Available online at http://scik.org
Adv. Fixed Point Theory, 8 (2018), No. 2, 144-173
https://doi.org/10.28919/afpt/3619
ISSN: 1927-6303

MODIFICATION OF VISCOSITY METHOD FOR STRICTLY PSEUDO-CONTRACTIVE MAPPING AND EQUILIBRIUM PROBLEM WITH SOME APPLICATIONS

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Abstract. For the purpose of this article, we are using the concept of equilibrium problem and prove the strong convergence theorem by the viscosity approximation methods for finding a common element of the set of fixed points of κ_i-strictly pseudo-contractive mappings and of a finite family of the set of solutions of equilibrium problems and variational inequality problems. Furthermore, we apply our main theorem for the numerical examples.

Keywords: viscosity approximation methods; strictly pseudo-contractive mapping; S-mapping; variational inequality problems; the combination of equilibrium problems.

2010 AMS Subject Classification: 47H09, 47H10, 47J20.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. A mapping $T : C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, for

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Received November 19, 2017

all $x,y \in C$. Recall that T is a κ-*strictly pseudo-contractive mapping* if there exists a constant $\kappa \in [0,1)$ such that

$$\|Tx-Ty\|^2 \leq \|x-y\|^2 + \kappa \|(I-T)x-(I-T)y\|^2, \ \forall x,y \in C. \tag{1.1}$$

If $\kappa = 0$, then (1.1) reduces to nonexpansive mappings.

A point $x \in C$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is denoted by $F(T) = \{x \in C : Tx = x\}$.

Recall that a mapping $f : C \to C$ is said to be *contractive* if there exists a constant $\eta \in (0,1)$ such that, for all $x,y \in H$

$$\|f(x)-f(y)\| \leq \eta \|x-y\|.$$

A mapping A of C into H is called α-inverse-strongly monotone if there exists a positive real number α such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha \|Ax-Ay\|^2, \ \forall x,y \in C. \tag{1.2}$$

Let $A : C \to H$. *The variational inequality problems* is to find a point $u \in C$ such that

$$\langle v-u, Au\rangle \geq 0, \ \forall v \in C. \tag{1.3}$$

The set of solutions of the variational inequality problems is denoted by $VI(C,A)$.

Variational inequalities were introduced and investigated by Stampacchia [8] in 1964. It is well known that variational inequalities cover as diverse disciplines as partial differential equations, optimal control, optimization, mathematical programming, mechanics and finance; see [9]−[11].

Let $F : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for F is to determine its equilibrium point, that is to find a point $x^* \in C$ such that $F(x^*,y) \geq 0$, for all $y \in C$.

The set of all solution of equilibrium problem is denoted by

$$EP(F) = \{x \in C : F(x^*,y) \geq 0, \forall y \in C\}. \tag{1.4}$$

The methods which are used to solve equilibrium problems have been applied in solving economic problem and some problems in pure and applied science; see [1, 2]. Many authors have studied an iterative scheme for the equilibrium problems; see, for example, [2]−[5].

In 2013, Suwannaut and Kangtunyakarn [15] introduced *the combination of equilibrium problem* which is to find $x \in C$ such that

$$\sum_{i=1}^{N} a_i F_i(x,y) \geq 0, \ \forall y \in C, \tag{1.5}$$

where $F_i : C \times C \to \mathbb{R}$ be bifunction and $a_i \in (0,1)$ with $\sum_{i=1}^{N} a_i = 1$, for every $i = 1,2,...,N$. The set of solution (1.5) is denoted by

$$EP\Big(\sum_{i=1}^{N} a_i F_i\Big) = \Big\{x \in C : \Big(\sum_{i=1}^{N} a_i F_i\Big)(x,y) \geq 0, \forall y \in C\Big\}.$$

If $F_i = F$, $\forall i = 1,2,...,N$, then (1.5) reduces to (1.4).

In 2007, Takahashi and Takahashi [5] proved the following theorem.

Theorem 1.1. *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $A1) - A4)$ and let S be a nonexpansive mapping of C into H such that $F(S) \cap EP(F) \neq \emptyset$. Let f be a contraction of H into itself, let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and*

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \ \forall y \in C,$$

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S u_n,$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset [0,1]$ satisfy some control conditions. Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap EP(F)$, where $z = P_{F(S) \cap EP(F)} f(z)$.

The explicit viscosity method for nonexpansive mappings generates a sequence $\{x_n\}$ through the iteration process:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T x_n, \ n \geq 0, \tag{1.6}$$

where I is the identity of H and $\{\alpha_n\}$ is a sequence in $(0,1)$. It is well known [6, 7] that under certain conditions, the sequence $\{x_n\}$ converges in norm to a fixed point q of T which solves the variational inequality

$$\langle (I - f)q, x - q \rangle \geq 0, \ x \in S, \tag{1.7}$$

where S is the set of fixed points of T, namely, $S = \{x \in H : Tx = x\}$.

Many authors proved a strong convergence theorem by using viscosity method; see, for instance, [5, 6].

In 2010, Kangtunyakarn [12] proved a strong convergence theorem of the iterative scheme (1.9) to a common fixed point of $q \in \bigcap_{i=1}^{N} F(T_i)$.

Theorem 1.2. *Let H be a Hilbert space, let f be an α-contraction on H and let A be a strongly positive linear bounded self-adjoint operator with coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \frac{\overline{\gamma}}{\lambda}$. Let $\{T_i\}_{i=1}^{N}$ be a finite family of κ_i-strict pseudo-contraction of H into itself, for some $\kappa_i \in [0,1)$ and $\kappa = max\{\kappa_i : i = 1,2,...,N\}$, with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let S_n be the S-mappings generated by $T_1, T_2, ..., T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, ..., \alpha_N^{(n)}$, where $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I, I = [0,1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\kappa < a \leq \alpha_1^{n,j}, \alpha_3^{n,j} \leq b < 1$, for all $j = 1,2,...,N-1, \kappa < c \leq \alpha_1^{n,N} \leq 1, \kappa \leq \alpha_3^{n,N} \leq d < 1, \kappa \leq \alpha_2^{n,j} \leq e < 1$, for all $j = 1,2,...,N$. For a point $u \in H$ and $x_1 \in H$, let $\{x_n\}$ and $\{y_n\}$ be the sequences defined iteratively by*

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n) S_n x_n, \\ x_{n+1} = \alpha_n \gamma\big(a_n u + (1-a_n) f(x_n)\big) + (1-\alpha_n A) y_n, \ n \geq 1, \end{cases} \tag{1.8}$$

where $\{\beta_n\}, \{\alpha_n\}$ and $\{a_n\}$ are sequences in $[0,1]$. Assume that the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} a_n = 0$;

(ii) $\sum_{n=1}^{\infty} \left|\alpha_1^{n+1,j} - \alpha_1^{n,j}\right| < \infty, \ \sum_{n=1}^{\infty} \left|\alpha_3^{n+1,j} - \alpha_3^{n,j}\right| < \infty$, *for all* $j \in \{1,2,...,N\}$ *and* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \ \sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$;

(iii) $0 \leq \kappa \leq \beta_n < \theta < 1$, *for all* $n \geq 1$, *for some* $\theta \in (0,1)$.

Then both $\{x_n\}$ and $\{y_n\}$ strongly converges to $q \in \bigcap_{i=1}^{N} F(T_i)$ which solves the following variational inequality

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \ \forall p \in \bigcap_{i=1}^{N} F(T_i). \tag{1.9}$$

From Theorem 1.1 [5] and [15], we modify the viscosity methods as following:

For every $i = 1,2,...,N$, let $F_i : C \times C \to \mathbb{R}$ be bifunction which satisfy $A1) - A4)$ and $a_i \in (0,1)$ with $\sum_{i=1}^{N} a_i = 1$, $T_i : C \to C$ be κ_i-strictly pseudo-contractive mapping, for all $i = 1,2,...,N$

and $\kappa = max\{\kappa_i : i = 1,2,...,N\}$. For each $j = 1,2,...,N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, for $x_1 \in C$ and sequence x_n generated by

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \beta_n \big(\alpha_n f(x_n) + (1-\alpha_n) S x_n \big) + (1-\beta_n) P_C (I - \lambda A) u_n, \forall n \geq 1, \end{cases} \tag{1.10}$$

where $A : C \to H$ is α-inverse-strongly monotone mapping and $S : C \to C$ is S-mapping generated by a finite family of strictly pseudo-contractive mappings and a finite real numbers under suitable conditions of the parameters $\{\beta_n\}, \{\alpha_n\}, \{r_n\} \in [0,1]$ and $\lambda \in (0, 2\alpha)$.

