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DATA DEPENDENCE FOR FOUR-STEP FIXED POINT ITERATIVE SCHEME ALLOCATING VIA CONTRACTIVE-LIKE OPERATORS

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Abstract. Generally, by the data dependence for an iterative scheme means that there exist an estimation by which one can find the fixed point of an unknown operator from the fixed point of given operator. In this paper, we have introduced and analyzed the four-step fixed point iterative scheme and established its convergence and data dependence results by using contractive-like operators. A numerical example is also given, in which instead of computing the fixed point of an operator, an approximation has been made about the fixed point of that operator through contractive-like one. This work is an extension and improvement of the corresponding works of Asaduzzaman *et al.* [10] and other several authors in literature.

Keywords: mann iterative scheme; ishikawa iterative scheme; noor iterative scheme; four-step fixed point iterative scheme; data dependence; contractive-like operators.

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1. Introduction

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The Mann iterative scheme (one-step iterative scheme) [20], invented in 1953 was used to prove the convergence of the sequence to a fixed point of many valued mapping for which the Banach fixed point theorem [12] failed. Later, in 1974 Ishikawa [13] devised a new iterative scheme known as two-step iterative scheme to establish the convergence of Lipschitzian pscudo-contractive map when Mann iterative scheme failed to converge. Noor [8, 9] introduced and analyzed Noor iterative scheme (three-step iterative scheme) to study the approximate solutions of variational inclusions (inequalities) in Hilbert spaces by using the techniques of updating the solution and the auxiliary principle. By inspiring the above mentioned works here, we have introduced and analyzed the four-step fixed point iterative scheme to study the solution of such problems which are impossible to study by above mentioned iterative schemes.

The data dependence abounds in literature of fixed point theory when dealing with Picard iterative scheme, but is quasi-inexistent when dealing with Mann-Ishikawa iterative scheme. In a paper of Soltuz [15] established a data dependence result concerning Mann-Ishikawa iterative scheme. There, he established the data dependence result of Ishikawa iterative scheme for contraction mappings. In [16] Soltuz et al. again established a data dependence result of Ishikawa iterative scheme, but there they used contractive-like operators replacing contraction mappings. In [10] Asaduzzaman et al. studied the data dependence of Noor iterative scheme by using contractive-like operators. Normally, Four-step fixed point iterative scheme is more complicated but nevertheless more stable from Mann iterative scheme or Ishikawa iterative scheme or Noor iterative scheme. It is clear from collected works that in which Mann or Ishikawa or Noor iterative scheme does not converge while Four-step fixed point iterative scheme does. From this point of view here, the Four-step fixed point iterative scheme has been considered and established a data dependence result of that iterative scheme for contractive-like operators.

2. Preliminary Notes

In this section, some definitions and a lemma have been discussed which are used as the tools of main works.

Throughout this paper **N** denotes the set of all natural numbers. Let B be a real Banach space and X be a nonempty, closed and convex subset of B. Let $T, S : X \to X$ be two maps. For given

$x_0, u_0 \in X$ *Four-step fixed point iterative scheme* for T and S as follows:

$$\left.\begin{aligned} x_{n+1} &= (1-a_n)x_n + a_n T y_n; \\ y_n &= (1-b_n)x_n + b_n T z_n; \\ z_n &= (1-c_n)x_n + c_n T r_n; \\ r_n &= (1-d_n)x_n + d_n T x_n, \ \forall n \in \mathbf{N} \end{aligned}\right\} \tag{2.1}$$

$$\left.\begin{aligned} u_{n+1} &= (1-a_n)u_n + a_n S v_n; \\ v_n &= (1-b_n)u_n + b_n S w_n; \\ w_n &= (1-c_n)u_n + c_n S t_n; \\ t_n &= (1-d_n)u_n + d_n S u_n, \ \forall n \in \mathbf{N} \end{aligned}\right\} \tag{2.2}$$

where, the sequences $\{a_n\}_{n=0}^{\infty} \subseteq [0,1]$, $\{b_n\}_{n=0}^{\infty} \subseteq [0,1]$, $\{c_n\}_{n=0}^{\infty} \subseteq [0,1]$ and $\{d_n\}_{n=0}^{\infty} \subset [0,1]$ are convergent, such that

$$\lim_{n\to\infty} a_n = 0,\ \lim_{n\to\infty} b_n = 0,\ \lim_{n\to\infty} c_n = 0,\ \lim_{n\to\infty} d_n = 0\, and\ \sum_{n=0}^{\infty} a_n = \infty. \tag{2.3}$$

If we put $d_n = c_n = b_n = 0\ \forall n \in \mathbf{N}$ in the above stated equations (2.1) and (2.2) then we get Mann iterative scheme [20] for T and S respectively and if we put $d_n = c_n = 0\ \forall n \in \mathbf{N}$ in the above stated equations (2.1) and (2.2) then we get Ishikawa iterative scheme [13] for T and S respectively and if we put $d_n = 0\ \forall n \in \mathbf{N}$ in the above stated equations (2.1) and (2.2) then we get Noor iterative scheme [8, 9] for T and S respectively.

