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# NEW ITERATION PROCESS FOR TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN UNIFORMLY CONVEX HYPERBOLIC SPACES

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**Abstract.** In this paper, we establish a fixed point result for a total asymptotically nonexpansive mapping along with the iterative construction of common fixed point of a finite family of these mappings in uniformly convex hyperbollic spaces. Our convergence results can be viewed not only as an analog of various existing results but also improve and generalize various results in the current literature.

**Keywords:** total asymptotically nonexpansive mappings; common fixed point; hyperbolic space; modulus of uniform convexity; asymptotic center.

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## 1. Introduction

Let $(X,d)$ be a metric space and $x,y \in X$ with $l = d(x,y)$. A geodesic from $x$ to $y$ in $X$ is a mapping $\theta : [0,l] \to X$ such that

$$\theta(0) = x, \theta(l) = y \text{ and } d(\theta(s),\theta(t)) = |s-t| \text{ for all } s,t \in [0,l].$$

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The above characteristics is reffered to as a constant speed of $\theta$, the parametrization of $\theta$ with respect to the arc length or distance preservation of $\theta$. The points $x=\theta(0)$ and $y=\theta(l)$ are called the end points or the extreme (maximal or minimal) points of the segment. The metric space $(X,d)$ is called a geodesic space if for every pair of points $x,y\in X$, there is a geodesic segment from $x$ to $y$. Moreover $(X,d)$ is uniquely geodesic if for all $x,y\in X$ there is exactly one geodesic from $x$ to $y$.

The class of hyperbolic spaces introduced by Kohlenbach [1] is an important example of a uniquely geodesic spaces . It is worth to mention that this class is prominent among various other notions of hyperbolic spaces in the current literature, for convenience of the reader; see [2-5]. The study of hyperbolic spaces has been largely motivated and dominated by questions about hyperbolic groups, one of the main objects of study in geometric group theory. We remark that the non-positively curved spaces, such as hyperbolic spaces, play a significant role in many branches of applied mathematics.

Throughout this paper, we work in the setting of hyperbolic spaces introduced by Kohlenbach [1].

A hyperbolic space [1] is a metric space $(X,d)$ together with a mapping $W:X^2\times[0,1]\to X$ satisfying

($W_1$) $d(u,W(x,y,\alpha))\leq \alpha d(u,x)+(1-\alpha)d(u,y)$,

($W_2$) $d(W(x,y,\alpha),W(x,y,\beta))=|\alpha-\beta|d(x,y)$,

($W_3$) $W(x,y,\alpha)=W(y,x,(1-\alpha))$,

($W_4$) $d(W(x,z,\alpha),W(y,w,\alpha))\leq \alpha d(x,y)+(1-\alpha)d(z,w)$

for all $x,y,z\in X$ and $\alpha,\beta\in[0,1]$.

The class of hyperbolic spaces in the sense of Kohlenbach [1] contains all normed linear spaces and convex subsets thereof as well as Hadamard manifolds and CAT(0) spaces in the sense of Gromov. An important example of a hyperbolic space due to Goebel and Reich [3] is stated as follows.

Let $B_H$ be the open unit ball in a general complex Hilbert space $(H,\langle .,.\rangle)$ and let $k_{B_H}$ be a metric on $B_H$ (also known as Kobayashi distance) defined as

$$\mathrm{k}_{B_H}(x,y): tanh^{-1}(1-\sigma(x-y))^{\frac{1}{2}}, \text{ where } \sigma(x-y)=\frac{(1-\|x\|^2)(1-\|y\|^2)}{|1-\langle x,y\rangle|^2} \text{ for all } x,y\in B_H.$$

The open unit ball $B_H$ together with the metric $k_{B_H}$ is coined as a Hilbert ball. Since $(B_H,k_{B_H})$ is a unique geodesic space, one can define a convexity mapping $W$ for the corresponding hyperbolic space $(B_H,k_{B_H},W)$. This space is of significant importance for the fixed point theory of holomorphic mappings as the said class of mappings is in $k_{B_H}$ nonexpansive $(B_H,k_{B_H},W)$. A metric space $(X,d)$ satisfying only $(W_1)$ is a convex metric space introduced by Takahashi [6]. A subset $K$ of a hyperbolic space $X$ is convex if $W(x,y,\alpha) \in K$ for all $x,y \in K$ and $\alpha \in [0,1]$. For more on hyperbolic spaces and a comparision between different notions of hyperbolic spaces in the current literature, we refer to [1], p.384.