Motivated by the above related literature, we prove a strong convergence theorem by modifying the viscosity methods for finding a common element of the set of solutions of equilibrium problems and variational inequality problems. Moreover, we apply our main result to obtain a strong convergence theorem for finding a common element of the set of fixed point of κ_i-strictly pseudo-contractive mappings. Finally, we also give a numerical examples to support our main theorem.

2. **Preliminaries**

In this section, we use some lemmas that will be used for our main result in the next section.

Let C be a nonempty closed convex subset of a real Hilbert space H. We denote weak and strong convergence by $\text{``} \rightharpoonup \text{''}$ and $\text{``} \to \text{''}$, respectively, and let P_C be the metric projection of H onto C, that is, for $x \in H$, $P_C x \in C$ satisfies the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|$$

and it is well-known that, for all $x, y \in H$ and $t \in [0,1]$,

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$$

and P_C is a *firmly nonexpansive mapping* of H onto C, that is,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \; \forall x, y \in H.$$

Lemma 2.1. *([19]) For given $z \in H$ and $u \in C$,*

$$u = P_C z \Leftrightarrow \langle u - z, v - u \rangle \geq 0, \ \forall v \in C.$$

Lemma 2.2. *([16]) Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.3. *([18]) Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n)s_n + \delta_n, \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1): $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2): $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4. *([19]) Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then, for $\lambda > 0$,*

$$u = P_C(I - \lambda A)u \Leftrightarrow u \in VI(C,A),$$

where P_C is the metric projection of H onto C.

Lemma 2.5. *([21]) Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ be a self-mapping of C. If S is a κ-strict pseudo-contractive mapping, then S satisfies the Lipschitz condition*

$$\|Sx - Sy\| \leq \frac{1+\kappa}{1-\kappa} \|x - y\|, \forall x, y \in C.$$

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that $F : C \times C \to \mathbb{R}$ satisfy the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x,y)+F(y,x)\leq 0$ for all $x,y\in C$;

(A3) For each $x,y,z\in C$,

$$\lim_{t\downarrow 0} F\big(tz+(1-t)x,y\big)\leq F(x,y);$$

(A4) For each $x\in C, y\mapsto F(x,y)$ is convex and lower semicontinuous.

Lemma 2.6. *([15]) Let C be a nonempty closed convex subset of a real Hilbert space H. For $i=1,2,...,N$, let $F_i : C\times C\to\mathbb{R}$ be bifunctions satisfying $(A1)-(A4)$ with $\bigcap_{i=1}^{N} EP(F_i)\neq\emptyset$. Then*

$$EP\Big(\sum_{i=1}^{N} a_iF_i\Big)=\bigcap_{i=1}^{N} EP(F_i),$$

where $a_i\in(0,1)$, for every $i=1,2,...,N$ and $\sum_{i=1}^{N} a_i=1$.

Lemma 2.7. *[14]) Let C be a nonempty close convex subset of H and F be a bifunction of $C\times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$. Let $r>0$ and $x\in H$, then there exists $z\in C$ such that*

$$F(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\forall y\in C.$$

Lemma 2.8. *([17]) Assume that $F : C\times C\to\mathbb{R}$ satisfies $(A1)-(A4)$. For $r>0$, define a mapping $T_r : H\to C$ as follows:*

$$T_r(x)=\Big\{z\in C : F(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\forall y\in C\Big\},$$

for all $x\in H$. Then, the following hold:

(i) T_r is single-valued;

(ii) T_r is firmly nonexpansive, i.e., for any $x,y\in H$,

$$\|T_r(x)-T_r(y)\|^2\leq\langle T_r(p)-T_r(y),x-y\rangle;$$

(iii) $F(T_r)=EP(F)$;

(iv) $EP(F)$ is closed and convex.

Remark 2.9. From Lemma 2.6 and 2.8, ([15]) prove the following results;

(i) $\sum_{i=1}^{N} a_iF_i$ satisfying $A1)-A4)$;

(ii) $F(T_r)=\bigcap_{i=1}^{N} EP(F_i)$, where $r>0$ and $a_i \in (0,1)$, for every $i=1,2,...,N$ with $\sum_{i=1}^{N} a_i = 1$.

In 2009, Kangtunyakarn and Suantai ([20]) introduced the S-mapping generated by a finite family of κ_i-strictly pseudo-contractions and a finite real numbers. The definition can be seen below:

Definition 2.1. *Let C be a nonempty convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^{N}$ be a finite family of κ_i-strictly pseudo-contractions of C into itself. For each $j=1,2,...,N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I=[0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S: C \to C$ as follows:*

$$
\begin{aligned}
U_0 &= I,\\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I,\\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I,\\
U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I,\\
&\;\;.\\
&\;\;.\\
&\;\;.\\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I,\\
S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned}
$$

This mapping is called an S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$.

Lemma 2.10. *([22]) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{N}$ be a finite family of κ_i-strictly pseudo-contractive mapping of C into itself with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and $\kappa = \max\{\kappa_i : i=1,2,...,N\}$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I=[0,1], \alpha_1^j + \alpha_2^j + \alpha_3^j = 1, \alpha_1^j, \alpha_2^j \in (\kappa, 1)$, for all $i=1,2,...,N-1$ and $\alpha_1^N \in (\kappa, 1], \alpha_3^N \in (\kappa, 1], \alpha_2^j \in (\kappa, 1]$, for all $j=1,2,...,N$, let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Then $F(S) = \bigcap_{i=1}^{N} F(T_i)$ and S is a nonexpansive mapping.*

3. Main result

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1,2,...,N$, let $F_i : C \times C \to \mathbb{R}$ be bifunction with satisfy $A1) - A4)$, $T_i : C \to C$ be κ_i-strictly pseudo-contractive mapping and let $A : C \to H$ be α-inverse strongly monotone mapping with $\mathscr{F} = \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} F(T_i) \cap VI(C,A) \neq \emptyset$. Let S be S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$, where $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I, I = [0,1]$ with $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$ and $\kappa < \alpha_1^j, \alpha_3^j < 1$, for all $i = 1,2,...,N-1$, $\kappa < \alpha_1^N \leq 1$, $\kappa \leq \alpha_3^N < 1$, $\kappa \leq \alpha_2^j < 1$, for all $j = 1,2,...,N$, where $\kappa = max\{\kappa_i : i = 1,2,...,N\}$. Let the sequence $\{x_n\}$ generated by $x_1 \in C$ and*

$$
\begin{cases}
\sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\
x_{n+1} = \beta_n \big(\alpha_n f(x_n) + (1-\alpha_n) S x_n \big) + (1-\beta_n) P_C (I - \lambda A) u_n, \forall n \geq 1,
\end{cases}
\tag{3.1}
$$

where $\{\beta_n\}, \{\alpha_n\} \subseteq [0,1]$ and $\lambda \in (0, 2\alpha)$. Suppose the following conditions hold:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$,

(ii) $0 < a \leq \beta_n, r_n \leq b < 1$, *for all* $n \geq 1$,

(iii) $f : C \to C$ *be* η*-contraction,*

(iv) $\sum_{n=1}^{N} a_i = 1$, *where* $a_i > 0$, *for all* $i = 1,2,...,N$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,
$\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z = P_{\mathscr{F}} f(z)$.

Proof. First, we show that $(I - \lambda A)$ is a nonexpansive mapping. Let $x, y \in C$. Since A is α-inverse strongly monotone and $\lambda < 2\alpha$, we have

$$
\begin{aligned}
\|(I-\lambda A)x - (I-\lambda A)y\|^2 &= \|x-y\|^2 - 2\lambda \langle x-y, Ax-Ay \rangle + \lambda^2 \|Ax-Ay\|^2 \\
&\leq \|x-y\|^2 - 2\alpha\lambda \|Ax-Ay\|^2 + \lambda^2 \|Ax-Ay\|^2 \\
&= \|x-y\|^2 + \lambda(\lambda - 2\alpha) \|Ax-Ay\|^2 \\
&\leq \|x-y\|^2.
\end{aligned}
$$

Then $(I-\lambda A)$ is a nonexpansive mapping. We will divide our proof into 5 steps.

