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CHAOS AND TOPOLOGICAL CONJUGATION OF GENERALIZED DISCRETE DYNAMICAL SYSTEM

XIAOQIAN SONG*, LIANGWEI WANG, JINKUI LIU

College of Mathematics and Statistics, Chongqing Three-Gorges University, wanzhou, 404100, China

Abstract: In this paper, by introducing several new concepts, Chaos and topological conjugation of generalized discrete dynamical system are discussed. We show that generalized discrete dynamical system has similar properties as in the case of classical discrete dynamical system. Moreover, we provide a sufficient condition for chaos in the sense of modified Devaney of generalized discrete dynamical system. These results further extend the scope of the research on discrete dynamical system and generalize the existing results to a very general case.

Keywords: discrete dynamical system; chaos; topological conjugation; topological transitivity.

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1. Introduction

In recent years there has been an increasing interest in the study of discrete dynamical system, which can describe organism, information processing, numerical simulation and many other problems in society. Considerable literatures have been appeared to study their properties such as chaos [1], stability [2], bifurcation [3], pseudo-random-distribution [4] and so on. Note that these results on dynamical properties of systems are restricted to systems with fixed parameters, for example,

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*Corresponding author

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one-dimensional discrete dynamical systems of the form

$$x_{n+1} = f(\mu, x_n), n = 0,1,\cdots, x_n \in I, \quad (1)$$

where I is a subset of $R = (-\infty, +\infty)$ and μ is a fixed parameter.

However, systems with variable parameters have attracted great interest in various scientific communities due to broad applications in many areas including vibration [5] and synchronization [6], just to name a few. In 2006, Tian and Chen proposed the notion of variable parametric discrete systems in [7, 8], i.e. systems of the form

$$x_{n+1} = g(\mu_n, x_n) = f(n+1, x_n), n = 0,1,\cdots, \quad (2)$$

we call it generalized discrete dynamical system in this paper. Let us briefly recall their work. They gave several new specific chaotic systems and presented a simple method for constructing discrete chaotic system with variable parameters. Recall that vibration, stability and pseudo-random of system (2) have received a great deal of attention in the past. To the best of our knowledge, chaoses in the sense of modified Devaney and topological conjugation of system (2) have rarely been discussed. As system (1) is the special case of system (2), we think that whether the basic theories of system (1) can be popularized to study the corresponding theories of system (2)？

In the present paper, we will restrict our attention to generalized discrete dynamical system. The remainder of the paper is organized as follows. Section 2 introduces some basic definitions and some notations, also several new concepts are given in this part. Section 3 is devoted to study the topological conjugation of generalized discrete dynamical system. In section 4, we discuss chaos in the sense of modified Devaney for system (2). These results further extend the scope of the research on discrete system.

2. Preliminaries and some new definitions

Definition 2.1. *Let* (X,d) *be a compact metric space and* $F=\{f_k\}_{k=1}^{\infty}:X\to X$ *be a sequence of continuous maps for. For any* $x_0\in X$ *, define a sequence of points as follows:*

$$x_1=f_1(x_0),x_2=f_2(x_1)\dots,\quad x_{n+1}=f_{n+1}(x_n),\quad \dots,\quad n=0,1,2,\dots$$

Then the system (X,F) *is said to be a generalized discrete dynamical system and the sequence* $O(x_0)=\{x_n\}_{n=0}^{\infty}$ *to be an orbit of the sequence* $F=\{f_k\}_{k=1}^{\infty}$ *of the maps.*

For convenience, denote maps $F_k:X\to X$ by

$$F_k(x)=f_k(f_{k-1}(\dots f_1(x)))=f_k\circ f_{k-1}\circ\cdots\circ f_1(x),k=1,2,\dots.$$

It's obvious that

$$x_1=f_1(x_0)=F_1(x_0),\quad x_2=f_2\circ f_1(x_0)=F_2(x_0),\quad \cdots,x_n=f_n\circ f_{n-1}\circ\cdots\circ f_1(x_0)=F_n(x_0).$$

In addition, for any $(x_1,x_2)\in X\times X$, we denote maps

$F\times F=\{f_k\times f_k\}_{k=1}^{\infty}:X\times X\to X\times X$ by $(f_k\times f_k)(x_1,x_2)=(f_k(x_1),f_k(x_2)),k=1,2,\dots.$

Then, it can be verified that $F_k\times F_k=(f_k\times f_k)\circ(f_{k-1}\times f_{k-1})\circ\cdots\circ(f_1\times f_1)$.