It is worth to mention that the fixed point theory of nonexpansive mappings (i.e., $d(T_x,T_y) \leq d(x,y)$ for $x,y \in K$) and its various generalization majorly depends on the geometrical characteristics of the underlying space. The class of nonexpansive mappings enjoys the fixed point property (FPP) and the approximate fixed point property (AFPP) in various settings of spaces, see for example [7] for the later property for the class of nonexpansive mappings. Moreover, it is natural to extend such powerful results to generalized nonexpansive mappings as a mean of testing the limit of the theory of nonexpansive mappings. It is remarked that the FPP and even AFPP, in a nonlinear domain, of various generalizations of nonexpansive mappings are still developing. The class of hyperbolic spaces is endowed with rich geometric structures for different results with applications in topology, graph theory, multivalued analysis and metric fixed point theory. An important ingredient for metric fixed point theory of nonexpansive mappings is uniform convexity.

A hyperbolic space $X,d,W$ is uniformly convex [8] if for all $u,x,y \in X, r > 0$, and $\varepsilon \in (0,2]$, there exists $\delta \in (0,1]$ such that

$$\text{d(W(x, y,}\tfrac{1}{2}\text{), u)} \leq (1-\delta)r$$

whenever $d(x,u) \leq r, d(y,u) \leq r$ and $d(x,y) \geq r\varepsilon$.

A mapping $\eta : (0,\infty) \times (0,2] \to (0,1)$ providing such $\delta = \eta(r,\varepsilon)$ for given $r > 0$ and $\varepsilon \in (0,2]$ is called modulus of uniform convexity. We call $\eta$ monotone if it decreases with $r$ (for a fixed $\varepsilon), i.e., \forall \varepsilon > 0, \forall r_2 \geq r_1 > 0(\eta(r_2,\varepsilon) \leq \eta(r_1,\varepsilon))$. The CAT(0) spaces are uniformly convex hyperboic spaces with modulus of uniform convexity $\eta(r,\varepsilon) = \frac{\varepsilon^2}{8}$ [8]. Therefore, the class of uniformly convex hyperbolic spaces includes CAT (0) spaces as a special case.

Metric fixed point theory of nonlinear mappings in a general setup of hyperbolic spaces is a fascinating field of research in nonlinear functional analysis. Moreover, iteration schemes are the only main tool to study fixed point problems of nonexpansive mappings and its various generalization in spaces of non-positive sectional curvature. In 2006, Alber et al. [9] introduced a unified and generalized notion of a class of nonlinear mappings in Banach spaces, which can be introduced in the general setup of hyperbolic spaces as followes.

A self mapping $T : K \to K$ is called

(1) an asymptotically nonexpansive mapping if there exists a nonnegative real sequence $\{K_n\}$ with $k_n \to 0$ such that

$$d(T^n x, T^n y) \leq d(x,y) + k_n d(x,y) \text{ for all } x,y \in K, n \geq 1$$

and

(2) total asymptotically nonexpansive mapping if there exists a nonnegative real sequence $\{K_n\}$ and $\{\varphi_n\}$ with $K_n \to 0$ and $\varphi_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\xi : R^+ \to R^+$ with $\xi(0) = 0$ such that

$$d(T^n x, T^n y) \leq d(x,y) + k_n \xi(d(x,y)) + \varphi_n \text{ for all } x, y \in K, n \geq 1. \tag{1.1}$$

**Example 1.1**

(i) Let $X = R, K = [0,\infty)$ and $T : K \to K$ be defined by $T_x = sinx$. Then $T$ is a total asymptotically nonexpansive.

(ii) Let $X = R, K = [\frac{-1}{\pi}, \frac{1}{\pi})$ and $T : K \to K$ be defined by $T_x = kxsin\frac{1}{x}$, where $k \in (0,1)$. Then $T$ is a total asymptotically nonexpansive.