Step 1: we show that the sequence $\{x_n\}$ is bounded. Since

$$\sum_{i=1}^{N} a_i F_i(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \geq 0, \forall y\in C.$$

Form Remark 2.9, we have $u_n = T_{r_n}x_n$ and $\bigcap_{i=1}^{N} EP(F_i) = F(T_{r_n})$.

Let $z\in F$. By nonexpansiveness of $(I-\lambda A)$ and T_{r_n}, we obtain

$$\begin{aligned}
\|x_{n+1}-z\| &= \left\|\beta_n\big(\alpha_n f(x_n)+(1-\alpha_n)Sx_n\big)+(1-\beta_n)P_C(I-\lambda A)u_n - z\right\| \\
&= \left\|\beta_n\big(\alpha_n f(x_n)+(1-\alpha_n)Sx_n - z\big)+(1-\beta_n)\big(P_C(I-\lambda A)u_n - z\big)\right\| \\
&\leq \beta_n\left\|\alpha_n\big(f(x_n)-z\big)+(1-\alpha_n)(Sx_n-z)\right\| \\
&\quad +(1-\beta_n)\|P_C(I-\lambda A)u_n-z\| \\
&\leq \beta_n\big(\alpha_n\|f(x_n)-f(z)\|+\alpha_n\|f(z)-z\|+(1-\alpha_n)\|Sx_n-z\|\big) \\
&\quad +(1-\beta_n)\|P_C(I-\lambda A)u_n-z\| \\
&\leq \beta_n\big(\alpha_n\eta\|x_n-z\|+\alpha_n\|f(z)-z\|+(1-\alpha_n)\|x_n-z\|\big) \\
&\quad +(1-\beta_n)\|u_n-z\| \\
&= \beta_n\Big(\big(1-\alpha_n(1-\eta)\big)\|x_n-z\|+\alpha_n\|f(z)-z\|\Big)+(1-\beta_n)\|x_n-z\| \\
&\leq \max\left\{\|x_1-z\|,\frac{\|f(z)-z\|}{1-\eta}\right\}.
\end{aligned}$$

By induction we can prove that $\{x_n\}$ is bounded and so is $\{u_n\}$.

Step 2: we will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. By definition of x_n, we have

$$\begin{aligned}
\|x_{n+1}-x_n\| = \Big\| &\Big(\beta_n\big(\alpha_n f(x_n)+(1-\alpha_n)Sx_n\big)+(1-\beta_n)P_C(I-\lambda A)u_n\Big) \\
&-\Big(\beta_{n-1}\big(\alpha_{n-1}f(x_{n-1})+(1-\alpha_{n-1})Sx_{n-1}\big) \\
&+(1-\beta_{n-1})P_C(I-\lambda A)u_{n-1}\Big)\Big\|
\end{aligned}$$

(3.2)

$$
\begin{aligned}
&\leq \beta_n \left\| \left(\alpha_n f(x_n) + (1-\alpha_n)Sx_n\right) - \left(\alpha_{n-1} f(x_{n-1}) + (1-\alpha_{n-1})Sx_{n-1}\right) \right\| \\
&\quad + |\beta_n - \beta_{n-1}| \left\| \alpha_{n-1} f(x_{n-1}) + (1-\alpha_{n-1})Sx_{n-1} \right\| \\
&\quad + (1-\beta_n) \left\| P_C(I-\lambda A)u_n - P_C(I-\lambda A)u_{n-1} \right\| \\
&\quad + |\beta_{n-1} - \beta_n| \left\| P_C(I-\lambda A)u_{n-1} \right\| \\
&\leq \beta_n \big(\alpha_n \left\| f(x_n) - f(x_{n-1}) \right\| + |\alpha_n - \alpha_{n-1}| \left\| f(x_{n-1}) \right\| \\
&\quad + (1-\alpha_n) \left\| Sx_n - Sx_{n-1} \right\| + |\alpha_{n-1} - \alpha_n| \left\| Sx_{n-1} \right\| \big) \\
&\quad + |\beta_n - \beta_{n-1}| \big(\alpha_{n-1} \left\| f(x_{n-1}) \right\| + (1-\alpha_{n-1}) \left\| Sx_{n-1} \right\| \big) \\
&\quad + (1-\beta_n) \left\| P_C(I-\lambda A)u_n - P_C(I-\lambda A)u_{n-1} \right\| \\
&\quad + |\beta_{n-1} - \beta_n| \left\| P_C(I-\lambda A)u_{n-1} \right\| \\
&\leq \beta_n \big(\alpha_n \eta \left\| x_n - x_{n-1} \right\| + |\alpha_n - \alpha_{n-1}| \left\| f(x_{n-1}) \right\| \\
&\quad + (1-\alpha_n) \left\| x_n - x_{n-1} \right\| + |\alpha_{n-1} - \alpha_n| \left\| Sx_{n-1} \right\| \big) \\
&\quad + |\beta_n - \beta_{n-1}| \big(\alpha_{n-1} \left\| f(x_{n-1}) \right\| + (1-\alpha_{n-1}) \left\| Sx_{n-1} \right\| \big) \\
&\quad + (1-\beta_n) \left\| u_n - u_{n-1} \right\| + |\beta_{n-1} - \beta_n| \left\| P_C(I-\lambda A)u_{n-1} \right\| \\
&\leq \beta_n \Big(|\alpha_n - \alpha_{n-1}| M + |\alpha_{n-1} - \alpha_n| M + \big(1 - \alpha_n(1-\eta)\big) \left\| x_n - x_{n-1} \right\| \Big) \\
&\quad + |\beta_n - \beta_{n-1}| \big(\alpha_{n-1} M + (1-\alpha_{n-1}) M \big) + (1-\beta_n) \left\| u_n - u_{n-1} \right\| \\
&\quad + |\beta_{n-1} - \beta_n| M \\
&= \beta_n \Big(2M |\alpha_n - \alpha_{n-1}| + \big(1 - \alpha_n(1-\eta)\big) \left\| x_n - x_{n-1} \right\| \Big) \\
&\quad + 2M |\beta_n - \beta_{n-1}| + (1-\beta_n) \left\| u_n - u_{n-1} \right\|, \qquad (3.3)
\end{aligned}
$$

where $M = \max_{n \in \mathbb{N}} \left\{ \left\| f(x_n) \right\|, \left\| Sx_n \right\|, \left\| P_C(I-\lambda A)u_n \right\| \right\}$.

Since $u_n = T_{r_n} x_n$ and definition of T_{r_n}, we obtain

$$
\sum_{i=1}^{N} a_i F_i(T_{r_n} x_n, y) + \frac{1}{r_n} \left\langle y - T_{r_n} x_n, T_{r_n} x_n - x_n \right\rangle \geq 0, \forall y \in C \qquad (3.4)
$$

and

$$
\sum_{i=1}^{N} a_i F_i(T_{r_{n+1}} x_{n+1}, y) + \frac{1}{r_{n+1}} \left\langle y - T_{r_{n+1}} x_{n+1}, T_{r_{n+1}} x_{n+1} - x_{n+1} \right\rangle \geq 0. \qquad (3.5)
$$

From (3.4) and (3.5). It follow that

$$\sum_{i=1}^{N} a_iF_i(T_{r_n}x_n, T_{r_{n+1}}x_{n+1}) + \frac{1}{r_n}\left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_n}x_n - x_n\right\rangle \geq 0 \tag{3.6}$$

and

$$\sum_{i=1}^{N} a_iF_i(T_{r_{n+1}}x_{n+1}, T_{r_n}x_n) + \frac{1}{r_{n+1}}\left\langle T_{r_n}x_n - T_{r_{n+1}}x_{n+1}, T_{r_{n+1}}x_{n+1} - x_{n+1}\right\rangle \geq 0. \tag{3.7}$$

From (3.6),(3.7) and the fact that $\sum_{i=1}^{N} a_iF_i$ satisfies (A2), we have

$$\begin{aligned}&\frac{1}{r_n}\left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_n}x_n - x_n\right\rangle \\ &+ \frac{1}{r_{n+1}}\left\langle T_{r_n}x_n - T_{r_{n+1}}x_{n+1}, T_{r_{n+1}}x_{n+1} - x_{n+1}\right\rangle \geq 0.\end{aligned}$$

Which implies that

$$\left\langle T_{r_n}x_n - T_{r_{n+1}}x_{n+1}, \frac{T_{r_{n+1}}x_{n+1} - x_{n+1}}{r_{n+1}} - \frac{T_{r_n}x_n - x_n}{r_n}\right\rangle \geq 0.$$