The map T is called *Kannan mapping* [11], if there exists $\beta \in (0,\ 1/2)$ such that

$$\|Tx - Ty\| \le \beta\left(\|x - Tx\| + \|y - Ty\|\right), \tag{2.4}$$

for all $x, y \in X$.

Similar mapping is called *Chatterjea mapping* [14], if there exists $\gamma \in (0,\ 1/2)$ such that

$$\|Tx - Ty\| \le \gamma(\|x - Ty\| + \|y - Tx\|), \tag{2.5}$$

for all $x, y \in X$.

In [17] Zamfirescu collected these classes and introduced the following definition:

Definition 2.1. [1, 17] The operator $T : X \to X$ is called a *Zamfirescu operator* if it satisfies the condition **Z** (Zamfirescu condition) i.e., if and only if there exist the real numbers α, β, γ satisfying $0<\alpha<1$, $0<\beta, \gamma<1/2$ such that for each pair $x, y\in X$, at least one of the following three conditions is true:

$$\left.\begin{aligned}&(z_1)\ \|Tx-Ty\|\leq\alpha\|x-y\|;\\&(z_2)\ \|Tx-Ty\|\leq\beta\left(\|x-Tx\|+\|y-Ty\|\right);\\&(z_3)\ \|Tx-Ty\|\leq\gamma(\|x-Ty\|+\|y-Tx\|);\end{aligned}\right\}\tag{2.6}$$

In [2] Rhoades proved that (z_1), (z_2) and (z_3) are independent conditions.

Consider $x, y\in X$. Since T is a Zamfirescu operator, therefore, at least one of the conditions (z_1), (z_2) and (z_3) is satisfied by T. If (z_2) holds, then

$$\begin{aligned}\|Tx-Ty\|&\leq\beta\left(\|x-Tx\|+\|y-Ty\|\right)\\&\leq\beta\left(\|x-Tx\|+\left(\|y-x\|+\|x-Tx\|+\|Tx-Ty\|\right)\right).\end{aligned}$$

$$\Rightarrow\|Tx-Ty\|\leq\frac{\beta}{(1-\beta)}\|x-y\|+\frac{2\beta}{(1-\beta)}\|x-Tx\|.\tag{2.7}$$

If (z_3) holds, then similarly we obtain

$$\|Tx-Ty\|\leq\frac{\gamma}{(1-\gamma)}\|x-y\|+\frac{2\gamma}{(1-\gamma)}\|x-Tx\|.\tag{2.8}$$

Let us denote

$$\lambda=max\left\{\alpha,\frac{\beta}{(1-\beta)},\frac{\gamma}{(1-\gamma)}\right\}.\tag{2.9}$$

Then we have $0\leq\lambda<1$ and in view of (z_1), (2.7) and (2.8) we get the following inequality

$$\|Tx-Ty\|\leq\lambda\|x-y\|+2\lambda\|x-Tx\|,\tag{2.10}$$

holds $\forall x, y\in X$.

Formula (2.10) was obtained as in [18].

Osilike *et al.* introduced in [7] a more general definition of a *quasi-contractive operator*, they considered the operator for which there exists $L\geq 0$ and $q\in(0, 1)$ such that

$$\|Tx-Ty\|\leq q\|x-y\|+L\|x-Tx\|,\forall x, y\in X.\tag{2.11}$$

In 2003, Imoru *et al.* [4] considered the following more general type of contractive operator but they are failed to name it. Later in 2008, Soltuz *et al.* [16] used it as contractive-like operators.

Definition 2.2.[16] The operator T is called *contractive-like operator* if there exist a constant $q \in (0, 1)$ and a strictly increasing and continuous function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that for each $x, y \in X$

$$\|Tx - Ty\| \leq q\|x - y\| + \phi\left(\|x - Tx\|\right), \forall x, y \in X. \tag{2.12}$$

The inequality $(1 - x) \leq exp(x), \forall x \geq 0$ leads the following remark.

Remark 2.3. [10] Let $\{e_n\}$ be a non-negative sequence such that $e_n \in (0, 1], \forall n \in \mathbf{N}$. If

$$\sum_{n=1}^{\infty} e_n = \infty,$$

then

$$\prod_{n=1}^{\infty} (1 - e_n) = 0.$$

Proof of remark 2.3. According to the assumption, we suppose that

$$e_1 = 1, e_2 = \frac{1}{2}, e_3 = \frac{1}{3}, e_4 = \frac{1}{4}, \cdots, e_n = \frac{1}{n}, \cdots$$

That is the given sequence is

$$\{e_n\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n}, \cdots\right\}$$

Hence, it is clear that

$$\sum_{n=1}^{\infty} e_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots = \infty$$

Now, we have to show that

$$\prod_{n=1}^{\infty} (1 - e_n) = 0.$$

We have

$$\prod_{n=1}^{\infty} (1 - e_n) = (1 - e_1) . (1 - e_2) . (1 - e_3) . (1 - e_4) \cdots$$

But according to the assumption one of the values of $e_1, e_2, e_3, e_4, \cdots$ must be equal to 1.

