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APPROXIMATION OF FIXED POINT FOR MULTI-VALUED NONEXPANSIVE MAPPING IN BANACH SPACES

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Abstract. This paper we deals with the approximation of fixed point for multi-valued nonexpansive mappings through a new iterative process which is independent and faster than the iterative processes discussed by Khan and Yildirim [7], Panyank [14], Sastry and Babu [15], Shahzad and Zegeye [18], Song and Wang [19], and Song and Cho [20], in uniformly convex Banach spaces. Thus, our results extend and improve the results which appears on multi-valued and single valued mappings in the contemporary literature.

Keywords: multi-valued nonexpansive mappings; iterative scheme; uniformly convex Banach spaces; strong and weak convergence theorems.

2010 AMS Subject Classification: 47H10.

1. Introduction

In this paper, $\mathbb{N}$ denotes the set of all positive integers and $F(T)$ denotes the set of all fixed points of T that is $F(T) = \{Tx = x; x \in K\}$, where T is a mapping from a nonempty subset of K of a normed space X. A mapping $T : K \to K$ is said to be

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(i) *nonexpansive*, if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in K$,

(ii) *quasi-nonexpansive*, if $\|Tx - p\| \leq \|x - p\|$, for all $x \in K$ and $p \in F(T)$.

Many nonlinear equations are naturally formulated as fixed point problems,

$$x = Tx, \tag{1.1}$$

where T, the fixed point mapping, may be nonlinear. A solution x^* of the problem (1.1) is called a fixed point of the mapping T. Consider a *fixed point iteration*, which is given by

$$x_{n+1} = Tx_n. \tag{1.2}$$

The iterative method (1.2) is also known as *Picard iteration*. For the Banach contraction mapping theorem, the Picard iteration converges at unique fixed point of T, but it fails to approximate fixed point for nonexpansive mappings, even when the existence of a fixed point of T is known. Thus, when a fixed point of nonexpansive mappings exists, other approximation techniques are needed to approximate it. In the last fifty years the numerous numbers of researchers have been attracted in this direction and developed iterative processes have been investigated to approximate fixed point for not only nonexpansive mapping, but also for some wider class of nonexpansive mappings (see e.g., Agarwal *et al.* [2], Ishikawa [5], Krasnosel'skiĭ [8], Mann [9], Noor [12], Schaefer [16]), and compare which one is faster.

In 2007, Agarwal et al. [2] introduced a new iterative process whose rate of convergence is faster than the iteration processes cited in above paragraph. Motivation of the results of Agarwal et al. [2] several researches work in this direction and developed new iteration processes whose rate of convergence are faster than S iteration process (see e.g., Abbas *et al.* [1] Kadioglu and Yildirim [6], Thakur et al. [21],) and approximate the fixed points, for nonexpansive in Banach spaces.

Recently, the following iteration process developed by Thakur et al.[21] for approximating the fixed point for nonexpansive mapping and establish some strong and weak convergence theorems in uniformly convex Banach spaces (see e. g. Thakur et al.[21, Theorem 2.3]).

For K a convex subset of normed space X and a nonlinear mapping T of K into itself, for each $x_0 \in K$, the sequence $\{x_n\}$ in K is defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n \\ y_n = (1-\beta_n)z_n + \beta_n Tz_n \\ z_n = (1-\gamma_n)x_n + \gamma_n Tx_n,\ n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are the real sequences in $(0,1)$.

On the other hand the approximation of fixed points for multi-valued nonexpansive maps using Hausdorff metric was initiated by Markin [10] (see also [11]). Later, an interesting and rich fixed point theory for such maps was developed which has applications in control theory, convex optimization, differential inclusion and economics (see [4] and references cited therein). The theory of multi-valued nonexpansive mappings are harder than the corresponding theory of single valued nonexpansive maps. Different iterative processes have been used to approximate the fixed points of multi-valued nonexpansive mappings (see e.g, Khan and Yildirim [7], Panyank [14], Sastry and Babu [15], Shahzad and Zegeye [18], Song and Wang [19], and Song and Cho [20]).

A subset $K \subset X \neq \phi$ is said to be *proximal*, if for each $x \in X$, there exists an element $y \in K$ such that

$$d(x,y) = \text{dist}(x,K) = \inf\{\|x-z\| : z \in K\}.$$

It is well known that each weakly compact convex subset of a Banach space is proximal, as well as each closed convex subset of a uniformly convex Banach space is also proximal.