In this paper, we will use (X,F) to denote a generalized discrete dynamical system with a metric d and a sequence of continuous maps $F=\{f_k\}_{k=1}^{\infty}$ on X .

Definition 2.2. *Let* (X,F) *be a generalized discrete dynamical system. If, for any point* $x_0\in X$ *, there is an integer* $k>0$ *for which* $x_{i+k}=x_i$, $i=0,1,2,\cdots$. *Then the point* x_0 *is said to be a periodic point of* $F=\{f_k\}_{k=1}^{\infty}$. *Especially, if* $k=1$, *then* x_0 *is said to be a fixed point of the sequence* $F=\{f_k\}_{k=1}^{\infty}$ *of the maps. We use* $Fix(F)$ *and* $P(F)$ *to denote the set of all fixed points and the set of all periodic points respectively.*

Definition 2.3. *Let* (X,F) *be a generalized discrete dynamical system. If, for any point* $x_0\in X$ *, there exists a positive increasing integer sequences* n_i *such that*

$\lim_{i\to\infty} F_{n_i}(x_0) = x_0$, *or equivalently, for any* $\varepsilon > 0$, *there exists an positive integer* $n > 0$ *such that* $F_n(x_0) \in V(x_0, \varepsilon)$ *(the* ε *-open ball), then the point* x_0 *is said to be an recurrent point of* F *,we denote the set of all recurrent points by* $R(F)$.

Definition 2.4. *Let* (X, F) *be a generalized discrete dynamical system. If, for any non empty open subsets* U *and* V *in* X , *there is a positive integer* n *for which* $F_n(U) \bigcap V \neq \varnothing$, *then the sequence* $F = \{ f_k \}_{k=1}^{\infty}$ *of maps is said to be transitive.*

Definition 2.5. *Let* (X, F) *be generalized discrete dynamical system. If* $F \times F$ *is transitive,then the sequence* $F = \{ f_k \}_{k=1}^{\infty}$ *of maps is said to be weakly mixing.*

Definition 2.6. *Let* (X, F) *be a generalized discrete dynamical system. If for any non empty open subsets* U *and* V *in* X *, there is a positive integer* N *such that*

$$F_n(U) \bigcap V \neq \varnothing, \text{ for every } n \geq N ,$$

then the sequence $F = \{ f_k \}_{k=1}^{\infty}$ *of maps is said to be mixing.*

The following concept can be found in the classical mathematical theory. For the sake of completeness, they are listed as follows.

The function $h : X \to Y$ is called a homeomorphism from a metric space (X, d) into a metric space $(Y, \tilde{d})$ if, it is one-to-one and onto, and both h and h^{-1} are continuous.

Definition 2.7. *Let* (X, F) *and* (Y, G) *be compact metric spaces, where* $F = \{ f_k \}_{k=1}^{\infty}$ *and* $G = \{ g_k \}_{k=1}^{\infty}$. *Suppose* $h : X \to Y$ *is a homeomorphism. If for any* $x \in X$, $h \circ f_k(x) = g_k \circ h(x)$, $k \geq 1$ *,then* $F = \{ f_k \}_{k=1}^{\infty}$ *and* $G = \{ g_k \}_{k=1}^{\infty}$ *are said to be conjugate (with respect to the map* h *), denoted by* $F \,\square\, G$. *If the map* h *is only continuous, then* $F = \{ f_k \}_{k=1}^{\infty}$ *and* $G = \{ g_k \}_{k=1}^{\infty}$ *are said to be semi-conjugate.*

For any $x \in X$, we use $h \circ F = G \circ h$ denote $h \circ f_k(x) = g_k \circ h(x)$, $k = 1, 2, \cdots$.

Definition 2.8. *Let* (X, F) *be a generalized discrete dynamical system with metric* d

on X. If there is a constant $k>0$ such that, for any $x\in X$ and any neighborhood U_x of x, there is a point $y\in U_x$ and a positive integer n such that

$$d(F_n(x),F_n(y))>\delta ,$$

then $F=\{f_k\}_{k=1}^{\infty}$ is said to have sensitive dependence on initial conditions.