It follows that

$$\left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_n}x_n - T_{r_{n+1}}x_{n+1} + T_{r_{n+1}}x_{n+1} - x_n - \frac{r_n}{r_{n+1}}(T_{r_{n+1}}x_{n+1} - x_{n+1})\right\rangle \geq 0. \tag{3.8}$$

From (3.8), we obtain

$$\begin{aligned}\left\|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\right\|^2 &\leq \left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_{n+1}}x_{n+1} - x_n - \frac{r_n}{r_{n+1}}(T_{r_{n+1}}x_{n+1} - x_{n+1})\right\rangle \\ &= \left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(T_{r_{n+1}}x_{n+1} - x_{n+1})\right\rangle \\ &\leq \left\|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\right\|\Big[\|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}| \\ &\quad \times \left\|T_{r_{n+1}}x_{n+1} - x_{n+1}\right\|\Big] \\ &= \left\|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\right\|\Big[\|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n| \\ &\quad \times \left\|T_{r_{n+1}}x_{n+1} - x_{n+1}\right\|\Big] \\ &\leq \left\|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\right\|\Big[\|x_{n+1} - x_n\| + \frac{1}{d}|r_{n+1} - r_n| \\ &\quad \times \left\|T_{r_{n+1}}x_{n+1} - x_{n+1}\right\|\Big],\end{aligned}$$

which yields

$$\|u_{n+1}-u_n\| \leq \|x_{n+1}-x_n\| + \frac{1}{d}|r_{n+1}-r_n| \, \|u_{n+1}-x_{n+1}\| . \tag{3.9}$$

From (3.9), we have

$$\|u_n-u_{n-1}\| \leq \|x_n-x_{n-1}\| + \frac{1}{d}|r_n-r_{n-1}| \, \|u_n-x_n\| . \tag{3.10}$$

By substituting (3.10) into (3.2), we have

$$\begin{aligned}
\|x_{n+1}-x_n\| \leq & \beta_n\Big(2M|\alpha_n-\alpha_{n-1}| + \big(1-\alpha_n(1-\eta)\big)\,\|x_n-x_{n-1}\|\Big) \\
& + 2M|\beta_n-\beta_{n-1}| + (1-\beta_n)\,\|u_n-u_{n-1}\| \\
\leq & \beta_n\Big(2M|\alpha_n-\alpha_{n-1}| + \big(1-\alpha_n(1-\eta)\big)\,\|x_n-x_{n-1}\|\Big) \\
& + 2M|\beta_n-\beta_{n-1}| + (1-\beta_n)\big(\,\|x_n-x_{n-1}\| + \frac{1}{d}|r_n-r_{n-1}|\,\|u_n-x_n\|\,\big) \\
= & \big(1-\beta_n\alpha_n(1-\eta)\big)\,\|x_n-x_{n-1}\| + 2M|\alpha_n-\alpha_{n-1}| \\
& + 2M|\beta_n-\beta_{n-1}| + (1-\beta_n)\frac{1}{d}|r_n-r_{n-1}|\,\|u_n-x_n\| . \qquad (3.11)
\end{aligned}$$

From (3.11), conditions (i),(v) and lemma 2.3, we obtain

$$\lim_{n\to\infty} \|x_{n+1}-x_n\| = 0. \tag{3.12}$$

Step 3: We will show that $\lim_{n\to\infty}\|u_n-x_n\| = \lim_{n\to\infty}\|P_C(I-\lambda A)u_n-x_n\| = \lim_{n\to\infty}\|Sx_n-x_n\| = 0$. Since T_{r_n} is a firmly nonexpansive mapping, then we obtain

$$\begin{aligned}
\|z-T_{r_n}x_n\|^2 = & \|T_{r_n}z-T_{r_n}x_n\|^2 \\
\leq & \langle T_{r_n}z-T_{r_n}x_n, z-x_n\rangle \\
= & \frac{1}{2}\big(\,\|T_{r_n}x_n-z\|^2 + \|x_n-z\|^2 - \|T_{r_n}x_n-x_n\|^2\,\big),
\end{aligned}$$

which yields

$$\|u_n-z\|^2 \leq \|x_n-z\|^2 - \|u_n-x_n\|^2 . \tag{3.13}$$

By nonexpansiveness of $P_C(I-\lambda A)$,(3.13) and definition of x_n, we have

$$\begin{aligned}
\|x_{n+1}-z\|^2 =& \left\|\beta_n\big(\alpha_n f(x_n)+(1-\alpha_n)Sx_n-z\big)+(1-\beta_n)\big(P_C(I-\lambda A)u_n-z\big)\right\|^2 \\
\leq& \beta_n\left\|\alpha_n\big(f(x_n)-z\big)+(1-\alpha_n)(Sx_n-z)\right\|^2 \\
&+(1-\beta_n)\|P_C(I-\lambda A)u_n-z\|^2 \\
\leq& \beta_n\alpha_n\|f(x_n)-z\|^2+\beta_n(1-\alpha_n)\|x_n-z\|^2+(1-\beta_n)\|u_n-z\|^2 \\
\leq& \beta_n\alpha_n\|f(x_n)-z\|^2+\beta_n(1-\alpha_n)\|x_n-z\|^2 \\
&+(1-\beta_n)\big(\|x_n-z\|^2-\|u_n-x_n\|^2\big) \\
\leq& \beta_n\alpha_n\|f(x_n)-z\|^2+\|x_n-z\|^2-(1-\beta_n)\|u_n-x_n\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
(1-\beta_n)\|u_n-x_n\|^2 \leq& \beta_n\alpha_n\|f(x_n)-z\|^2+\|x_n-z\|^2-\|x_{n+1}-z\|^2 \\
\leq& \beta_n\alpha_n\|f(x_n)-z\|^2+\big(\|x_n-z\|+\|x_{n+1}-z\|\big)\|x_{n+1}-x_n\|. \qquad (3.14)
\end{aligned}$$

By (3.12),(3.14), conditions (i) and (ii), we have

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \qquad (3.15)$$

Put $w_n=\alpha_n f(x_n)+(1-\alpha_n)Sx_n$. By definition of x_n, we have

$$\begin{aligned}
\|x_{n+1}-z\|^2 =& \|\beta_n w_n+(1-\beta_n)P_C(I-\lambda A)u_n-z\|^2 \\
=& \left\|\beta_n(w_n-z)+(1-\beta_n)\big(P_C(I-\lambda A)u_n-z\big)\right\|^2 \\
\leq& \beta_n\|w_n-z\|^2+(1-\beta_n)\|P_C(I-\lambda A)u_n-z\|^2-\beta_n(1-\beta_n) \\
&\times\|w_n-P_C(I-\lambda A)u_n\|^2
\end{aligned}$$

$$
\begin{aligned}
&=\beta_n \left\|\alpha_n f(x_n)+(1-\alpha_n)Sx_n-z\right\|^2+(1-\beta_n)\left\|P_C(I-\lambda A)u_n-z\right\|^2\\
&\quad-\beta_n(1-\beta_n)\left\|w_n-P_C(I-\lambda A)u_n\right\|^2\\
&\leq\beta_n\big(\alpha_n\left\|f(x_n)-z\right\|^2+(1-\alpha_n)\left\|Sx_n-z\right\|^2\big)+(1-\beta_n)\left\|u_n-z\right\|^2\\
&\quad-\beta_n(1-\beta_n)\left\|w_n-P_C(I-\lambda A)u_n\right\|^2\\
&=\beta_n\alpha_n\left\|f(x_n)-z\right\|^2+\beta_n(1-\alpha_n)\left\|x_n-z\right\|^2+(1-\beta_n)\left\|u_n-z\right\|^2\\
&\quad-\beta_n(1-\beta_n)\left\|w_n-P_C(I-\lambda A)u_n\right\|^2\\
&\leq\beta_n\alpha_n\left\|f(x_n)-z\right\|^2+\beta_n(1-\alpha_n)\left\|x_n-z\right\|^2+(1-\beta_n)\left\|x_n-z\right\|^2\\
&\quad-\beta_n(1-\beta_n)\left\|w_n-P_C(I-\lambda A)u_n\right\|^2\\
&=\beta_n\alpha_n\left\|f(x_n)-z\right\|^2+(1-\beta_n\alpha_n)\left\|x_n-z\right\|^2-\beta_n(1-\beta_n)\\
&\quad\times\left\|w_n-P_C(I-\lambda A)u_n\right\|^2\\
&\leq\beta_n\alpha_n\left\|f(x_n)-z\right\|^2+\left\|x_n-z\right\|^2-\beta_n(1-\beta_n)\left\|w_n-P_C(I-\lambda A)u_n\right\|^2.
\end{aligned}
$$