Therefore,$(1 - e_1) . (1 - e_2) . (1 - e_3) . (1 - e_4) \cdots \infty = 0$.

Which implies that

$$\prod_{n=1}^{\infty} (1 - e_n) = 0.$$

This completes the proof.

The following lemma is collected from the paper of Park [6].

Lemma 2.4. [6] *Let $\{x_n\}$ be a nonnegative sequence for which one supposes there exists $n_0 \in \boldsymbol{N}$, such that for all $n \geq n_0$ one has satisfied the following inequality:*

$$x_{n+1} \leq (1-\delta_n)x_n + \delta_n\sigma_n. \tag{2.13}$$

where,

$$\delta_n \in (0,1), \sum_{n=1}^{\infty}\delta_n = \infty, \textit{and } \sigma_n \geq 0, \forall n \in \boldsymbol{N}.$$

Then

$$0 \leq \lim_{n\to\infty}\sup x_n \leq \lim_{n\to\infty}\sup \sigma_n. \tag{2.14}$$

Proof. There exists $n_1 \in \mathbf{N}$ such that $\sigma_n \leq \lim_{n\to\infty}\sup \sigma_n, \forall n \geq n_1$. Set $n_2 = \{n_0, n_1\}$ such that the following inequality holds, for all $n \geq n_1$:

$$x_{n+1} \leq (1-\delta_n)(1-\delta_{n-1})\cdots(1-\delta_{n_1})x_{n_1} + \lim_{n\to\infty}\sup \sigma_n. \tag{2.15}$$

Using the above Remark 2.3 with $e_n = \delta_n$, we get the conclusion. In order to prove (2.14), consider (2.13) and the induction step:

$$\begin{aligned} x_{n+2} &\leq (1-\delta_{n+1})x_{n+1} + \delta_{n+1}\sigma_{n+1} \\ &\leq (1-\delta_{n+1})(1-\delta_n)(1-\delta_{n-1})\cdots(1-\delta_{n_1})x_{n_1} + (1-\delta_{n+1})\lim_{n\to\infty}\sup \sigma_n + \delta_{n+1}\sigma_{n+1} \\ &= (1-\delta_{n+1})(1-\delta_n)(1-\delta_{n-1})\cdots(1-\delta_{n_1})x_{n_1} + (1-\delta_{n+1})\lim_{n\to\infty}\sup \sigma_{n+1}. \end{aligned} \tag{2.16}$$

This completes the lemma.

3. Convergence of Four-Step Fixed Point Iterative Scheme

In this section, a convergence theorem for Four-step fixed point iterative scheme has been stated and proved.

Theorem 3.1. *Let B be a real Banach space, X a nonempty, closed and convex subset of B, and $T: X \to X$ a contractive-like map with m^* being the fixed point. Then for all $x_0 \in X$, the*

Four-step fixed point iterative scheme $\{x_n\}$ *defined by (2.1) converges to the unique fixed point of* T.

Proof. First we prove the uniqueness of the fixed point of T . If possible let the mapping T has two distinct fixed points m^* and n^*. Then by using the definition 2.2 we get

$$\|m^* - n^*\| = \|Tm^* - Tn^*\| \leq q\|m^* - n^*\| + \phi\left(\|m^* - Tm^*\|\right) = q\|m^* - n^*\|. \tag{3.1}$$

This implies that $m^* = n^*$.

Hence the fixed point of T is unique.