Definition 2.9. *A generalized discrete system (X,F) is called Devaney chaotic if it satisfies the following three conditions:*

1. *F is transitive,*
2. *The set of periodic points of F is dense in X,*
3. *F has sensitive dependence on initial conditions.*

In particular, if the sequence $F=\{f_k\}_{k=1}^{\infty}$ of maps satisfy the condition 1 and 3, then F is said to be chaotic in the sense of modified Devaney

3. Topological conjugation of generalized discrete dynamical system

Lemma 3.1. *Let (X,F) and (Y,G) be generalized discrete dynamical systems. If F and G are semi-conjugate, then F_n and G_n are also semi-conjugate for every $n=1,2,\cdots$.*

Proof. Since F and G are semi-conjugate, there exists a continuous onto map $h:X\to Y$ such that $h\circ f_n=g_n\circ h$ for $n\ge 1$. Therefore

$$h\circ F_n=h\circ f_n\circ f_{n-1}\circ\cdots\circ f_1=g_n\circ h\circ f_{n-1}\circ\cdots\circ f_1$$

$$=g_n\circ g_{n-1}\circ h\circ\cdots\circ f_1=\cdots=g_n\circ g_{n-1}\circ\cdots\circ g_1\circ h=G_n\circ h.$$

Thus F_n and G_n are semi-conjugate for all $n=1,2,\cdots$.

Corollary 3.1. *Let (X,F) and (Y,G) be generalized discrete dynamical systems, where $F=\{f_k\}_{k=1}^{\infty}$ and $G=\{g_k\}_{k=1}^{\infty}$. If F and G are conjugate, then F_n and G_n are also conjugate, for all $n=1,2,\cdots$.*

The following lemma 3.2 holds obviously.

Lemma 3.2. *Let* (X,F) *and* (Y,G) *be generalized discrete dynamical systems. Suppose* h *is a conjugate function from* F *to* G, *then* h^{-1} *is a conjugate function from* G *to* F.

Theorem 3.1. *Let* (X,F) *and* (Y,G) *be generalized discrete dynamical systems, where* $F=\{f_k\}_{k=1}^{\infty}$ *and* $G=\{g_k\}_{k=1}^{\infty}$. *Suppose* F *and* G *are conjugate, If, for a point* $x\in X$, *there exists a positive increasing integer sequences* n_i, *such that*

$$\lim_{i\to\infty} F_{n_i}(x)=x_0\in X,$$

then

$$\lim_{i\to\infty} G_{n_i}(h(x))=y_0\in Y, \text{and } y_0=h(x_0).$$

Proof. By lemma 3.1, h is a semi-conjugate function from F_n to G_n, so we have $h\circ F_{n_i}=G_{n_i}\circ h$. Since $\lim_{i\to\infty} F_{n_i}(x)=x_0\in X$ and h is continuous, the following equation holds,

$$h(x_0)=h(\lim_{i\to\infty} F_{n_i}(x))=\lim_{i\to\infty} h(F_{n_i}(x))=\lim_{i\to\infty} G_{n_i}(h(x)),$$

denote $h(x_0)$ by y_0, then the proof is completed.

Theorem 3.2. *Let* (X,F) *and* (Y,G) *be generalized discrete dynamical systems, where* $F=\{f_k\}_{k=1}^{\infty}$ *and* $G=\{g_k\}_{k=1}^{\infty}$. *Suppose* h *is a conjugate function from* F *to* G. *Then*

i) $h(Fix(F))=Fix(G)$,

ii) $h(P(F))=P(G)$,

iii) $h(R(F))=R(G)$.

Proof. i) First, we are going to prove that $h(Fix(F))\subset Fix(G)$. For any $x_0\in Fix(F)$, by definition 2.2, $x_{i+1}=x_i$, $i\geq 0$, i.e., $F_{i+1}(x_0)=F_i(x_0)$, thus $h\circ F_{i+1}(x_0)=h\circ F_i(x_0)$, for any $i \geq 0$. By corollary 3.1, $h \circ F_{i+1} = G_{i+1} \circ h$, $h \circ F_i = G_i \circ h$ for any $i \geq 0$. So we have

$$G_{i+1}(h(x_0)) = G_i(h(x_0)) \qquad \text{for any } i \geq 0,$$

it follows that $h(x_0) \in Fix(G)$, this implies that $h(Fix(F)) \subset Fix(G)$.

Second, it is to prove that $h(Fix(F)) \supset Fix(G)$. By Lemma 3.2, h^{-1} is a semi-conjugate function from G to F, and it can be proved similarly to the above section that $h^{-1}(Fix(G)) \subset Fix(F)$, thus $Fix(G) \subset h(Fix(F))$.