Let $CB(K)$ and $P(K)$, the collection of all nonempty and closed bounded subsets, and the collection of all nonempty proximal bounded and closed subsets of K, respectively.

Let $H(.,.)$ be the Hausdorff distance on $CB(K)$, is define by

$$H(A,B) = \max\left\{\sup_{x\in A} \text{dist}(x,B), \sup_{y\in B} \text{dist}(y,A)\right\},\ \text{for all } A,B \in CB(X).$$

Let $T : K \to 2^K$ be multi-valued mapping. An element $x \in K$ is said to be fixed point of T, if $x \in Tx$. The set of fixed points of T will be denoted by $\Omega(T)$.

Definition 1.1. A multi-valued mapping $T : K \to CB(K)$ is said to be

(i) *nonexpansive*, if

$$H(Tx,Ty) \le d(x,y), \text{ for all } x,y \in K,$$

(ii) *quasi-nonexpansive*, if $\Omega(T) \neq \phi$ and

$$H(Tx,Tp)) \le d(x,p), \text{ for all } x \in K, \text{ and all } p \in \Omega(T).$$

In this paper, we define a multi-valued version of iterative scheme (1.3) and deals with the approximation of fixed point for multi-valued nonexpansive mappings through a new iterative process which is independent and faster than the iterative schemes discussed by Khan and Yildirim [7], Panyank [14], Sastry and Babu [15], Shahzad and Zegeye [18], Song and Wang [19], and Song and Cho [20], in uniformly convex Banach spaces. Thus, our results extend and improve the results appears on multi-valued and single valued mappings in the contemporary literature.

2. **Preliminaries**

Definition 2.1. Let a sequence $\{x_n\}$ in X is said to be *Fejer monotone* with respect to subset K of X if

$$\|x_{n+p} - p\| \le \|x_n - p\|,$$

for all $p \in K$, $n \ge 1$.

Definition 2.2. A Banach space X is said to satisfy Opial's condition [13], if for any sequnce x_n in X, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|,$$

for all $y \in E$ with $y \neq x$.

Examples of Banach spaces satisfying the Opial's condition are Hilbert spaces and all ℓ^p spaces $(1 < p < \infty)$. On the other hand, $L^p[0,2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial's condition.

Definition 2.3. A multi-valued mapping $T : K \to P(X)$ is called demiclosed at $y \in K$ if for any sequence $\{x_n\}$ in K weakly convergent to an element x and $y_n \in Tx_n$ strongly convergent to y, we have $y \in Tx$.

The following is the multi-valued version of *condition(I)* of Senter and Dotson [17].

Definition 2.4. A multi-valued nonexpansive mapping $T : K \to CB(K)$, where K a subset of X is said to satisfy *condition (I)* if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that $d(x,Tx) \geq f(d(x,F(T)))$ for all $x \in K$.

Lemma 2.5. [3, Lemma 2.3] *Let $T : K \to P(K)$ be a multi-valued mapping with $\Omega(T) \neq \phi$ and let $P_T : K \to 2^K$ be a multi-valued mapping defined by*

$$P_T(x) = \{y \in Tx : \|x-y\| = d(x,Tx)\},\ x \in K. \tag{2.1}$$

Then the following conclusions is hold:

(1) $\Omega(T) = \Omega(P_T)$;
(2) $P_T(p) = \{p\}$, *for each* $p \in \Omega(T)$;
(3) for each $x \in K$, $P_T(x)$ *is a closed subset of* $T(x)$ *and so it is compact;*
(4) $d(x,Tx) = d(x,P_T(x))$, *for each* $x \in K$;
(5) P_T *is a multi-valued mapping from* K *to* $P(K)$.

Lemma 2.6. [7, Lemma 3] *Let X be a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of X such that $\limsup_{n\to\infty} \|x_n\| \leq r$ and $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

3. **Main results**

We now define multi-valued version of iterative process of (1.3).

Let K be a nonempty closed and convex subset of X and $T : K \to P(K)$ be a multi-valued mappings. Let $\{x_n\}$ in K defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)y_n + \alpha_n v_n \\ y_n = (1-\beta_n)z_n + \beta_n w_n \\ z_n = (1-\gamma_n)x_n + \gamma_n u_n, \ n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $u_n \in P_T(x_n), v_n \in P_T(y_n)$ and $w_n \in P_T(z_n)$ and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$.

We start with the following couple of Lemmas.