Now, from (2.1) and (2.12) we obtain

$$\begin{aligned}
\|x_{n+1} - m^*\| &= \|\left(1-a_n\right)x_n + a_nTy_n - m^*\| \\
&\leq \left(1-a_n\right)\|x_n - m^*\| + a_n\|Ty_n - Tm^*\| \\
&\leq \left(1-a_n\right)\|x_n - m^*\| + qa_n\|y_n - m^*\| \\
&\leq \left(1-a_n\right)\|x_n - m^*\| + qa_n\left(1-b_n\right)\|x_n - m^*\| + qa_nb_n\|Tz_n - Tm^*\| \\
&\leq \left(1-a_n\right)\|x_n - m^*\| + qa_n\left(1-b_n\right)\|x_n - m^*\| + q^2a_nb_n\|z_n - m^*\| \\
&\leq \left(1-a_n\right)\|x_n - m^*\| + qa_n\left(1-b_n\right)\|x_n - m^*\| + q^2a_nb_n\left(1-c_n\right)\|x_n - m^*\| \\
&\quad + q^2a_nb_nc_n\|Tr_n - Tm^*\| \\
&\leq \left(1-a_n\right)\|x_n - m^*\| + qa_n\left(1-b_n\right)\|x_n - m^*\| + q^2a_nb_n\left(1-c_n\right)\|x_n - m^*\| \\
&\quad + q^3a_nb_nc_n\|r_n - m^*\| \\
&\leq \left(1-a_n\right)\|x_n - m^*\| + qa_n\left(1-b_n\right)\|x_n - m^*\| + q^2a_nb_n\left(1-c_n\right)\|x_n - m^*\| \\
&\quad + q^3a_nb_nc_n\left(1-d_n\right)\|x_n - m^*\| + q^4a_nb_nc_nd_n\|x_n - m^*\| \\
&= \left(\left(1-a_n\right) + qa_n\left(1-b_n\right) + q^2a_nb_n\left(1-c_n\right) + q^3a_nb_nc_n\left(1-d_n\right)\right. \\
&\quad \left. + q^4a_nb_nc_nd_n\right)\|x_n - m^*\| \\
&= \left(1 - a_n\left(1 - q\left(1-b_n\right) - q^2b_n\left(1-c_n\right) - q^3b_nc_n\left(1-d_n\right) - q^4b_nc_nd_n\right)\right) \\
&\quad \|x_n - m^*\|
\end{aligned}$$

$$= \left(1 - a_n \left(1 - q \left(1 - b_n + q b_n (1 - c_n) + q^2 b_n c_n (1 - d_n) + q^3 b_n c_n d_n\right)\right)\right) \|x_n - m^*\|$$

$$= \left(1 - a_n \left(1 - q \left(1 - b_n \left(1 - q (1 - c_n) - q^2 c_n (1 - d_n) - q^3 c_n d_n\right)\right)\right)\right) \|x_n - m^*\|$$

$$= \left(1 - a_n \left(1 - q \left(1 - b_n \left(1 - q \left(1 - c_n + q c_n (1 - d_n) + q^2 c_n d_n\right)\right)\right)\right)\right) \|x_n - m^*\|$$

$$= \left(1 - a_n \left(1 - q \left(1 - b_n \left(1 - q \left(1 - c_n \left(1 - q (1 - d_n) - q^2 d_n\right)\right)\right)\right)\right)\right) \|x_n - m^*\|$$

$$= (1 - a_n (1 - q (1 - b_n (1 - q (1 - c_n (1 - q (1 - d_n + q d_n))))))) \|x_n - m^*\|$$

$$= (1 - a_n (1 - q (1 - b_n (1 - q (1 - c_n (1 - q (1 - d_n (1 - q)))))))) \|x_n - m^*\|$$

$$i.e. \|x_{n+1} - m^*\| \leq (1 - a_n (1 - q (1 - b_n (1 - q (1 - c_n (1 - q (1 - d_n (1 - q)))))))) \|x_n - m^*\|. \tag{3.2}$$

But, according to the supposition we can write

$$(1 - a_n (1 - q (1 - b_n (1 - q (1 - c_n (1 - q (1 - d_n (1 - q)))))))) \leq (1 - a_n (1 - q)). \tag{3.3}$$

Combining (3.2) and (3.3) we get

$$\begin{aligned} \|x_{n+1} - m^*\| &\leq (1 - a_n (1 - q)) \|x_n - m^*\| \\ &\leq (1 - a_n (1 - q)) (1 - a_{n-1} (1 - q)) \|x_{n-1} - m^*\| \\ &\leq (1 - a_n (1 - q)) (1 - a_{n-1} (1 - q)) (1 - a_{n-2} (1 - q)) \|x_{n-2} - m^*\| \\ &\leq \cdots \leq \left[\prod_{k=0}^{n} (1 - a_k (1 - q))\right] \|x_0 - m^*\|. \end{aligned} \tag{3.4}$$

Since, $\sum_{n=1}^{\infty} a_n = \infty$, hence $\sum_{n=1}^{\infty} a_n (1 - q) = \infty \; \forall \, q \in (0, 1)$. Now, by the Remark 2.3 we can write

$$\lim_{n \to \infty} \left[\prod_{k=0}^{n} (1 - a_k (1 - q))\right] = 0. \tag{3.5}$$

From (3.4) and (3.5) we get

$$\lim_{n \to \infty} \|x_{n+1} - m^*\| = 0 \Rightarrow \lim_{n \to \infty} \|x_n - m^*\| = 0.$$

This completes the proof.

4. Data Dependence for Four-Step Fixed Point Iterative Scheme in Banach Spaces

In this section, the data dependence result for Four-step fixed point iterative scheme has been established.