(iii) Let $K = \{x := (x_1, x_2, ..., x_n, ...) | x_1 \leq 0, x_i \in R, i \geq 2\}$ be a nonempty subset of $X = l^2$ with the norm $\|.\|$ defined as

$\|x\| = \sqrt{\sum_{i=1}^{\infty} x_i^2}$.

if $T : K \to K$ is defined by

$T(x) = (0, 4x_2, 0, 0, 0, ...)$,

Then $T$ is an asymptoticaly nonexpansive.

(iv) Let $X = R$ and $K = [0,2)$. Let $T : K \to K$ be a mapping defined by

$$Tx = \begin{cases} 1, x \in [0,1], \\ \frac{1}{\sqrt{3}}\sqrt{4-x^2}, x \in [1,2]. \end{cases}$$

Then $T$ is a total asymptotically nonexpansive mapping with $F(T) = \{1\}$. However, $T$ is not a Lipschitzian and hence it is not an asymptotically nonexpansive mapping.

The class of total asymptotically nonexpansive mapping and asymptotically nonexpansive mappings have been studied extensively in the literature [10-16] and the references cited therein. It is worth mentioning that the results established for total asymptotically nonexpansive mappings are applicable to the mappings associated with the class of asymptotically nonexpansive mappings and which are extensions of nonexpansive mappings.

It is remarked that the iterative construction of common fixed points of a finite family of asymptotically quasi-nonexpansive mappings in a Banach space through higher arity of an iteration is essentially due to Khan et al. [18] (see also [19] for the case of nonexpansive mappings). This iteration was further generalized by Khan and Ahmad [19] and Khan et al. [20] to the setup of convex metric spaces and CAT(0) spaces, respectively.

The purpose of this paper, we establish a fixed point result for a total asymptotically nonexpansive mapping along with the iterative construction of common fixed point of a finite family of these mappings in uniformly convex hyperbollic spaces. Our convergence results can be viewed not only as an analog of various existing results but also improve and generalize various results in the current literature.

## 2. Preliminaries and some auxiliary lemmas

We start this section with the notion of asymptotic center - essentially due to Edelstein [21] - of a sequence which is not only useful in proving a fixed point result but also plays a key role to define the concept of $\Delta$-convergence in hyperbolic spaces. In 1976, Lim [22] introduced the concept of $\Delta$-convergence in the general setting of metric spaces. In 2008, Kirk and Panyanak [23] further analyzed this concept in geodesic spaces. They showed that many Banach space results involving weak convergence have a precise analog version of $\Delta$-convergence in geodesic spaces.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space $X$. For $x \in X$, define a continuous functional $r(.,\{x_n\}) : X \to [0,\infty)$ by

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n).$$

The asymptotic radius and asymptotic center of the bounded sequence $\{x_n\}$ with respect to a subset $K$ of $X$ is defined and denoted thus:

$$r_K(\{x_n\}) = inf\{r(x,\{x_n\}) : x \in K\}$$

and

$$A_K(\{x_n\}) = \{x \in K : r(x,\{x_n\}) \leq r(y,\{x_n\}) \; for \; all \; y \in K\},$$

respectively.

Recall that a sequence $\{x_n\}$ in $X$ is said to $\Delta$-convergence to $x \in X$ if $x$ is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta - lim_n x_n = x$ and call $x$ the $\Delta$- limit of $\{x_n\}$.

A mapping $T : K \to K$ is semi-compact if every bounded sequence $\{x_n\} \subset K$ satisfying $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, has a convergent subsequnce.

We now list some useful lemmas as well as establish some auxiliary results required in the sequel.

**Lemma 2.1.** [(24)] *Let $(X,d,W)$ be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity $\eta$. Then every bounded sequence $\{x_n\}$ in $X$ has a unique asymptotic center with respect to any nonempty closed convex subset $K$ of $X$.*

**Proposition 2.2.** [(25)] *Let $(X,d,W)$ be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity $\eta$. The intersection of any decreasing sequence of nonempty bounded closed convex subsets of $X$ is nonempty.*

**Lemma 2.3.** [(26)] *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. If $a_{n+1} \leq (1+b_n)a_n + c_n, n \geq 1$, then $lim_{n\to\infty} a_n$ exists.*

