} \subset X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = ga = t \in X$. We claim that $ga = fa$. Suppose not. Then $d(ga, fa) > 0$. Now, from Condition (1), we have

$F(d(fa,fx_n),d(ga,gx_n),d(ga,fa),d(gx_n,fx_n),d(ga,fx_n),d(gx_n,fa)) \leq 0$

Taking limit as $n \to \infty$ gives

$F(d(fa,t),d(ga,t),d(ga,fa),d(t,t),d(ga,t),d(t,fa)) \leq 0$ or

$F(d(fa,ga),d(t,t),d(ga,fa),d(t,t),d(t,t),d(ga,fa)) \leq 0$ or

$F(d(fa,ga),0,d(ga,fa),0,0,d(ga,fa)) \leq 0$, which by (ψ_{22}) implies

that $d(fa,ga) \leq 0$. Therefore, $fa = ga$. This proves (a).

Now we establish (b). Suppose that f and g are weakly compatible. Then we have $gt = gfa = fga = ft$. We claim that $ft = t$. Suppose not. Then $d(ft,t) > 0$. From Condition (1), we have

$F(d(ft,fa),d(gt,ga),d(gt,ft),d(ga,fa),d(gt,fa),d(ga,ft)) \leq 0$ or

$F(d(ft,t),d(ft,t),d(gt,ft),d(t,t),d(ft,t),d(t,ft)) \leq 0$ or

$F(d(ft,t),d(ft,t),0,0,d(ft,t),d(t,ft)) \leq 0$, which is a contradiction to (ψ_3). Therefore $d(ft,t) = 0$. Hence t is a common fixed point of the mappings f and g. Further, we claim that t is unique. Suppose not and $s \neq t$ is also a common fixed point of the mappings. Then from Condition (1) we have

$F(d(ft,fs),d(gt,gs),d(gt,ft),d(gs,fs),d(gt,fs),d(gs,ft)) \leq 0$ or

$F(d(t,s),d(t,s),d(t,t),d(s,s),d(t,s),d(s,t)) \leq 0$ or

$F(d(t,s),d(t,s),0,0,d(t,s),d(s,t)) \leq 0$ which contradicts (ψ_3).

Therefore $d(s,t) = 0$. Hence $s = t$. This completes the proof.

The following Corollary is a generalization of single-valued versions of Theorems 1 and 2 in [6, 10].

Corollary 3.1. *Let* (X,d) *be a metric space and* $f,g : X \to X$ *be mappings such that :*

(i) f *and* g *satisfy the* (CLRg) *property,*

(ii) $\forall x,y \in X$,

$$d(fx,fy) \leq h\max\{d(gx,gy),d(gx,fx), d(gy,fy),\frac{1}{2}[d(gx,fy)+d(gy,fx)]\}$$

where $h \in (0,1)$.

Then

(a) *the pair* (f,g) *has a point of coincidence,*

(b) *the pair* (f,g) *has a common fixed point provided it is weakly compatible.*

Example 3.1. Let $X = [-1,1]$ equipped with the usual metric and f and g as defined in Example 2.1. Clearly, the mappings are weakly compatible as $fg0 = gf0$, and 0 is their point of coincidence. In fact, 0 is their unique common fixed point. Consider a continuous function $F(t_1,t_2,...,t_6) = t_1 - k\max\{t_2,t_3,t_4,\frac{1}{2}(t_5+t_6)\}$, where $k \in (0,1)$, it can be easily established that f and g satisfy Condition (1) for $k = \frac{12}{13}$.

Remark 3.1. Notice that Theorem 2.1. does not apply to Example 3.1. because the mappings f and g are discontinuous. Also, since $g(X) = (-\frac{1}{2},\frac{1}{2}) \cup \{\frac{1}{3},\frac{4}{5}\}$ is not a complete (closed) subspace of X, then Theorem 2.2. is not applicable too.

Conflict of Interests

The author declares that there is no conflict of interests.

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