Theorem 4.1. *Let B be a real Banach space, X be a nonempty, closed and convex subset of B and let $\varepsilon > 0$ be a fixed number. Suppose $T : X \to X$ is a contractive-like operator with the fixed point x^* and $S : X \to X$ is an operator with the fixed point u^*, (supposed nearest to x^*). If Four-step fixed point iterative schemes (2.1) and (2.2) are defined for T and S respectively, and the following relation is satisfied:*

$$\|Tz - Sz\| \leq \varepsilon, \; \forall \, z \in X, \tag{4.1}$$

then,

$$\|x^* - u^*\| \leq \frac{\varepsilon}{(1-q)}, \; \forall \, q \in (0, 1). \tag{4.2}$$

Proof. From the definition of Four-step fixed point iterative scheme defined by (2.1) and (2.2), we can write

$$x_{n+1} - u_{n+1} = (1-a_n)(x_n - u_n) + a_n (Ty_n - Sv_n). \tag{4.3}$$

Now, taking norm on both sides of (4.3), we have

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| &= \| (1-a_n)(x_n - u_n) + a_n (Ty_n - Sv_n) \| \\
&\leq (1-a_n)\|x_n - u_n\| + a_n \|Ty_n - Sv_n\| \\
&\leq (1-a_n)\|x_n - u_n\| + a_n \|Ty_n - Tv_n + Tv_n - Sv_n\| \\
&\leq (1-a_n)\|x_n - u_n\| + a_n \|Tv_n - Sv_n\| + a_n \|Tv_n - Ty_n\| \\
&\leq (1-a_n)\|x_n - u_n\| + a_n \varepsilon + q a_n \|y_n - v_n\| + a_n \phi (\|y_n - Ty_n\|)
\end{aligned}$$

$$\leq (1-a_n)\|x_n-u_n\|+a_n\varepsilon+qa_n(1-b_n)\|x_n-u_n\|+qa_nb_n\|Tz_n-Sw_n\|$$
$$+a_n\phi(\|y_n-Ty_n\|)$$
$$\leq (1-a_n)\|x_n-u_n\|+a_n\varepsilon+qa_n(1-b_n)\|x_n-u_n\|$$
$$+qa_nb_n\|Sw_n-Tw_n+Tw_n-Tz_n\|+a_n\phi(\|y_n-Ty_n\|)$$
$$\leq (1-a_n)\|x_n-u_n\|+a_n\varepsilon+qa_n(1-b_n)\|x_n-u_n\|+qa_nb_n\varepsilon+qa_nb_n\|Tz_n-Tw_n\|$$
$$+a_n\phi(\|y_n-Ty_n\|)$$
$$\leq (1-a_n)\|x_n-u_n\|+a_n\varepsilon+qa_n(1-b_n)\|x_n-u_n\|+qa_nb_n\varepsilon+q^2a_nb_n\|z_n-w_n\|$$
$$+qa_nb_n\phi(\|z_n-Tz_n\|)+a_n\phi(\|y_n-Ty_n\|)$$
$$\leq (1-a_n)\|x_n-u_n\|+a_n\varepsilon+qa_n(1-b_n)\|x_n-u_n\|+qa_nb_n\varepsilon$$
$$+q^2a_nb_n(1-c_n)\|x_n-u_n\|+q^2a_nb_nc_n\|Tr_n-St_n\|+qa_nb_n\phi(\|z_n-Tz_n\|)$$
$$+a_n\phi(\|y_n-Ty_n\|)$$
$$\leq (1-a_n)\|x_n-u_n\|+a_n\varepsilon+qa_n(1-b_n)\|x_n-u_n\|+qa_nb_n\varepsilon$$
$$+q^2a_nb_n(1-c_n)\|x_n-u_n\|+q^2a_nb_nc_n\|St_n-Tt_n+Tt_n-Tr_n\|+qa_nb_n\phi(\|z_n-Tz_n\|)$$
$$+a_n\phi(\|y_n-Ty_n\|)$$
$$\leq (1-a_n)\|x_n-u_n\|+a_n\varepsilon+qa_n(1-b_n)\|x_n-u_n\|+qa_nb_n\varepsilon$$
$$+q^2a_nb_n(1-c_n)\|x_n-u_n\|+q^2a_nb_nc_n\varepsilon+q^2a_nb_nc_n\|Tr_n-Tt_n\|+qa_nb_n\phi(\|z_n-Tz_n\|)$$
$$+a_n\phi(\|y_n-Ty_n\|)$$
$$\leq (1-a_n)\|x_n-u_n\|+a_n\varepsilon+qa_n(1-b_n)\|x_n-u_n\|+qa_nb_n\varepsilon$$
$$+q^2a_nb_n(1-c_n)\|x_n-u_n\|+q^2a_nb_nc_n\varepsilon+q^3a_nb_nc_n\|r_n-t_n\|+q^2a_nb_nc_n\phi(\|r_n-Tr_n\|)$$
$$+qa_nb_n\phi(\|z_n-Tz_n\|)+a_n\phi(\|y_n-Ty_n\|)$$
$$\leq (1-a_n)\|x_n-u_n\|+a_n\varepsilon+qa_n(1-b_n)\|x_n-u_n\|+qa_nb_n\varepsilon$$
$$+q^2a_nb_n(1-c_n)\|x_n-u_n\|+q^2a_nb_nc_n\varepsilon+q^3a_nb_nc_n(1-d_n)\|x_n-u_n\|$$
$$+q^3a_nb_nc_nd_n\|Tx_n-Su_n\|+q^2a_nb_nc_n\phi(\|r_n-Tr_n\|)+qa_nb_n\phi(\|z_n-Tz_n\|)$$
$$+a_n\phi(\|y_n-Ty_n\|)$$