Lemma 3.1. *Let X be a normed space and K be a nonempty closed convex subset of X. Let $T : K \to P(K)$ be a multi-valued mapping such that $\Omega(T) \neq \phi$ and P_T is a nonexpansive mapping. Let $\{x_n\}$ be the sequence as defined in (3.1), then $\{x_n\}$ is a Fejer monotone with respect to $\Omega(T)$ and $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \Omega(T)$.*

Proof. Let $p \in \Omega(T)$, then by using Lemma 2.5, we have $p \in P_T(p) = \{p\}$. It follows from (3.1), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n)y_n + \alpha_n v_n - p\| \\ &\leq (1-\alpha_n)\|y_n - p\| + \alpha_n\|v_n - p\| \\ &\leq (1-\alpha_n)\|y_n - p\| + \alpha_n H(P_T(y_n), P_T(p)) \\ &\leq (1-\alpha_n)\|y_n - p\| + \alpha_n\|y_n - p\|, \\ &\leq \|y_n - p\|, \end{aligned} \tag{3.2}$$

from (3.1), we have

$$\begin{aligned} \|y_n - p\| &= \|(1-\beta_n)z_n + \beta_n w_n - p\| \\ &\leq (1-\beta_n)\|z_n - p\| + \beta_n\|w_n - p\| \\ &\leq (1-\beta_n)\|z_n - p\| + \beta_n H(P_T(z_n), P_T(p)) \\ &\leq (1-\beta_n)\|z_n - p\| + \beta_n\|z_n - p\| \\ &\leq \|z_n - p\|, \end{aligned} \tag{3.3}$$

again from (3.1), we have

$$\begin{aligned}
\|z_n - p\| &= \|(1-\gamma_n)x_n + \gamma_n u_n - p\| \\
&\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|u_n - p\| \\
&\leq (1-\gamma_n)\|x_n - p\| + \gamma_n H(P_T(x_n), P_T(p)) \\
&\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|x_n - p\| \\
&\leq \|x_n - p\|. \qquad (3.4)
\end{aligned}$$

Hence, from (3.2), (3.3) and (3.4), we have

$$\|x_{n+1} - p\| \leq \|x_n - p\|. \qquad (3.5)$$

This shows that $\{x_n\}$ is a Fejer monotone with respect to $\Omega(T)$. Notice from (3.5) that $\|x_{n+1} - p\| \leq \|x_n - p\|$ for all $n \geq 1$. This implies that $\{\|x_n - p\|\}$ is bounded and decreasing. Hence $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in \Omega(T)$. □

Lemma 3.2. *Let X be uniformly convex Banach space and K be a nonempty closed convex subset of X. Let $T : K \to P(K)$ be a multi-valued mapping such that $\Omega(T) \neq \phi$ and P_T is nonexpansive mapping. Let $\{x_n\}$ be the sequence as defined in (3.1), then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.*

Proof. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists, for $p \in \Omega(T)$. Let it be $\lim_{n\to\infty} \|x_n - p\| = c \geq 0$. If $c = 0$, then

$$d(x_n, Tx_n) \leq \|x_n - y_n\|$$

The conclusion holds for $c = 0$. If $c > 0$, taking lim sup both the sides of (3.4), we have

$$\limsup_{n\to\infty} \|z_n - p\| \leq c, \qquad (3.6)$$

taking lim sup both the sides of (3.3), we have

$$\limsup_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|z_n - p\| \leq c. \qquad (3.7)$$

In addition to

$$\begin{aligned}\limsup_{n\to\infty}\|y_n-p\| &\le \limsup_{n\to\infty}H(P_T(x_n),P_T(p))\\ &\le \limsup_{n\to\infty}\|x_n-p\|\le c,\end{aligned}\tag{3.8}$$

and using (3.7), we have

$$\begin{aligned}\limsup_{n\to\infty}\|v_n-p\| &\le \limsup_{n\to\infty}H(P_T(y_n),P_T(p))\\ &\le \limsup_{n\to\infty}\|y_n-p\|\le c.\end{aligned}\tag{3.9}$$