Which yields

$$
\begin{aligned}
\beta_n(1-\beta_n)\left\|w_n-P_C(I-\lambda A)u_n\right\|^2 &\leq\beta_n\alpha_n\left\|f(x_n)-z\right\|^2+\left\|x_n-z\right\|^2-\left\|x_{n+1}-z\right\|^2\\
&\leq\beta_n\alpha_n\left\|f(x_n)-z\right\|^2\\
&\quad+\big(\left\|x_n-z\right\|+\left\|x_{n+1}-z\right\|\big)\left\|x_{n+1}-x_n\right\|.
\end{aligned}
\tag{3.16}
$$

By (3.12),(3.16), conditions (i) and (ii), we have

$$
\lim_{n\to\infty}\left\|w_n-P_C(I-\lambda A)u_n\right\|=0. \tag{3.17}
$$

By the definition of x_n, we obtain

$$
\begin{aligned}
x_{n+1}-P_C(I-\lambda A)u_n &=\beta_n w_n-\beta_n P_C(I-\lambda A)u_n\\
&=\beta_n\big(w_n-P_C(I-\lambda A)u_n\big).
\end{aligned}
\tag{3.18}
$$

By (3.18), we have

$$\begin{aligned}\|x_n - P_C(I-\lambda A)x_n\| &= \|x_n - x_{n+1} + x_{n+1} - P_C(I-\lambda A)u_n + P_C(I-\lambda A)u_n - P_C(I-\lambda A)x_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - P_C(I-\lambda A)u_n\| \\ &\quad + \|P_C(I-\lambda A)u_n - P_C(I-\lambda A)x_n\| \\ &\le \|x_n - x_{n+1}\| + \beta_n \|w_n - P_C(I-\lambda A)u_n\| + \|u_n - x_n\|.\end{aligned}$$

Form (3.12),(3.15) and (3.17), we have

$$\lim_{n\to\infty} \|x_n - P_C(I-\lambda A)x_n\| = 0. \tag{3.19}$$

Since

$$\begin{aligned}\|x_n - P_C(I-\lambda A)u_n\| &= \|x_n - P_C(I-\lambda A)x_n + P_C(I-\lambda A)x_n - P_C(I-\lambda A)u_n\| \\ &\le \|x_n - P_C(I-\lambda A)x_n\| + \|P_C(I-\lambda A)x_n - P_C(I-\lambda A)u_n\| \\ &\le \|x_n - P_C(I-\lambda A)x_n\| + \|x_n - u_n\|.\end{aligned}$$

From (3.15) and (3.19), we have

$$\lim_{n\to\infty} \|x_n - P_C(I-\lambda A)u_n\| = 0. \tag{3.20}$$

By the definition of x_n, we obtain

$$\begin{aligned}x_{n+1} - x_n =& \beta_n\alpha_n\big(f(x_n) - x_n\big) + \beta_n(1-\alpha_n)(Sx_n - x_n) \\ &+ (1-\beta_n)\big(P_C(I-\lambda A)u_n - x_n\big).\end{aligned} \tag{3.21}$$

It follows that

$$\begin{aligned}\beta_n(1-\alpha_n)\|Sx_n - x_n\| \le& \beta_n\alpha_n \|f(x_n) - x_n\| \\ &+ (1-\beta_n)\|P_C(I-\lambda A)u_n - x_n\| + \|x_{n+1} - x_n\|.\end{aligned}$$

By (3.12),(3.20), conditions (i) and (ii), we have

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0. \tag{3.22}$$

Step 4: We will show that $\limsup_{n\to\infty} \langle f(z)-z, x_n - z\rangle \le 0$, where $z = P_{\mathscr{F}} f(z)$.
To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle f(z)-z, x_n - z\rangle = \limsup_{k\to\infty} \langle f(z)-z, x_{n_k} - z\rangle . \tag{3.23}$$

Without loss of generality, we can assume that $x_{n_k} \rightharpoonup \omega$ as $k\to\infty$, where $\omega \in C$.
From (3.15), we obtain $u_{n_k} \rightharpoonup \omega$ as $k\to\infty$.
Assume that $\omega \notin VI(C,A)$. Since $VI(C,A) = F\left(P_C(I-\lambda A)\right)$, we have $\omega \neq P_C(I-\lambda A)\omega$.
By nonexpansiveness of $P_C(I-\lambda A)$,(3.19) and Opial's condition, we have

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - \omega\| &< \liminf_{k\to\infty} \|x_{n_k} - P_C(I-\lambda A)\omega\| \\
&= \liminf_{k\to\infty} \|x_{n_k} - P_C(I-\lambda A)x_{n_k} + P_C(I-\lambda A)x_{n_k} - P_C(I-\lambda A)\omega\| \\
&\le \liminf_{k\to\infty} \|x_{n_k} - P_C(I-\lambda A)x_{n_k}\| \\
&\quad + \liminf_{k\to\infty} \|P_C(I-\lambda A)x_{n_k} - P_C(I-\lambda A)\omega\| \\
&\le \liminf_{k\to\infty} \|x_{n_k} - \omega\| .
\end{aligned}$$

This is a contradiction. Then we have

$$\omega \in VI(C,A). \tag{3.24}$$

Next, we will show that $\omega \in \bigcap_{i=1}^{N} F(T_i)$.
By Lemma 2.10, we have $F(S) = \bigcap_{i=1}^{N} F(T_i)$. Assume that $\omega \neq S\omega$. Using Opial's condition, (3.22), we obtain

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - \omega\| &< \liminf_{k\to\infty} \|x_{n_k} - S\omega\| \\
&= \liminf_{k\to\infty} \|x_{n_k} - Sx_{n_k} + Sx_{n_k} - S\omega\| \\
&\le \liminf_{k\to\infty} \|x_{n_k} - Sx_{n_k}\| + \liminf_{k\to\infty} \|Sx_{n_k} - S\omega\| \\
&\le \liminf_{k\to\infty} \|x_{n_k} - \omega\| .
\end{aligned}$$

This is a contradiction. Then we have

$$\omega \in F(S) = \bigcap_{i=1}^{N} F(T_i). \tag{3.25}$$

Next, we will show that $\omega \in \bigcap_{i=1}^{N} EP(F_i)$.

Since $\sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C$ and $\sum_{i=1}^{N} a_i F_i$ satisfies condition (A1)-(A4), we obtain

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq \sum_{i=1}^{N} a_i F_i(y, u_n), \forall y \in C.$$

In particular, it follows that

$$\left\langle y - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}} \right\rangle \geq \sum_{i=1}^{N} a_i F_i(y, u_{n_k}), \forall y \in C. \tag{3.26}$$

From (3.15),(3.26) and (A4), we have

$$\sum_{i=1}^{N} a_i F_i(y, \omega) \leq 0, \forall y \in C. \tag{3.27}$$

Put $y_t := ty + (1-t)\omega$, for all $t \in (0,1]$, we have $y_t \in C$. By using (A1),(A4) and (3.27), we have

$$\begin{aligned} 0 &= \sum_{i=1}^{N} a_i F_i(y_t, y_t) \\ &= \sum_{i=1}^{N} a_i F_i\big(y_t, ty + (1-t)\omega\big) \\ &\leq t \sum_{i=1}^{N} a_i F_i(y_t, y) + (1-t) \sum_{i=1}^{N} a_i F_i(y_t, \omega) \\ &\leq t \sum_{i=1}^{N} a_i F_i(y_t, y). \end{aligned}$$

It implies that

$$0 \leq \sum_{i=1}^{N} a_i F_i\big(ty + (1-t)\omega, y\big), \tag{3.28}$$

for all $t \in (0,1]$ and $y \in C$.