$$
\begin{aligned}
&\leq (1-a_n)\|x_n-u_n\| + a_n\varepsilon + qa_n(1-b_n)\|x_n-u_n\| + qa_nb_n\varepsilon \\
&+ q^2a_nb_n(1-c_n)\|x_n-u_n\| + q^2a_nb_nc_n\varepsilon + q^3a_nb_nc_n(1-d_n)\|x_n-u_n\| \\
&+ q^3a_nb_nc_nd_n\|Tx_n-Tu_n+Tu_n-Su_n\| + q^2a_nb_nc_n\phi(\|r_n-Tr_n\|) + qa_nb_n\phi(\|z_n-Tz_n\|) \\
&+ a_n\phi(\|y_n-Ty_n\|) \\
&\leq \big(1-a_n+qa_n(1-b_n)+q^2a_nb_n(1-c_n)+q^3a_nb_nc_n(1-d_n)\big)\|x_n-u_n\| \\
&+ a_n\varepsilon + qa_nb_n\varepsilon + q^2a_nb_nc_n\varepsilon + q^3a_nb_nc_nd_n\varepsilon + q^4a_nb_nc_nd_n\|x_n-u_n\| \\
&+ q^3a_nb_nc_nd_n\phi(\|x_n-Tx_n\|) + q^2a_nb_nc_n\phi(\|r_n-Tr_n\|) \\
&+ qa_nb_n\phi(\|z_n-Tz_n\|) + a_n\phi(\|y_n-Ty_n\|) \\
&\leq \big(1-a_n+qa_n(1-b_n)+q^2a_nb_n(1-c_n)+q^3a_nb_nc_n(1-d_n)+q^4a_nb_nc_nd_n\big)\|x_n-u_n\| \\
&+ a_n\varepsilon + qa_nb_n\varepsilon + q^2a_nb_nc_n\varepsilon + q^3a_nb_nc_nd_n\varepsilon \\
&+ q^3a_nb_nc_nd_n\phi(\|x_n-Tx_n\|) + q^2a_nb_nc_n\phi(\|r_n-Tr_n\|) \\
&+ qa_nb_n\phi(\|z_n-Tz_n\|) + a_n\phi(\|y_n-Ty_n\|) \\
&\leq (1-a_n(1-q(b_n(1-q(1-c_n(1-q(1-d_n(1-q))))))))\|x_n-u_n\| \\
&+ q^3a_nb_nc_nd_n\phi(\|x_n-Tx_n\|) + q^2a_nb_nc_n\phi(\|r_n-Tr_n\|) + qa_nb_n\phi(\|z_n-Tz_n\|) \\
&+ a_n\phi(\|y_n-Ty_n\|) + a_n\varepsilon + qa_nb_n\varepsilon + q^2a_nb_nc_n\varepsilon + q^3a_nb_nc_nd_n\varepsilon \\
&\leq (1-a_n(1-q))\|x_n-u_n\| \\
&+ a_n\big(q^3b_nc_nd_n\phi(\|x_n-Tx_n\|) + q^2b_nc_n\phi(\|r_n-Tr_n\|) + qb_n\phi(\|z_n-Tz_n\|) + \phi(\|y_n-Ty_n\|)\big) \\
&+ a_n\big(\varepsilon + qb_n\varepsilon + q^2b_nc_n\varepsilon + q^3b_nc_nd_n\varepsilon\big) \\
&\leq (1-a_n(1-q))\|x_n-u_n\| \\
&+ a_n(1-q)\frac{\left\{\begin{array}{l} q^3b_nc_nd_n\phi(\|x_n-Tx_n\|) + q^2b_nc_n\phi(\|r_n-Tr_n\|) + qb_n\phi(\|z_n-Tz_n\|) \\ +\phi(\|y_n-Ty_n\|) + \varepsilon + qb_n\varepsilon + q^2b_nc_n\varepsilon + q^3b_nc_nd_n\varepsilon \end{array}\right\}}{(1-q)}.
\end{aligned} \tag{4.4}
$$

It is clear that

$$\lim_{n\to\infty}\phi(\|x_n-Tx_n\|)=\lim_{n\to\infty}\phi(\|y_n-Ty_n\|)=\lim_{n\to\infty}\phi(\|z_n-Tz_n\|)=\lim_{n\to\infty}\phi(\|r_n-Tr_n\|)=0,$$

because of ϕ is a continuous function and $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{r_n\}$ all are converge to the fixed point of T.