**Lemma 2.4.** [(27)] *Let $(X,d,W)$ be a uniformly convex hyperbolic space with monotone modulus of uniform convexity $\eta$. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a,b]$ for some $a,b \in (0,1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that $\limsup_{n\to\infty} d(x_n,x) \leq c$, $\limsup_{n\to\infty} d(y_n,x) \leq c$ and $\lim_{n\to\infty} d(W(x_n,y_n,\alpha_n),x) = c$, for some $c \geq 0$, then $\lim_{n\to\infty} d(x_n,y_n) = 0$.*

**Lemma 2.5.** [(27)] *Let $K$ be a nonempty, closed, and convex subset of a uniformly convex hyperbolic space $X$ and $\{x_n\}$ a bounded sequence in $K$ such that $A_K(\{x_n\}) = \{y\}$ and $r_K(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in $K$ such that $\lim_{m\to\infty} r(y_m,\{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.*

# 3. Main Results

**Theorem 3.1.** *Let $C$ be a nonempty, closed and convex subset of a hyperbolic space $X$ and $T : C \to C$ be a $(\{\mu_n\},\{v_n\},\xi)$ uniformly lipschitzian total asymptotically nonexpansive if the following condition are satisfied:*

(i) $\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} v_n < \infty$;

(ii) *there exists a constant $M > r$ such that $\xi(r) \leq Mr$, $r \geq 0$; then sequence $\{x_n\}$ defined by*

$$\begin{aligned} x_{n+1} &= W(T^n y_n, 0, 0) \\ y_n &= W(T^n x_n, T^n z_n, \alpha_n) \\ z_n &= W(T^n x_n, x_n, \beta_n) \end{aligned} \tag{3.1}$$

*for all $n \in N$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ then it is bounded and $lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F(T)$.*

*Proof.* Since $T$ is lipschitz continous then $F(T) \neq \phi$, so for $p \in F(T)$ and from (3.1), we have

$$\begin{aligned} d(z_n,p) &= d(W((T^n x_n, x_n, \beta_n), p) \\ &\leq (1-\beta_n)d(x_n,p) + \beta_n d(T^n x_n, p) \\ &\leq (1-\beta_n)d(x_n,p) + \beta_n[d(x_n,p) + \mu_n \xi(d(x_n,p)) + v_n] \end{aligned} \tag{3.2}$$

$$
\begin{aligned}
&\leq (1-\beta_n)d(x_n,p)+\beta_n d(x_n,p)+\beta_n\mu_n\xi(d(x_n,p))+\beta_n v_n\\
&\leq d(x_n,p)+\beta_n\mu_n\xi(d(x_n,p))+v_n\\
&\leq d(x_n,p)+\beta_n\mu_n M d(x_n,p)+v_n\\
&\leq (1+\beta_n\mu_n M)d(x_n,p)+v_n\\
&\leq (1+\mu_n M)d(x_n,p)+v_n. \qquad (3.3)
\end{aligned}
$$

using (3.1) and (3.2), we have

$$
\begin{aligned}
d(y_n,p) &= d(W((T^n x_n,T^n z_n,\alpha_n),p)\\
&\leq (1-\alpha_n)d(T^n x_n,p)+\alpha_n d(T^n z_n,p)\\
&\leq (1-\alpha_n)[d(x_n,p)+\mu_n\xi(d(x_n,p))+v_n]+\alpha_n[d(z_n,p)+\mu_n\xi(d(z_n,p))+v_n]\\
&\leq (1-\alpha_n)[(1+\mu_n M)d(x_n,p)+v_n]+\alpha_n[(1+\mu_n M)d(z_n,p)+v_n]\\
&\leq (1+\mu_n M)[(1-\alpha_n)d(x_n,p)+\alpha_n)d(z_n,p)]+v_n\\
&\leq (1+\mu_n M)[(1-\alpha_n)d(x_n,p)+\alpha_n((1+\mu_n M)d(x_n,p)+v_n]+v_n\\
&\leq (1+\mu_n M)[(1-\alpha_n)+\alpha_n(1+\mu_n M)]d(x_n,p)+v_n+(1+\mu_n M)v_n\alpha_n\\
&\leq (1+\mu_n M)^2 d(x_n,p)+v_n[2+v_n M] \qquad (3.4)
\end{aligned}
$$