From (3.28), taking $t \to 0^+$ and using (A3), we can conclude that

$$\begin{aligned} 0 &\leq \lim_{t \to 0^+} \left(\sum_{i=1}^{N} a_i F_i\big(ty + (1-t)\omega, y\big) \right) \\ &\leq \sum_{i=1}^{N} a_i F_i(\omega, y), \forall y \in C. \end{aligned}$$

Therefore, $\omega \in EP\Big(\sum_{i=1}^{N} a_i F_i\Big)$. By Lemma 2.6, we obtain $EP\Big(\sum_{i=1}^{N} a_i F_i\Big) = \bigcap_{i=1}^{N} EP(F_i)$. It follows that

$$\omega \in \bigcap_{i=1}^{N} EP(F_i). \tag{3.29}$$

From (3.24),(3.25) and (3.29), we can deduce that $\omega \in \mathscr{F}$.
Since $x_{n_k} \rightharpoonup \omega \in \mathscr{F}$ and Lemma 2.1, we can conclude that

$$\begin{aligned}
\limsup_{n\to\infty} \langle f(z) - z, x_n - z\rangle &= \limsup_{k\to\infty} \langle f(z) - z, x_{n_k} - z\rangle \\
&= \langle f(z) - z, \omega - z\rangle \\
&\leq 0,
\end{aligned} \tag{3.30}$$

where $z = P_{\mathscr{F}} f(z)$.

Step 5: Finally, we will show that the sequence $\{x_n\}$ converges strongly to $z = P_{\mathscr{F}} f(z)$.
By nonexpansive of S and $P_C(I - \lambda A)$, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \big\|\beta_n\big(\alpha_n f(x_n) + (1-\alpha_n)Sx_n\big) + (1-\beta_n)P_C(I-\lambda A)u_n - z\big\|^2 \\
&= \big\|\beta_n\alpha_n\big(f(x_n) - z\big) + \beta_n(1-\alpha_n)(Sx_n - z) + (1-\beta_n)\big(P_C(I-\lambda A)u_n - z\big)\big\|^2 \\
&\leq \big\|\beta_n(1-\alpha_n)(Sx_n - z) + (1-\beta_n)\big(P_C(I-\lambda A)u_n - z\big)\big\|^2 \\
&\quad + 2\beta_n\alpha_n\langle f(x_n) - z, x_{n+1} - z\rangle \\
&\leq \big(\beta_n(1-\alpha_n)\|Sx_n - z\| + (1-\beta_n)\|P_C(I-\lambda A)u_n - z\|\big)^2 \\
&\quad + 2\beta_n\alpha_n\langle f(x_n) - z, x_{n+1} - z\rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq \big((1-\beta_n\alpha_n)\|x_n-z\|\big)^2+2\beta_n\alpha_n\langle f(x_n)-f(z),x_{n+1}-z\rangle \\
&\quad +2\beta_n\alpha_n\langle f(z)-z,x_{n+1}-z\rangle \\
&\leq \big((1-\beta_n\alpha_n)\|x_n-z\|\big)^2+2\beta_n\alpha_n\|f(x_n)-f(z)\|\,\|x_{n+1}-z\| \\
&\quad +2\beta_n\alpha_n\langle f(z)-z,x_{n+1}-z\rangle \\
&\leq (1-\beta_n\alpha_n)\|x_n-z\|^2+2\beta_n\alpha_n\eta\|x_n-z\|\,\|x_{n+1}-z\| \\
&\quad +2\beta_n\alpha_n\langle f(z)-z,x_{n+1}-z\rangle \\
&\leq (1-\beta_n\alpha_n)\|x_n-z\|^2+\beta_n\alpha_n\eta\|x_n-z\|^2+\beta_n\alpha_n\eta\|x_{n+1}-z\|^2 \\
&\quad +2\beta_n\alpha_n\langle f(z)-z,x_{n+1}-z\rangle .
\end{aligned}
$$

Which implies that

$$
\begin{aligned}
\|x_{n+1}-z\|^2 &\leq \frac{1-\beta_n\alpha_n\eta-\beta_n\alpha_n+2\beta_n\alpha_n\eta}{1-\beta_n\alpha_n\eta}\|x_n-z\|^2 \\
&\quad +\frac{2\beta_n\alpha_n}{1-\beta_n\alpha_n\eta}\langle f(z)-z,x_{n+1}-z\rangle \\
&=\left(1-\frac{\beta_n\alpha_n}{1-\beta_n\alpha_n\eta}\right)\|x_n-z\|^2+\frac{2\beta_n\alpha_n\eta}{1-\beta_n\alpha_n\eta}\|x_n-z\|^2 \\
&\quad +\frac{2\beta_n\alpha_n}{1-\beta_n\alpha_n\eta}\langle f(z)-z,x_{n+1}-z\rangle \\
&=\left(1-\frac{\beta_n\alpha_n}{1-\beta_n\alpha_n\eta}\right)\|x_n-z\|^2+\frac{\beta_n\alpha_n}{1-\beta_n\alpha_n\eta}\big(2\eta\|x_n-z\|^2 \\
&\quad +2\langle f(z)-z,x_{n+1}-z\rangle\big).
\end{aligned}
$$

Applying the conditions (ii),(3.30) and Lemma 2.3, we have the sequence $\{x_n\}$ converges strongly to $z=P_{\mathscr{F}}f(z)$. From (3.15), we obtain $\{u_n\}$ converges strongly to $z=P_{\mathscr{F}}f(z)$. This completes the proof. □

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F: C\times C\to\mathbb{R}$ be a bifunction with satisfy $A1)-A4)$, $T_i: C\to C$ be κ_i-strictly pseudo-contractive mapping, for all $i=1,2,...,N$ and let $A: C\to H$ be α-inverse strongly monotone mapping with $\mathscr{F}=EP(F)\cap\bigcap_{i=1}^N F(T_i)\cap VI(C,A)\neq\emptyset$. Let S be S-mapping generated by $T_1,T_2,...,T_N$ and $\alpha_1,\alpha_2,...,\alpha_N$, where $\alpha_j=(\alpha_1^j,\alpha_2^j,\alpha_3^j)\in I\times I\times I, I=[0,1]$ with $\alpha_1^j+\alpha_2^j+\alpha_3^j=1$ and $\kappa<$*

$\alpha_1^j, \alpha_3^j < 1$, *for all* $i = 1,2,...,N-1, \kappa < \alpha_1^N \leq 1, \kappa \leq \alpha_3^N < 1, \kappa \leq \alpha_2^j < 1$, *for all* $j = 1,2,...,N$, *where* $\kappa = \max\{\kappa_i : i = 1,2,...,N\}$. *Let the sequence* $\{x_n\}$ *generated by* $x_1 \in C$ *and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \beta_n\big(\alpha_n f(x_n) + (1-\alpha_n)Sx_n\big) + (1-\beta_n)P_C(I - \lambda A)u_n, \forall n \geq 1, \end{cases} \tag{3.31}$$

where $\{\beta_n\}, \{\alpha_n\} \subseteq [0,1]$ *and* $\lambda \in (0, 2\alpha)$. *Suppose the following conditions hold:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$,

(ii) $0 < a \leq \beta_n, r_n \leq b < 1$, *for all* $n \geq 1$,

(iii) $f : C \to C$ *be* η*-contraction,*

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,
$\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ *converges strongly to* $z = P_{\mathscr{F}} f(z)$.

Proof. Take $F = F_i, \forall i = 1,2,...,N$. By Theorem 3.1, we obtain the desired conclusion. □

Corollary 3.3. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. $T_i : C \to C$ *be* κ_i*-strictly pseudo-contractive mapping, for all* $i = 1,2,...,N$ *and let* $A : C \to H$ *be* α*-inverse strongly monotone mapping with* $\mathscr{F} = EP(F) \cap \bigcap_{i=1}^N F(T_i) \cap VI(C,A) \neq \emptyset$. *Let* S *be S-mapping generated by* $T_1, T_2, ..., T_N$ *and* $\alpha_1, \alpha_2, ..., \alpha_N$, *where* $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I, I = [0,1]$ *with* $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$ *and* $\kappa < \alpha_1^j, \alpha_3^j < 1$, *for all* $i = 1,2,...,N-1, \kappa < \alpha_1^N \leq 1, \kappa \leq \alpha_3^N < 1, \kappa \leq \alpha_2^j < 1$, *for all* $j = 1,2,...,N$, *where* $\kappa = max\{\kappa_i : i = 1,2,...,N\}$. *Let the sequence* $\{x_n\}$ *generated by* $x_1 \in C$ *and*

$$x_{n+1} = \beta_n\big(\alpha_n f(x_n) + (1-\alpha_n)Sx_n\big) + (1-\beta_n)P_C(I - \lambda A)x_n, \forall n \geq 1, \tag{3.32}$$

where $\{\beta_n\}, \{\alpha_n\} \subseteq [0,1]$ *and* $\lambda \in (0, 2\alpha)$. *Suppose the following conditions hold:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$,

(ii) $0 < a \leq \beta_n, r_n \leq b < 1$, *for all* $n \geq 1$,

(iii) $f : C \to C$ *be* η*-contraction,*

(iv) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty, \sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty,$
$\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty.$

Then $\{x_n\}$ *converges strongly to* $z=P_{\mathscr{F}}f(z)$.