Now, if we put

$$\delta_n = a_n(1-q)$$

$$\sigma_n = \frac{\left\{\begin{array}{l} q^3 b_n c_n d_n \phi\left(\|x_n - Tx_n\|\right) + q^2 b_n c_n \phi\left(\|r_n - Tr_n\|\right) + q b_n \phi\left(\|z_n - Tz_n\|\right) \\ + \phi\left(\|y_n - Ty_n\|\right) + \varepsilon + q b_n \varepsilon + q^2 b_n c_n \varepsilon + q^3 b_n c_n d_n \varepsilon \end{array}\right\}}{(1-q)}.$$

then, from the Lemma 2.4 we get

$$\begin{aligned} &\lim_{n\to\infty} sup\|x_n-u_n\| \\ &\leq \lim_{n\to\infty} sup \frac{\left\{\begin{array}{l} q^3 b_n c_n d_n \phi\left(\|x_n - Tx_n\|\right) + q^2 b_n c_n \phi\left(\|r_n - Tr_n\|\right) + q b_n \phi\left(\|z_n - Tz_n\|\right) \\ + \phi\left(\|y_n - Ty_n\|\right) + \varepsilon + q b_n \varepsilon + q^2 b_n c_n \varepsilon + q^3 b_n c_n d_n \varepsilon \end{array}\right\}}{(1-q)}. \end{aligned}$$

Hence from (4.4), we get

$$\|x^* - u^*\| \leq \frac{\varepsilon}{(1-q)}.$$

This completes the theorem.

Corollary 4.2.

(i) *If* $d_n = 0, \forall n \in \boldsymbol{N}$, *arise in Theorem 4.1, then the data dependence for Noor iterative scheme [8, 9] obtain.*

(ii) *If* $c_n = d_n = 0, \forall n \in \boldsymbol{N}$, *arise in Theorem 4.1, then the data dependence for Ishikawa iterative scheme [13] obtain.*

(iii) $b_n = c_n = d_n = 0, \forall n \in \boldsymbol{N}$, *arise in Theorem 4.1, then the data dependence for Mann iterative scheme [20] obtain.*

(iv) *If is replaceable the contractive-like operator by Zamfirescu operator or Chatterjea mapping or Kannan mapping in the Theorem 4.1 then the Theorem 4.1, remains true for all of these operators.*

Proof of Corollary 4.2. From the definition of Four-step fixed point iterative scheme defined by (2.1), we have

$$\left.\begin{aligned} x_{n+1} &= (1-a_n)x_n + a_n T y_n; \\ y_n &= (1-b_n)x_n + b_n T z_n; \\ z_n &= (1-c_n)x_n + c_n T r_n; \\ r_n &= (1-d_n)x_n + d_n T x_n, \ \forall n \in \mathbf{N} \end{aligned}\right\}$$

where, the sequences $\{a_n\}_{n=0}^{\infty} \subseteq [0,1]$, $\{b_n\}_{n=0}^{\infty} \subseteq [0,1]$, $\{c_n\}_{n=0}^{\infty} \subseteq [0,1]$ and $\{d_n\}_{n=0}^{\infty} \subset [0,1]$ are convergent, such that

$$\lim_{n\to\infty} a_n = 0, \lim_{n\to\infty} b_n = 0, \lim_{n\to\infty} c_n = 0, \lim_{n\to\infty} d_n = 0 \, and \sum_{n=0}^{\infty} a_n = \infty.$$

Now, if we put $d_n = 0, \forall n \in \mathbf{N}$, $c_n = d_n = 0, \forall n \in \mathbf{N}$, $b_n = c_n = d_n = 0, \forall n \in \mathbf{N}$ in the above equation, then we obtain the Noor iterative scheme

$$\left.\begin{aligned} x_{n+1} &= (1-a_n)x_n + a_n T y_n; \\ y_n &= (1-b_n)x_n + b_n T z_n; \\ z_n &= (1-c_n)x_n + c_n T x_n, \ \forall n \in \mathbf{N} \end{aligned}\right\} \tag{4.5}$$

where, the sequences $\{a_n\}_{n=0}^{\infty} \subseteq [0,1]$, $\{b_n\}_{n=0}^{\infty} \subseteq [0,1]$, and $\{c_n\}_{n=0}^{\infty} \subseteq [0,1]$ are convergent, such that

$$\lim_{n\to\infty} a_n = 0, \lim_{n\to\infty} b_n = 0, \lim_{n\to\infty} c_n = 0, \, and \sum_{n=0}^{\infty} a_n = \infty,$$

the Ishikawa iterative scheme

$$\left.\begin{aligned} x_{n+1} &= (1-a_n)x_n + a_n T y_n; \\ y_n &= (1-b_n)x_n + b_n T x_n, \ \forall n \in \mathbf{N} \end{aligned}\right\} \tag{4.6}$$