Since $\lim_{n\to\infty}\|x_{n+1}-p\|=c$, therefore from (3.8), (3.9) and applying Lemma (2.6), we have

$$\lim_{n\to\infty}\|y_n-v_n\| = 0.\tag{3.10}$$

On the other hand, from (3.1), we have

$$\begin{aligned}\|x_{n+1}-p\| &= \|(1-\alpha_n)y_n+\alpha_n v_n-p\|\\ &\le (1-\alpha_n)\|y_n-p\|+\alpha_n\|v_n-p\|\\ &\le (1-\alpha_n)\|x_n-p\|+\alpha_n\|y_n-p\|\\ \|x_n-p\| &\le \frac{\|x_n-p\|-\|x_{n+1}-p\|}{\alpha_n}+\|y_n-p\|,\end{aligned}$$

taking $\liminf$ both the sides, we get

$$c \le \liminf_{n\to\infty}\|y_n-p\|.\tag{3.11}$$

Hence from (3.7) and (3.11), we have

$$\lim_{n\to\infty}\|y_n-p\| = c,$$

by Lemma 2.6, we have

$$\lim_{n\to\infty}\|z_n-w_n\| = 0.\tag{3.12}$$

Since

$$\begin{aligned}\|y_n - p\| &= \|(1-\beta_n)z_n + \beta_n w_n - p\| \\ &\leq (1-\beta_n)\|z_n - p\| + \beta_n\|w_n - p\| \\ &\leq \|z_n - p\| + \beta_n\|w_n - z_n\|,\end{aligned}$$

applying (3.12) and taking lim inf both the sides, we get

$$c \leq \liminf_{n\to\infty}\|z_n - p\|, \tag{3.13}$$

using (3.6) and (3.13), we have

$$\lim_{n\to\infty}\|z_n - p\| = c,$$

hence, applying Lemma 2.6, we have

$$\lim_{n\to\infty}\|x_n - y_n\| = 0, \tag{3.14}$$

since

$$d(x_n, Tx_n) \leq \|x_n - y_n\|,$$

therefore, taking lim as $n \to \infty$ both the sides of the above inequality, we have

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

□

Next, we approximate fixed points of the mapping T through weak convergence of the sequence $\{x_n\}$ defined by (3.1).

Theorem 3.3. *Let X be a uniformly convex Banach space satisfying Opial's condition and K a nonempty closed convex subset of X. Let $T : K \to P(K)$ be a multi-valued mapping such that $\Omega(T) \neq \phi$ and P_T is a nonexpansive mapping. Let $\{x_n\}$ be the sequence defined in (3.1). Let $I - P_T$ be demiclosed with respect to zero, then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Let $p \in \Omega(T) = \Omega(P_T)$, from Lemma 3.1, we have $\lim_{n\to\infty}\|x_n - p\|$ exists. Now we prove that $\{x_n\}$ has a unique weak sequential limit in $\Omega(T)$. For this let z_1 and z_2 be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ respectively. By (3.14) there exists $y_n \in Tx_n$ such that $\|x_n - y_n\| = 0$. Since $I - P_T$ is demiclosed with respect to zero, therefore we obtained $z_1 \in \Omega(P_T) = \Omega(T)$. In the same way, we can prove that $z_2 \in \Omega(T)$.

Next, we prove that uniqueness. For this, suppose that $z_1 \neq z_2$. Then by Opial's condition, we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - z_1\| &= \lim_{n_i\to\infty} \|x_{n_i} - z_1\| \\
&< \lim_{n_i\to\infty} \|x_{n_i} - z_2\| \\
&= \lim_{n\to\infty} \|x_n - z_2\| \\
&= \lim_{n_j\to\infty} \|x_{n_j} - z_2\| \\
&< \lim_{n_j\to\infty} \|x_{n_j} - z_1\| \\
&= \lim_{n\to\infty} \|x_n - z_1\|,
\end{aligned}$$

which is a contradiction. Hence $\{x_n\}$ converges weakly to a point in $\Omega(T)$. This completes the proof. □

Next we give some strong convergence theorems, our first strong convergence theorem is valid in the setting of general Banach spaces, then we apply this theorem to obtain a strong convergence theorem in the setting of uniformly convex Banach spaces.