Proof. Put $F_i=0, \forall i=1,2,...,N$. Then we have $u_n=P_Cx_n=x_n, \forall n\in\mathbb{N}$. Therefore the conclusion of Corollary 3.3 can be obtained by Theorem 3.1. □

4. **Application**

In this section, we apply our main theorem to prove strong convergence theorems involving optimization problem.

Let us recall the standard constrained convex optimization problem as follows:

$$\text{find } x^\star\in C \text{ such that } g(x^\star)=\min_{x\in C}g(x), \tag{4.1}$$

where $g:C\to\mathbb{R}$ is a convex, *Frechet* differentiable function, C is closed-convex subset of H. The set of all solutions of (4.1) is denoted by Ω_g.

The following lemmas is important to prove Theorem 4.2.

Lemma 4.1. *(*[23]*) (Optimality condition) A necessary condition of optimality for a point* $x^\star\in C$ *to be a solution of the minimization problem (4.1) is that* $x^\star$ *solves the variational inequality*

$$\langle\nabla g(x^\star), x-x^\star\rangle\geq 0, \ \forall x\in C. \tag{4.2}$$

Equivalently, $x^\star\in C$ *solves the fixed point equation*

$$x^\star=P_C\big(x^\star-\lambda\nabla g(x^\star)\big),$$

for every constant $\lambda>0$*. if, in addition, g is convex, then the optimality condition (4.2) is also sufficient.*

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. For every* $i=1,2,...,N$*, let* $F_i:C\times C\to\mathbb{R}$ *be bifunction with satisfy* $A1)-A4)$*,* $g:C\to\mathbb{R}$ *be a real value convex function with gradient* ∇g *is* $\frac{1}{L}$*-inverse strongly monotone and continuous function for* *all $L \geq 0$. Assume that $\mathscr{F} = \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} F(T_i) \cap \Omega_g \neq \emptyset$. Let S be S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$, where $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I, I = [0,1]$ with $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$ and $\kappa < \alpha_1^j, \alpha_3^j < 1$, for all $i = 1,2,...,N-1, \kappa < \alpha_1^N \leq 1, \kappa \leq \alpha_3^N < 1, \kappa \leq \alpha_2^j < 1$, for all $j = 1,2,...,N$, where $\kappa = \max\{\kappa_i : i = 1,2,...,N\}$. Let the sequence $\{x_n\}$ generated by $x_1 \in C$ and*

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \beta_n\big(\alpha_n f(x_n) + (1-\alpha_n) S x_n\big) + (1-\beta_n) P_C(I - \lambda \nabla g) u_n, \forall n \geq 1, \end{cases} \tag{4.3}$$

where $\{\beta_n\}, \{\alpha_n\} \subseteq [0,1]$ and $\lambda \in (0, \frac{2}{L})$. Suppose the following conditions hold:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$,

(ii) $0 < a \leq \beta_n, r_n \leq b < 1$, *for all* $n \geq 1$,

(iii) $f : C \to C$ *be* η*-contraction,*

(iv) $\sum_{n=1}^{N} a_i = 1$, *where* $a_i > 0$, *for all* $i = 1,2,...,N$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,
$\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z = P_{\mathscr{F}} f(z)$.

Proof. The conclusion of Theorem 4.2 can be obtained from Theorem 3.1 and Lemma 4.1. □

5. Example and Numerical Results

In this section, two examples are given to support Theorem 3.1 and Theorem 4.2, repectively.

Example 5.1. Let $\mathbb{R}$ be the set of real numbers and let the mapping $A : \mathbb{R} \to \mathbb{R}$ defined by $Ax = \frac{2x}{3}, \forall x \in \mathbb{R}$. For all $i = 1,2,...,N$, let the mapping $T_i : \mathbb{R} \to \mathbb{R}$ defined by

$T_i x = \frac{6i}{6i+1} x, \ \forall x \in \mathbb{R}$

and let $F_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$F_i(x,y) = i(-7x^2 + xy + 6y^2), \forall x, y \in \mathbb{R}$.

Furthermore, let $a_i = \frac{6}{7^i} + \frac{1}{N7^N}$, for every $i = 1,2,...,N$. Then we have

$$\sum_{i=1}^{N} a_i F_i(x,y) = \sum_{i=1}^{N} \Big(\frac{6}{7^i} + \frac{1}{N7^N}\Big) i(-7x^2+xy+6y^2) = E(-7x^2+xy+6y^2),$$

where $E = \sum_{i=1}^{N} \Big(\frac{6}{7^i} + \frac{1}{N7^N}\Big) i$, it is easy to check that $\sum_{i=1}^{N} a_i F_i$ satisfies all the conditions of Theorem 3.1. By the definition of F_i, we have

$$\begin{aligned}
0 \leq & \sum_{i=1}^{N} a_i F_i(u_n,y) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \\
=& E(-7x^2+xy+6y^2) + \frac{1}{r_n}(y-u_n)(u_n-x_n) \\
=& E(-7x^2+xy+6y^2) + \frac{1}{r_n}(yu_n - yx_n - u_n^2 - u_n x_n) \\
\Leftrightarrow & \\
0 \leq & Er_n(-7x^2+xy+6y^2) + (yu_n - yx_n - u_n^2 - u_n x_n) \\
=& 6Er_n y^2 + Eu_n r_n y - 7Eu_n^2 r_n + yu_n - yx_n - u_n^2 - u_n x_n \\
=& 6Er_n y^2 + (u_n - x_n + Eu_n r_n)y + (-7Eu_n^2 r_n - u_n^2 - u_n x_n).
\end{aligned}$$

Let $G(y) = 6Er_n y^2 + \big(u_n(1+Er_n) - x_n\big)y - 7Eu_n^2 r_n - u_n^2 - u_n x_n$. $G(y)$ is a quadratic function of y with coefficient $a = 6Er_n$, $b = u_n(1+Er_n) - x_n$ and $c = -7Eu_n^2 r_n - u_n^2 - u_n x_n$. Determine the discriminant Δ of G as follows:

$$\begin{aligned}
\Delta =& b^2 - 4ac \\
=& \big(u_n(1+Er_n) - x_n\big)^2 - 4(6Er_n)(-7Eu_n^2 r_n - u_n^2 - u_n x_n) \\
=& u_n^2(1+Er_n)^2 - 2u_n x_n(1+Er_n) + x_n^2 + 168E^2u_n^2r_n^2 + 24Eu_n^2 r_n - 24Eu_n x_n r_n \\
=& u_n^2 + 2Eu_n^2 r_n + E^2u_n^2r_n^2 - 2u_n x_n - 2Eu_n x_n r_n + x_n^2 + 168E^2u_n^2r_n^2 + 24Eu_n^2 r_n \\
& - 24Eu_n x_n r_n \\
=& u_n^2 + 26Eu_n^2 r_n + 169E^2u_n^2r_n^2 - 2u_n x_n - 26Eu_n x_n r_n + x_n^2 \\
=& (u_n + 13Eu_n r_n)^2 - 2x_n(u_n + 13Eu_n r_n) + x_n^2 \\
=& (u_n + 13Eu_n r_n - x_n)^2.
\end{aligned}$$

We know that $G(y) \geq 0, \forall y \in \mathbb{R}$. If it has at most one solution in $\mathbb{R}$, then $\Delta \leq 0$, so we obtain

$$u_n = \frac{x_n}{1+13\sum_{i=1}^{N}\left(\frac{6}{7^i}+\frac{1}{N7^N}\right)ir_n}, \ for\ all\ n \in \mathbb{N}. \tag{5.1}$$

For every $j = 1,2,...,N$, let $\alpha_1^j = \frac{1}{2j}$, $\alpha_2^j = \frac{3j-1}{16j}$, $\alpha_3^j = \frac{13j-7}{16j}$. Then $\alpha_j = \left(\frac{1}{2j}, \frac{3j-1}{16j}, \frac{13j-7}{16j}\right)$, for all $j = 1,2,...,N$. Let S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. From the definition T_i, A and F_i, we have

$$\{0\} = \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} F(T_i) \cap VI(C,A).$$