Hence from (3.1) and (3.3), we have

$$
\begin{aligned}
d(x_{n+1},p) &= d(W((T^n y_n,0,0),p)\\
&\leq d(T^n y_n,p)\\
&\leq d(y_n,p)+\mu_n\xi(d(y_n,p))+v_n\\
&\leq (1+\mu_n M)d(y_n,p)+v_n\\
&\leq (1+\mu_n M)[(1+\mu_n M)^2 d(x_n,p)+v_n(2+\mu_n M)]+v_n\\
&\leq (1+\mu_n M)^3 d(x_n,p)+v_n[(1+\mu_n M)(2+\mu_n M)+1]\\
&\leq (1+\mu_n M)^3 d(x_n,p)+v_n[2+2\mu_n M+\mu_n M+\mu_n^2 M^2+1]\\
&\leq (1+\mu_n M)^3 d(x_n,p)+v_n[3+3\mu_n M+\mu_n^2 M^2]\\
&\leq (1+\mu_n^3 M^3+3\mu_n M^2+3\mu_n^2 M^2)d(x_n,p)+v_n(3+3\mu_n M+\mu_n^2 M^2)\\
&\leq [1+\mu_n M(3M+3\mu_n+\mu_n^2 M^2)]d(x_n,p)+v_n(3+3\mu_n M+\mu_n^2 M^2)\\
&\leq (1+\sigma_n)d(x_n,p)+\rho_n \qquad (3.5)
\end{aligned}
$$

Where $\sigma_n = \mu_n M(3M + 3\mu_n + \mu_n^2 M^2)$, $\rho_n = v_n(3 + 3\mu_n M + \mu_n^2 M^2)$. It follows that the lemma 2.3, sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, p)$ exists. This completes the proof.

**Theorem 3.2.** *Let $C$ be a nonempty, closed and convex subset of a uniformly convex hyperbolic space $X$ with monotone modulus of uniform convexity $n$ and $T : C \to C$ be a uniformly lipschitzian and $(\{\mu_n\}, \{v_n\}, \xi)$ - total asymptotically nonexpansive mapping with $\{\mu_n\}$and$\{v_n\}, n \in N$ such that $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} v_n < \infty$. Let the sequence $\{x_n\}$ defined by (3.1) then it has approximate fixed point property for $T$ i.e.* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

*Proof.* From theorem 3.1, we have $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F(T)$, without loss of generatily we assume that $\lim_{n\to\infty} d(x_n, p) = r \geq 0$. If $r = 0$ then it is obvious from (3.2), we have

$$\begin{aligned}
\limsup_{n\to\infty} d(z_n, p) &\leq \limsup_{n\to\infty} [(1 + \mu_n M) d(x_n, p) + v_n] \\
&\leq \limsup_{n\to\infty} d(x_n, p) \\
&\leq r
\end{aligned} \tag{3.6}$$

Again from equation (3.1) and (3.5), we have

$$\begin{aligned}
\limsup_{n\to\infty} d(T^n z_n, p) &\leq \limsup_{n\to\infty} [d(z_n, p) + \mu_n \xi(d(z_n, p)) + v_n] \\
&\leq \limsup_{n\to\infty} [(1 + \mu_n M) d(z_n, p) + v_n] \\
&\leq \limsup_{n\to\infty} d(z_n, p) \\
&\leq r
\end{aligned} \tag{3.7}$$

From (3.3) we have

$$\begin{aligned}
\limsup_{n\to\infty} d(y_n, p) &\leq \limsup_{n\to\infty} [(1 + \mu_n M)^2 d(x_n, p)) + v_n(2 + \mu_n M)] \\
&\leq \limsup_{n\to\infty} d(x_n, p) \\
&\leq r
\end{aligned} \tag{3.8}$$

But

$$\begin{aligned}
\limsup_{n\to\infty} d(T^n y_n, p) &\leq \limsup_{n\to\infty} [(1 + \mu_n M)(d(y_n, p)) + v_n] \\
&\leq \limsup_{n\to\infty} d(y_n, p) \\
&\leq r.
\end{aligned} \tag{3.9}$$

and

$\limsup_{n\to\infty} d(T^n x_n, p) \leq \limsup_{n\to\infty}[(1+\mu_n M)(d(x_n,p))+v_n]$
$\leq r.$