Theorem 3.4. *Let X be a real Banach space and K a nonepmty closed convex subset of X. Let $T : K \to P(K)$ be a multi-valued mapping such that $\Omega(T) \neq \phi$ and P_T is a nonexpansive mapping. Let $\{x_n\}$ be the sequence as defined in (3.1), then $\{x_n\}$ converges strongly to a point of $\Omega(T)$, if and only if $\liminf_{n\to\infty} d(x_n, \Omega(T)) = 0$.*

Proof. The necessity is obvious. Conversely, suppose that the

$$\liminf_{n\to\infty} d(x_n, \Omega(T)) = 0.$$

By Lemma 3.1, sequence $\{x_n\}$ is a Fejer monotone, that is

$$\|x_{n+1} - p\| \leq \|x_n - p\|,$$

it gives that

$$d(x_{n+1}, \Omega(T)) \leq d(x_n, \Omega(T)).$$

This implies that $\lim_{n\to\infty} d(x_n, \Omega(T))$ exists and so by hypothesis, $\liminf_{n\to\infty} d(x_n, \Omega(T)) = 0$. Therefore, we must have $\lim_{n\to\infty} d(x_n, \Omega(T))) = 0$.

Next, we suppose that $\{x_n\}$ is a Cauchy sequence in K . Let $\varepsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, \Omega(T)) = 0$, there exists a constant n_0 such that for all $n \geq n_0$, we have

$$d(x_n, \Omega(T)) < \frac{\varepsilon}{4}.$$

In particular, $\inf\{x_0 - p\| : p \in \Omega(T)\} < \frac{\varepsilon}{4}$. There must exists a $p^* \in \Omega(T)$ such that

$$\|x_{n_0} - p^*\| < \frac{\varepsilon}{2}.$$

Now for $m, n \geq n_0$, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p^*\| + \|x_n - p^*\| \\ &\leq 2\|x_{n_0} - p^*\| \\ &\leq 2\left(\frac{\varepsilon}{2}\right) = \varepsilon. \end{aligned}$$

Hence, $\{x_n\}$ is a cauchy sequence in a closed subset of K of a Banach space E, and so it must convergence in K. Let $\lim_{n\to\infty} x_n = q$. Now

$$\begin{aligned} d(q, P_T(q)) &\leq \|x_n - q\| + d(x_n, P_T(x_n)) + H(P_T(x_n), P_T(q)) \\ &\leq \|x_n - q\| + \|x_n - y_n\| + \|x_n - q\| \\ &\to 0 \text{ as } n \to \infty, \end{aligned}$$

which gives that $d(q, P_T(q)) = 0$. But P_T is a nonexpansive mappings, so that $F(P_T)$ is closed. Therefore, $q \in \Omega(P_T) = \Omega(T)$. This completes the proof. □

Theorem 3.5. *Let X be a real Banach space and K a nonepmty closed convex subset of X. Let $T : K \to P(K)$ be a multi-valued mapping such that $\Omega(T) \neq \phi$ and P_T is a nonexpansive mapping. If T satisfies condition (I) with respect to sequence $\{x_n\}$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $\lim_{n\to\infty} d(x_n, \Omega(T)) = 0$.*

Proof. By the assumption, we can find a nondecreasing function $f:[0,\infty)\to[0,\infty)$ with $f(0)=0$ and $f(r)>0$ for $r\in(0,\infty)$ such that $f(d(x_n,\Omega(T)))\leq d(x_n,Tx_n)$ for all $n\in\mathbb{N}$, so we have

$$\lim_{n\to\infty} d(x_n,\Omega(T))) \quad\leq\quad \lim_{n\to\infty} d(x_n,Tx_n)=0.$$

Therefore, $\lim_{n\to\infty} d(x_n,\Omega(T))=0$. This completes the proof. $\square$

Theorem 3.6. *Let X be uniformly convex Banach space and Ka nonempty closed convex subset of X. Let $T:K\to P(K)$ be a multi-valued mapping with $\Omega(T)\neq\phi$ and P_T is nonexpansive mapping. Assume that T satisfies condition(I), then Let $\{x_n\}$ be a sequence as defined by (3.1) converges strongly to a point $\Omega(T)$.*

Proof. By Lemma 3.1, sequence $\{x_n\}$ is bounded and by Lemma 3.2, we have $\lim_{n\to\infty} d(x_n,Tx_n)=0$. Since T satisfies condition(I) with respect to $\{x_n\}$. Then by Theorem 3.5, $\lim_{n\to\infty} d(x_n,\Omega(T))=0$, therefore the rest of the result follows by Theorem 3.4. This completes the proof. $\square$

Remark 3.7. Our Theorems 3.3 and 3.6 specially improves Theorems 1 and 2 of Khan and Yildirim [7], and iterative processes discussed by Panyank [14], Sastry and Babu [15], Shahzad and Zegeye [18], Song and Wang [19], and Song and Cho [20], in the sense of faster iterative processes.

Conflict of Interests

The authors declare that there is no conflict of interests.

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