Put $\alpha_n = \frac{1}{3n}$, $\beta_n = \frac{4n+2}{17n}$, $r_n = \frac{n}{2n+1}$, $f(x) = \frac{3x}{5}$ and $\lambda = 1$, $\forall n \in \mathbb{N}$. From (5.1) we rewrite (3.1) as follows:

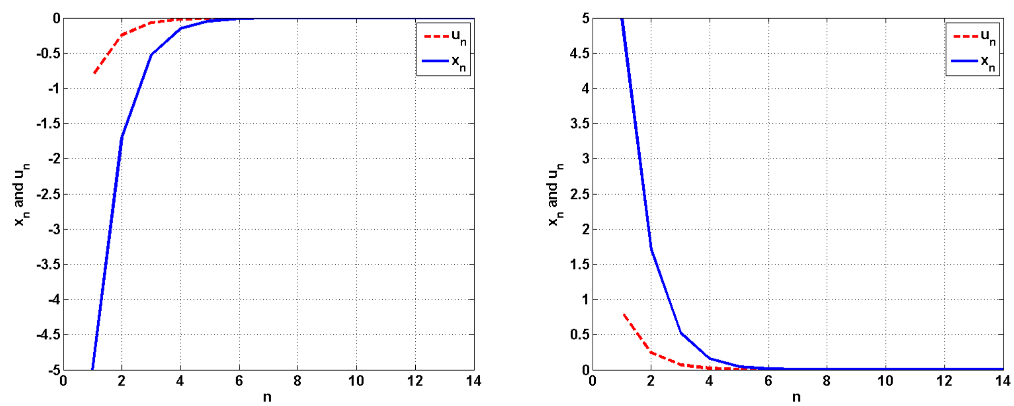
$$\begin{aligned} x_{n+1} =& \left(\frac{4n+2}{17n}\right)\left(\frac{1}{3n}f(x_n) + \left(1-\frac{1}{3n}\right)Sx_n\right) \\ &+ \left(1-\frac{4n+2}{17n}\right)(I-A)\frac{x_n}{1+13\sum_{i=1}^{N}\left(\frac{6}{7^i}+\frac{1}{N7^N}\right)ir_n}, \forall n \geq 1. \end{aligned} \tag{5.2}$$

It is clear that the sequence $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy all the conditions of Theorem 3.1. From Theorem 3.1, we can conclude that the sequence $\{x_n\}$ and $\{u_n\}$ converges strongly to 0.

Table 1 shows that values of sequences $\{x_n\}$ and $\{u_n\}$, where $x_1 = -5$ and $x_1 = 5$ and $n = N = 14$.

	$x_1=-5$		$x_1=5$	
n	u_n	x_n	u_n	x_n
1	-0.825688	-5.000000	0.825688	5.000000
2	-0.241546	-1.706927	0.241546	1.706927
3	-0.070026	-0.525198	0.070026	0.525198
4	-0.019977	-0.154636	0.019977	0.154636
5	-0.005630	-0.044447	0.005630	0.044447
⋮	⋮	⋮	⋮	⋮
8	-0.000120	-0.000980	0.000120	0.000980
⋮	⋮	⋮	⋮	⋮
11	-0.000002	-0.000020	0.000002	0.000020
12	-0.000001	-0.000006	0.000001	0.000006
13	-0.000000	-0.000002	0.000000	0.000002
14	-0.000000	-0.000000	0.000000	0.000000

TABLE 1. The values of $\{u_n\}$ and $\{x_n\}$ where $n=14$.



(A) $x_1=-5$ (B) $x_1=5$

FIGURE 1. The convergence comparison of the sequences $\{x_n\}$ and $\{u_n\}$ with different initial values x_1.

Example 5.2. In this example, we consider the same mappings and parameters as in Example 5.1 except the following mapping $g:\mathbb{R}\to\mathbb{R}$ be defined by $gx=2x^2+1$. It is clear that

$$\{0\}=\bigcap_{i=1}^{N}EP(F_i)\cap\bigcap_{i=1}^{N}F(T_i)\cap\Omega_g.$$

Put $\lambda = \frac{1}{8}$. From (5.1), we rewrite (4.3) as follows:

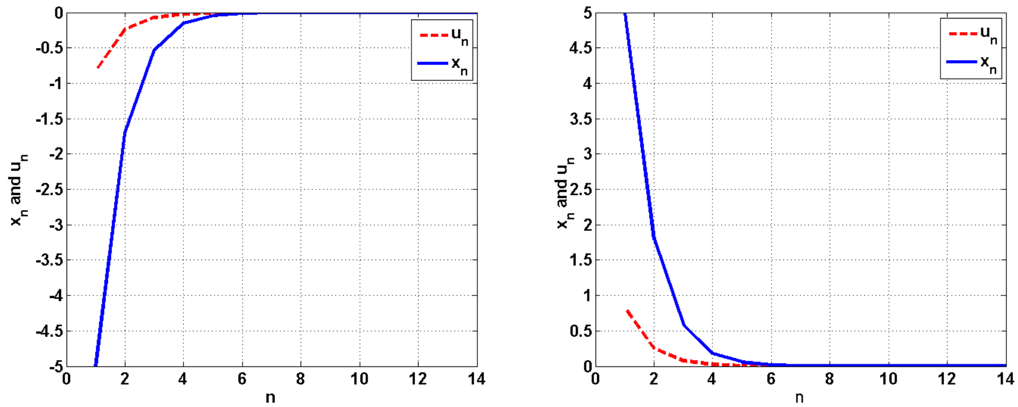
$$\begin{aligned} x_{n+1} =& \Big(\frac{4n+2}{17n}\Big)\Big(\frac{1}{3n}f(x_n) + \Big(1-\frac{1}{3n}\Big)Sx_n\Big) \\ &+ \Big(1-\frac{4n+2}{17n}\Big)(I-\frac{1}{8}\nabla g)\frac{x_n}{1+13\sum_{i=1}^{N}\Big(\frac{6}{7^i}+\frac{1}{N7^N}\Big)ir_n}, \forall n \geq 1. \end{aligned} \tag{5.3}$$

It is clear that the sequence $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy all the conditions of Theorem 4.2. From Theorem 4.2, we can conclude that the sequence $\{x_n\}$ and $\{u_n\}$ converges strongly to 0.

Table 2 shows that values of sequences $\{x_n\}$ and $\{u_n\}$, where $x_1 = -5$ and $x_1 = 5$ and $n = N = 14$.

	$x_1 = -5$		$x_1 = 5$	
n	u_n	x_n	u_n	x_n
1	-0.825688	-5.000000	0.825688	5.000000
2	-0.254147	-1.795972	0.254147	1.795972
3	-0.077666	-0.582496	0.077666	0.582496
4	-0.023370	-0.180898	0.023370	0.180898
5	-0.006949	-0.054859	0.006949	0.054859
⋮	⋮	⋮	⋮	⋮
8	-0.000175	-0.001422	0.000175	0.001422
⋮	⋮	⋮	⋮	⋮
11	-0.000004	-0.000035	0.000004	0.000035
12	-0.000001	-0.000010	0.000001	0.000010
13	-0.000000	-0.000003	0.000000	0.000003
14	-0.000000	-0.000001	0.000000	0.000001

TABLE 2. The values of $\{u_n\}$ and $\{x_n\}$ where $n = 14$.



(A) $x_1 = -5$ (B) $x_1 = 5$

FIGURE 2. The convergence comparison of the sequences $\{x_n\}$ and $\{u_n\}$ with different initial values x_1.

Conclusion

(1) Table 1 and Figure 1 show that $\{x_n\}$ and $\{u_n\}$ converges to 0, where $\{0\} \in \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} F(T_i) \cap VI(C,A)$. The convergence of $\{x_n\}$ and $\{u_n\}$ of Example 5.1 can be guaranteed by Theorem 3.1.

(2) Table 2 and Figure 2 show that $\{x_n\}$ and $\{u_n\}$ converges to 0, where $\{0\} \in \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} F(T_i) \cap \Omega_g$. The convergence of $\{x_n\}$ and $\{u_n\}$ of Example 5.2 can be guaranteed by Theorem 4.2.

(3) From these Example, we obtain that the sequence $\{x_n\}$ in Example 5.1 converges faster than the sequence $\{x_n\}$ in Example 5.2.

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgements This paper was supported by the the Research and Innovation Services of King Mongkut's Institute of Technology Ladkrabang.

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