where, the sequences $\{a_n\}_{n=0}^{\infty} \subseteq [0,1]$, and $\{b_n\}_{n=0}^{\infty} \subseteq [0,1]$ are convergent, such that

$$\lim_{n\to\infty} a_n = 0, \lim_{n\to\infty} b_n = 0, \, and \sum_{n=0}^{\infty} a_n = \infty,$$

and the Mann iterative scheme

$$x_{n+1} = (1-a_n)x_n + a_n T x_n, \ \forall n \in \mathbf{N} \tag{4.7}$$

where, the sequence $\{a_n\}_{n=0}^{\infty} \subseteq [0,1]$ is convergent, such that $\lim_{n\to\infty} a_n = 0$, and $\sum_{n=0}^{\infty} a_n = \infty$ respectively.

Hence, it is easily verified that, if we replace the Four-step fixed point iterative scheme by Noor iterative scheme defined by (4.5), Ishikawa iterative scheme defined by (4.6), Mann Iterative scheme defined by (4.7) in the Theorem 4.1, then we get the data dependence for Noor iterative scheme, Ishikawa iterative scheme and Mann iterative scheme respectively.

This proves (i), (ii) and (iii).

To prove (iv), First we observe the equations (2.4), (2.5), (2.6) and (2.12) and after that observation we can comment that, the contractive-like operator is a general operator among contractive-like operator, Zamfirescu operator, Chatterjea mapping and Kannan mapping. Now since, the theorem 4.1 is true for the general operator (contractive-like operator), hence, it will must be true for all other above mentioned operators.

5. Numerical example

In this section, a numerical example has been discussed which shows the application of Theorem 4.1.

Example 5.1. Let $T : \mathbf{R} \to \mathbf{R}$ be a mapping defined by

$$Tx = \begin{cases} 2; & when x \in (-\infty, 4] \\ -2.5; & when x \in (4, +\infty), \end{cases} \tag{5.1}$$

where, $\mathbf{R}$ denotes the set of real numbers. Then, it is clear that T is contractive-like operator with $q = 0.01$ and ϕ is an identity function and $x = 2$ is the unique fixed point of T.

Now, we consider an arbitrary mapping $S : \mathbf{R} \to \mathbf{R}$, which is defined as follows:

$$Sx = \begin{cases} 3; & when x \in (-\infty, 4] \\ -3.5; & when x \in (4, +\infty). \end{cases} \tag{5.2}$$

It is clear that $x = 3$ is the unique fixed point of S.

Now, we cheek it by applying Four-step fixed point iterative scheme defined by (2.2) and form an estimation by applying the Theorem 4.1.

If we take the initial approximation as $u_0 = 2$ and $a_n = b_n = c_n = d_n = 1/(n+2)$, then for S the

Four-step fixed point iterative scheme defined by (2.2) gives the following result:

Iterative Step (n)	Obtained value by Four-step fixed point iterative scheme
$n=1$	2.500000
$n=2$	2.666667
......	......
$n=5$	2.833333
......	......
$n=10$	2.909091
......	......
$n=100$	2.990099
......	......
$n=200$	2.995025
......	......
......	......
$n \to \infty$	3

TABLE 1. Computation of fixed point by Four-step fixed point iterative scheme.

The above computations have been obtained by using a MATLAB-7 program. This computation conclude that Four-step fixed point iterative scheme (2.2) defined for S converges to fixed point $u^* = 3$, which is the unique fixed point of S.

Take ε to be the distance between the two maps as follows:

$$\|Sx - Tx\| \leq 1, \forall x \in \mathbf{R}, \textit{i.e.}, \varepsilon = 1.$$

Ultimately, we can see that the distance between the fixed points of T and S is 1. Actually, without knowing the fixed point of S and without computing it, with the Theorem 4.1, we can formulate the following estimate for it:

$$\|x^* - u^*\| \leq \frac{1}{(1-0.01)} = \frac{1}{0.99} = 1.01010101.$$

That is the distance between the fixed points of T and S must be less than 1.01010101.

6. Conclusion

The result established in this paper is an extension and improvement of the results of Asaduzzaman *et al*. [10], the result of Soltuz *et al.* [15, 16] and Karakaya *et al.* [19]. Here, the data dependence for more general Four-step fixed point iterative scheme defined by (2.1)has been established. By applying the Theorem 4.1 one can estimate the unknown fixed point of an arbitrary mapping without computing fixed point of it. The data dependence of Four-step fixed point iterative scheme generate the data dependence of Mann, Ishikawa and Noor iterative schemes, because from the data dependence of Four-step fixed point iterative scheme, it can easily be accessible the data dependence of Mann, Ishikawa and Noor iterative schemes.

Conflict of Interests

The authors declare that there is no conflict of interests.

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