Now from (3.1)

$$\begin{aligned} d(x_{n+1},p) &= d(T^n y_n, p) \\ &\leq d(y_n,p)+\mu_n\xi(d(y_n,p))+v_n \\ &\leq (1+\mu_n M)d(y_n,p)+v_n \end{aligned} \tag{3.10}$$

taking limit inferior both the side, we have

$$\begin{aligned} \liminf_{n\to\infty} d(x_{n+1},p) &\leq \liminf_{n\to\infty}(d(y_n,p) \\ r &\leq \liminf_{n\to\infty}(d(y_n,p) \end{aligned} \tag{3.11}$$

Hence from (3.7) and (3.9), we have

$\lim_{n\to\infty} d(y_n,p) = \lim_{n\to\infty} d(w(T^n x_n, T^n z_n, \alpha_n), p) = r$

therefore, by lemma 2.4, we have

$$\lim_{n\to\infty} d(T^n z_n, T^n x_n) = 0 \tag{3.12}$$

again from

$$\begin{aligned} d(x_{n+1},p) &= d(T^n y_n,p) \\ &\leq (1+\mu_n M)d(y_n,p)+v_n \\ &\leq (1+\mu_n M)[d(W(T^n x_n, T^n z_n),p]+v_n \\ &\leq (1+\mu_n M)[(1-\alpha_n)d(T^n x_n,p)+\alpha_n d(T^n z_n,p)+v_n] \\ &\leq (1+\mu_n M)[(1-\alpha_n)(1+\mu_n M)d(x_n,p)+\alpha_n(1+\mu_n M)d(z_n,p)+(1-\alpha_n)v_n+\alpha_n v_n]+v_n \\ &\leq (1+\mu_n M)^2[(1-\alpha_n)d(x_n,p)+\alpha_n d(z_n,p)]+2v_n \\ d(x_n,p) &\leq \frac{(1+\mu_n M)^2 d(x_n,p)}{\alpha_n} - \frac{d(x_{n+1},p)}{\alpha_n} + d(z_n,p) + \frac{2v_n}{\alpha_n} \end{aligned}$$

taking limit inferior both the side we get

$$r \leq \liminf_{n\to\infty}(d(z_n,p)$$

(3.13)

From equation (3.5) and (3.12) we have

$\lim_{n\to\infty}(d(z_n,p) = \lim_{n\to\infty} d(W(x_n,T^n x_n,\beta_n),p) = r$

therefore, by lemma 2.4, we have

$$\lim_{n\to\infty}(d(x_n,T^n x_n) = 0$$

(3.14)

Next we compute

$$\begin{aligned} d(x_{n+1},x_n) &= d(T^n y_n,x_n) \\ &\leq d(T^n y_n,T^n z_n)+d(T^n z_n,T^n x_n)+d(T^n x_n,x_n) \\ &\leq d(y_n,z_n)+\mu_n\xi(d(y_n,z_n))+\nu_n+d(T^n z_n,T^n x_n)+d(T^n x_n,x_n) \\ &\leq (1+\mu_n M)(d(y_n,z_n))+\nu_n+d(T^n z_n,T^n x_n)+d(T^n x_n,x_n) \end{aligned} \tag{3.15}$$

Now we compute

$$\begin{aligned} d(y_n,z_n) &\leq d(y_n,x_n)+d(z_n,x_n) \\ &\leq d(W(T^n x_n,T^n z_n,\alpha_n),x_n)+d(W(x_n,T^n x_n,\beta_n),x_n) \\ &\leq (1-\alpha_n)d(T^n x_n,x_n)+\alpha_n d(T^n z_n,x_n)+(1-\beta_n)d(x_n,x_n)+\beta_n d(T^n x_n,x_n) \\ &\leq d(T^n x_n,x_n)+\alpha_n d(T^n z_n,T^n x_n)+\beta_n d(T^n x_n,x_n) \\ &\leq (1+\beta_n)d(x_n,T^n x_n)+\alpha_n d(T^n z_n,T^n x_n) \end{aligned} \tag{3.16}$$