From the above proof, we have $h(Fix(F)) = Fix(G)$.

ii) Now, using the same argument, it can be proved $h(P(F)) = P(G)$.

iii) Let $x_0 \in R(F)$ to prove $h(R(F)) \subset R(G)$. Then there is a positive increasing integer sequences n_i for which $\lim_{i \to \infty} F_{n_i}(x_0) = x_0$. Since h is a conjugate function from F to G, by theorem 3.1, $\lim_{i \to \infty} G_{n_i}(h(x_0)) = h(x_0)$, then by definition 2.3, $h(x_0) \in R(G)$, hence $h(R(F)) \subset R(G)$.The opposite inclusion we get from the fact that h^{-1} is a conjugate function from G to F, so we have $h^{-1}(R(G)) \subset R(F)$, thus $h(R(F)) \supset R(G)$. Therefore $h(R(F)) = R(G)$.

The proof is completed.

Lemma 3.3. *Let* (X, F), (X, G) *and* (X, Q) *be there generalized discrete dynamical systems, where* $F = \{f_k\}_{k=1}^{\infty}$, $G = \{g_k\}_{k=1}^{\infty}$ *and* $Q = \{q_k\}_{k=1}^{\infty}$ *respectively. Then*

i) $F \sim F$,

ii) $F \sim G \Rightarrow G \sim F$,

iii) $F \sim G, G \sim Q \Rightarrow F \sim Q$.

Proof. i) Clearly, the identified map is a conjugate function from F to F, hence $F \sim F$.

ii) Suppose $F \sim G$, and h is a conjugate function from F to G, then by lemma 3.2, h^{-1} is a conjugate function from G to F, this implies that $G \sim F$.

iii) Suppose F □ G, G □ Q, h is a conjugate function from F to G, and ϕ is a conjugate function from G to Q, then by definition 2.7, $h \circ f_k = g_k \circ h$ and $\phi \circ g_k = q_k \circ \phi$ for any $k \geq 1$, thus we can draw that $\phi \circ h \circ f_k = \phi \circ g_k \circ h = q_k \circ \phi \circ h$ for any $k \geq 1$. At the same time, it is easy to see that $\phi \circ h$ is a homeomorphism. So $\phi \circ h$ is a conjugate function from F to Q, then, F □ Q holds. This completes the proof.

The sequence $F = \{f_k\}_{k=1}^{\infty}$ of maps is said to be mutually exchangeable if $f_i \circ f_j = f_j \circ f_i$ for any i, j.

Theorem 3.3. *Let (X, F) and (X, G) be generalized discrete dynamical systems. Suppose h is a semi-conjugate function from F to G, $F = \{f_k\}_{k=1}^{\infty}$ and $G = \{g_k\}_{k=1}^{\infty}$ are mutually exchangeable respectively. We find that if F is topologically transitive (weak mixing, mixing), then G has the same properties.*

Proof. Let us begin with the transitive case. Suppose F is topological transitivity, U and V are non-empty open subsets of Y, it is easy to see that $h^{-1}(U)$ and $h^{-1}(V)$ are non-empty open subsets of X. Since both $F = \{f_k\}_{k=1}^{\infty}$ and $G = \{g_k\}_{k=1}^{\infty}$ are mutually exchangeable, then we have $F_n^{-1} = F_{-n}$ and $G_n^{-1} = G_{-n}$. And from theorem 1 in [9], we can conclude that there exists a positive integer n such that $F_{-n}(h^{-1}(U)) \bigcap h^{-1}(V) \neq \varnothing$. Notice that h is a semi-conjugate function from F to G, and from lemma 3.1, it follows that h is also a semi-conjugate function from F_k to G_k, i.e. for all $k \geq 1$, $h \circ F_k = G_k \circ h$. Immediately, we obtain the following results

$$F_{-n}(h^{-1}(U)) = F_n^{-1}(h^{-1}(U)) = (h \circ F_n)^{-1}(U)$$

$$= (G_n \circ h)^{-1}(U) = h^{-1}(G_n^{-1}(U)) = h^{-1}(G_{-n}(U)),$$

Hence

$$F_{-n}(h^{-1}(U))\bigcap h^{-1}(V)=h^{-1}(G_{-n}(U))\bigcap h^{-1}(V)=h^{-1}(G_{-n}(U)\bigcap V)\neq\varnothing,$$

this implies $G_{-n}(U)\bigcap V\neq\varnothing$, by theorem 1 in [9],we know $G_n(U)\bigcap V\neq\varnothing$,which shows that G is topologically transitive.

The proof of topologically weak mixing is similar to above section.