Hence from (3.14) and (3.15), we have

$$\begin{aligned} d(x_{n+1},x_n) &\leq (1+\mu_n M)[(1+\beta_n)d(x_n,T^n x_n)+\alpha_n d(T^n x_n,T^n z_n)]+d(T^n x_n,T^n z_n)+d(T^n x_n,x_n) \\ &\leq [(1+\mu_n M)(1+\beta_n)+1]d(x_n,T^n x_n)+[(1+\mu_n M)\alpha_n]d(T^n x_n,T^n z_n) \end{aligned}$$

taking $\lim_{n\to\infty}$ and applying (3.11) and (3.12), we have

$$\lim_{n\to\infty} d(x_{n+1},x_n) = 0$$

(3.17)

Using T is uniformly $(\mu_n, v_n, \xi)$ total asymptotically nonexpansive, finally we compute that
$d(x_n,Tx_n) \leq d(x_n,x_{n+1}) + d(x_{n+1},T^{n+1}x_{n+1}) + d(T^{n+1}x_{n+1},T^{n+1}x_n) + d(T^{n+1}x_n,Tx_n)$

$\leq d(x_n,x_{n+1}) + d(x_{n+1},T^{n+1}x_{n+1}) + d(T^{n+1}x_n,Tx_n) + d(x_n,x_{n+1}) + \mu_n\xi d(x_{n+1},x_n) + v_{n+1}$

by continuity of T, $d(x_n,Tx_n) \to 0$ as $n \to \infty$ implies that $d(Tx_n,T^{n+1}x_n) \to 0$ as $n \to \infty$. Hence, we have $d(x_n,Tx_n) \to 0$ as $n \to \infty$. It shows that the seqence $\{x_n\}$ has an approximate fixed point property for map T. This completes the proof.

**Theorem 3.3.** *Let C be a nonempty, closed and convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity n and $T: C \to C$ be a uniformly lipschitzian and $(\{\mu_n\},\{v_n\},\xi)$ - total asymptotically nonexpansive mapping with $\{\mu_n\}$and$\{v_n\}, n \in N$ such that $\sum_{n=1}^{\infty}\mu_n < \infty$, $\sum_{n=1}^{\infty} v_n < \infty$. Let the sequence $\{x_n\}$ defined by (3.1) converges strongly to a point in $T$ if and only if* $\lim_{n\to\infty} d(x_n, p) = 0$.

*Proof.* The necessity of the conditions is obvious. Thus we only prove the sufficiency. It follows From theorem 3.1 that $\lim_{n\to\infty} d(x_n, p)$ converges. Moreover, $\lim_{n\to\infty} d(x_n, p) = 0$ implies that $\lim_{n\to\infty} d(x_n, p) = 0$. This completes the proof.

**Remark 3.4.** *we conclude that our main results 3.1, 3.2 and 3.3 extend several results as (i) Theorem 3.1 improve and generalize Lemma 2.6 in [15] to the general setup of uniformly convex hyperbolic spaces.*
*(ii) Theorem 3.2 improve and generalize Lemma 2.7 in [15] to the setup of uniformly convex hyperbolic spaces.*
*(iii) Theorem 3.3 generalize Lemma 2.2 in [15] for the setting of uniformly convex hyperbolic spaces.*

**Conflict of Interests**

The authors declare that there is no conflict of interests.

## References

[1] U. Kohlenbach: Some logical metatheorems with applications in functional analysis. Trans. Amer. Math. Soc. 357 (2005), 89-128.

[2] K. Goebel, W.A. Kirk: Iteration processes for nonexpansive mappings.ln: S.P.Singh, S.Thomeier, B.Watson,(eds.) Topological Methods in nonlinear Functional Analysis. Contemporary Mathematics, vol.21, Amer. Math.Soc., Providence (1983) pp. 115-123.

[3] K. Goebel, S. Reich: Uniformly Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, Marcel Dekker, Inc., New York, 1984.

[4] S. Reich, AJ. Zasalvski: Genericity in Nonlinear Analysis. Springer, New York (2014).

[5] S. Reich, I. Shafrir: Nonexpansive iterations in Hyperbolic space, Nonlinear Anal. 15(1990), 537-558.

[6] W. Takahashi: A convexity in metric spaces and nonexpansive mappings. Kodai Math. Semin.Rep. 22(1970), 142-149.

[7] E. Matouskova, S. Reich: reflexivity and approximate fixed points Stud.Math. 159(2004), 403-415.

[8] L. Leustean: A quadratic rate of asymptotic regularity for CAT(0) spaces J. Math. Anal. Appl. 325(2007), 386-399.

[9] Y.I. Alber, C.E. Chidume, H. Zegeye: Approximating Fixed points of total asymptotically nonexpansive nonexpansive mappings, Fixed Points Theory Appl. 2006(2006). Article ID 10673.

[10] S.S. Chang, L.Wang, H.W.J. Lee, C.K. Chan, L. Yang: Demiclosed principle and $\Delta$-convergence theorems for total asymptotically nonexpansive mappings in CAT(0) spaces. Appl.Math. Comput. 219(2012), 2611-2617.

[11] H. Fukhar-ud-din, A.R. Khan: Approximating common fixed points of asymptotically nonexpansive maps in uniformly convex Banach spaces. Comput. Math. Appl. 53(2007), 1349-1360.

[12] H. Fukhar-ud-din, A.R. Khan, MAA: A new implicit algorithm of asymptotically quasi-nonexpansive mappings in uniformly convex Banach spaces. IAENG Int. J. Appl. Math. 42(3)(2012), 1-5.

[13] M.A.A. Khan: Convergence analysis of a multi-step iteration for a finite family of asymptotically quasi-nonexpansive mappings. J. Inequal. Appl. 2013 (2013), Article ID 423.

[14] M.A.A. Khan, H. Fukhar-ud-din: Convergence analysis of a general iteration schema of nonlinear mappings in hyperbolic spaces. Fixed Point Theory Appl. 2013 (2013), Article ID 238.

[15] M.A.A. Khan, H. Fukhar-ud-din and A Kalsoom: Existence and higher arity iteration for total asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces . Fixed Point Theory Appl. 2016(2016), Article ID 3.

[16] I. Yildirim, S.H. Khan: Convergence theorems for common fixed points of asymptotically quasi-nonexpansive mappings in convex metric spaces. Appl. Math. Comput. 218(2012), 4860-4866.

[17] A.R. Khan, A.A. Domlo, H. Fukhar-ud-din: Common fixed points Noor iteration for a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces. J. Math. Anal. Appl. 341(2008), 1-11.

[18] P.K.F. Kuhfittig: Common fixed points of nonexpansive mappings by iteration. Pac. J. Math. 97(1981), 137-139.

[19] A.R. Khan, M.A. Ahmad: Convergence of a general iterative scheme for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces and applications. Comput. Math. Appl. 59(2010), 2990-2995.

[20] A.R. Khan, M.A. Khamsi, H. Fukhar-ud-din: Strong convergence of a general iteration scheme in CAT(0) spaces. Nonlinear Anal. 74(2011), 783-791.

[21] M. Edelstein: The construction of an asymptotic center with a fixed point property. Bull. Am. Math. Soc. 78(1972), 206-208.

[22] T.C. Lim: Remarks on some fixed point theorems. Proc. Amer. Math. Soc. 60(1976), 179-182.

[23] W. Kirk, B. Panyanak: A concept of convergence in geodesic spaces. Nonlinear Anal. 68(2008), 3689-3696.

[24] L. Leustean: Nonexpansive iterations in uniformly convex W-hyperbolic spaces. In: A. Leizarowitz, B.S. Mordukhovich, I. Shafrir, A. Zaslavski(eds.) Nonlinear Analysis and Optimization I: Nonlinear Analysis. Contemporary Mathematics, vol. 513(2010), Am. Math. Soc pp. 193-209.

[25] U. Kohlenbach, L. Leustean: Asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces. J. Eur. Math. Soc. 12(2010), 71-92.

[26] S.S. Chang, Y.J. Cho, H. Zhou: Demiclosed principal and weak convergence problems for asymptotically nonexpansive mappings. J. Korean Math. Soc. 38(6)(2001), 1245-1260.

[27] A.R. Khan, H. Fukhar-ud-din, M.A.A. Khan: An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces. Fixed Point Theory Appl. 2012(2012), Article ID 54. doi:10.1186/1687-1812-2012-54.