For the rest of the proof , we suppose F is topologically mixing, by certification, we know that the map $h\times h: X\times X\to Y\times Y$ is a semi-conjugate function from $F\times F$ to $G\times G$, furthermore, both $F\times F=\{f_k\times f_k\}_{k=1}^{\infty}$ and $G\times G=\{g_k\times g_k\}_{k=1}^{\infty}$ are mutually exchangeable. Thus from the above section, the transitivity of $F\times F$ implies the transitivity of $G\times G$. By definition 2.6, the sequence $G=\{g_k\}_{k=1}^{\infty}$ of the maps is weak mixing.

Theorem 3.4. *Let* (X,F) *and* (X,G) *be generalized discrete dynamical systems. Suppose* h *is a conjugate function from* F *to* G *, then* F *is topologically transitive (weak mixing, mixing) if and only if* G *has the same properties.*

Proof. Consider first the case of transitivity. Suppose F is topologically transitive, U and V are non-empty open subsets of Y, then $h^{-1}(U)$ and $h^{-1}(V)$ are also non-empty open subsets of X. By the transitivity of F, there exists a positive integer n such that $F_n(h^{-1}(U))\bigcap h^{-1}(V)\neq\varnothing$. By Lemma 3.2, h^{-1} is a conjugate function from G to F , i.e. for all $k\geq 1$, $h^{-1}\circ g_k=f_k\circ h^{-1}$, thus

$$\begin{aligned}F_n(h^{-1}(U))&=f_n\circ f_{n-1}\circ\cdots\circ f_1\circ h^{-1}(U)\\&=f_n\circ f_{n-1}\circ\cdots\circ h^{-1}\circ g_1(U)\\&=\cdots=h^{-1}\circ g_n\circ g_n\circ\cdots\circ g_1(U)\\&=h^{-1}(G_n(U)).\end{aligned}$$

Therefore

$$h^{-1}\circ G_n(U)\bigcap h^{-1}(V)\neq\varnothing,$$

that is $h^{-1}(G_n(U)\bigcap V)\neq\varnothing$, this implies $G_n(U)\bigcap V\neq\varnothing$, hence G is topologically transitive. The proof of the reverse implication is entirely similar.

Next, consider the situation of weak mixing. Clearly, $h \times h : X \times X \to Y \times Y$ is a conjugate function from $F \times F$ to $G \times G$. And now, using the same argument, we know that $F \times F$ is topologically transitive if and only if $G \times G$ has the same property. Therefore F is weak mixing if and only if G is also weak mixing.

By entirely analogous reasoning, one can show that the situation of mixing holds.

4. Results for chaos in the sense of modified Devaney

Theorem 4.1. *Let (X, F) be generalized discrete dynamical system with metric d on X. Then F is topologically mixing implies that it is chaotic in the sense of modified Devaney.*

Proof. First, it is to prove that F is topologically transitive, see [9].

Next, only the sensitive dependence on initial conditions needs to be proved. Denote the diameter of X by $D(X) = \sup_{x,y \in X} \{d(x, y)\} = \delta > 0$. Next it is to show that $\frac{\delta}{4}$ is a sensitive constant. Suppose $x \in X$, choose $y, z \in X, \varepsilon > 0$ such that $d(V(y,\varepsilon), V(z,\varepsilon)) > \frac{\delta}{2}$. Since F is topologically mixing, there is a $N > 0$ for which $F_n(V(x,\varepsilon)) \bigcap V(y,\varepsilon) \neq \varnothing$, and $F_n(V(x,\varepsilon)) \bigcap V(z,\varepsilon) \neq \varnothing$, for each $n \geq N$.

Choose $x_1 \in F_n(V(x,\varepsilon)) \bigcap V(y,\varepsilon)$, and $x_2 \in F_n(V(x,\varepsilon)) \bigcap V(z,\varepsilon)$. Since $d(V(y,\varepsilon), V(z,\varepsilon)) > \frac{\delta}{2}$, then $D(F_n(V(x,\varepsilon))) > \frac{\delta}{2}$. Therefore for every fixed $n \geq N$, there exists $y \in V(x,\varepsilon)$ such that $d(F_n(x), F_n(y)) > \frac{\delta}{4}$, otherwise, $D(F_n(V(x,\varepsilon))) \leq \frac{\delta}{2}$, this conflicts with $D(F_n(V(x,\varepsilon))) > \frac{\delta}{2}$, by definition 2.8, F has sensitive dependence on initial conditions, hence F is topologically mixing implies that it is chaotic in the sense of modified Devaney.

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