Common fixed point theorems involving contractive conditions of rational type in dislocated quasi metric spaces

Yeshimabet Jira, Kidane Koyas, Aynalem Girma[*]

Department of Mathematics, Jimma University, Jimma, Ethiopia

Abstract: The purpose of this paper is to establish new common fixed point results for a pair of self- maps involving contractive condition of rational type in complete dislocated quasi metric space by extending the result of Rahman and Sarwar.

Keywords: dislocated quasi metric space; compatible maps; weakly compatible mappings; coincidence points and common fixed point.

2010 AMS Subject Classification: 47H10.

1. Introduction

Fixed point theory is one of the most dynamic research subjects in nonlinear analysis and can be used to many discipline branches such as; control theory, convex optimization, differential equation, integral equation, economics etc. In this area, the first important and remarkable result was presented by Banach in 1922 for a contraction mapping in a complete metric space. Dass and Gupta [17] generalized the Banach contraction principle in a metric space for some rational type

[*]Corresponding author

E-mail address: aynalemgirma46@yahoo.com

YESHIMABET JIRA, KIDANE KOYAS, AYNALEM GIRMA

contractive conditions. Hitzler and Seda [19] investigated the useful applications of dislocated topology in the context of logic programming semantics. Dislocated metric space has a key role in logic programming and electronics engineering. Furthermore, Zeyada et al. [3] generalized the results of Hitzler and Seda [19] and introduced the concept of complete dislocated quasi metric space. Aage and Salunke [18] derived some fixed point theorems in dislocated quasi metric spaces. In this paper, we prove fixed point result in the setting of dislocated quasi-metric spaces for a pair of self-mappings which generalize the result of Rahman and Sarwar [16].

2. Preliminaries

Definition 2.1[3]. Let X be a non- empty set and $d : X \times X \to [0,\infty)$ be a function satisfying the following conditions:

$$d_1) \ d(x, y) = d(y, x) = 0, \ \text{implies} \ x = y \ ;$$
$$d_2) \ d(x, y) \ \leq \ d(x, z) + d(z, y), \ \text{for all } x, y, z \in X.$$

Then d is called a dislocated quasi-metric (dq-metric) on X A pair (X, d) is called dq-metric space (dq-metric space).

Definition 2.2 [3]. A sequence $\{x_n\}$ in a dq-metric space (X, d) is called Cauchy sequence if for all $\varepsilon > 0,$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ or $d(x_m, x_n) < \varepsilon$ for all $m, n \geq n_0.$

Definition 2.3 [3]. A sequence $\{x_n\}$ in a dq-metric space (X, d) is said to be dislocated quasi-convergent (dq-convergent) to x if $\lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} d(x, x_n) = 0.$ In this case, x is called a dq-limit of sequence $\{x_n\}$ and we write $x_n \to x$ as $n \to \infty.$

Dentition 1.4 [3]. A dq-metric space (X, d) is called complete if every Cauchy sequence in it is dq-convergent.

Definition 2.5[3]. Let (X,d) be a dq-metric space. A self-map $T : X \to X$ said to be a contraction

map if there exists a constant $k \in [0,1)$ such that

$$d(Tx,Ty) \le kd(x,y) \text{ for all } x, y \text{ in } (X,d).$$

Lemma 2.6[3]. Limit of a convergent sequence in a dq-metric space is unique.

In the following theorem, Zeyada et al. [3] generalized the Banach contraction principle in

dislocated quasi metric space.

Theorem 2.7[3]. Let (X,d) be a complete dq-metric space, $T : X \to X$ be a contraction, then T

has a unique fixed point in X.

Dentition 2.8[7]. A map $\varphi : [0,\infty) \to [0,\infty)$ is called comparison function if it satisfies:

      i) φ is monotonically increasing;

      ii)   The sequence $\left\{ \varphi^n (t)_{n=0}^{\infty} \right\}$ converges to zero for all $t \in [0,\infty)$.

If φ satisfies

      iii) $\sum_{k=0}^{\infty} \varphi^k (t)$ Converges for all $t \in [0,\infty)$, then φ is called (c) comparison function.

Every comparison function is (c) comparison.

Prototype example for comparison function is $\varphi(t) = \alpha t, t \in [0,\infty)$ and $\alpha \in [0,1)$.

Lemma1.9[7]. For every comparison function $\varphi : [0,\infty) \to [0,\infty)$

$\varphi(t) < t$ for all $t > 0$, and $\varphi(t) = 0$ if and only if $t = 0$.

Theorem 1.10[8]. Let (X,d) be a complete dislocated quasi b-metric space. Let $T_1 : X \to X$

and $T_2 : X \to X$ are continuous functions on X, for $k \ge 1$ satisfying

$$d(T_1 x, T_2 y) \le \varphi d(x,y) \text{ for all } x, y \in X.$$

YESHIMABET JIRA, KIDANE KOYAS, AYNALEM GIRMA

Where φ is a comparison function, then T_1 and T_2 have a unique common fixed point in X.

Theorem 2.11[16]. Let (X,d) be a complete dislocated quasi metric space. And $T : X \to X$ be self-mapping satisfying

$$d(Tx,Ty) \leq a\varphi\big(d(x,y)\big) + b\varphi\big(\max\{d(x,Tx),d(x,y)\}\big) +$$
$$c\varphi\left(\frac{d(x,y)\left[1+\sqrt{d(x,y)d(x,Tx)}\right]^2}{\big(1+d(x,y)\big)^2}\right);$$

for all $x, y \in X$ and $a,b,c \geq 0$ with $a+b+c < 1$, and φ is a comparison function. Then T has a unique fixed point.

We denote $C(T,f) = \{x \in X : Tx = fx\}$ and $F(T,f) = \{x \in X : Tx = fx = x\}$.

Where $C(T,f)$ is the set of coincidence point and $F(T,f)$ is the set of common fixed point of T and f.

In the sequel we need the following dentitions.

Definition 2.12 [4]. Two self-maps T and f on a nonempty set X are said to be commuting if $Tfx = fTx$ for all $x \in X$.

Definition 2.13[5]. Two self-mappings T and f of a metric space (X,d) are called compatible if $\lim_{n\to\infty} d\big(fTx_n, Tfx_n\big) = 0$ whenever $\{x_n\}_n^\infty$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Definition 2.14[6]. Two self- maps T and f of a metric space (X,d) are called weakly compatible if they commute at their coincidence points. That is, $fu = Tu$ for $u \in X$, then $Tfu = fTu$, for $u \in X$.

3. Main results

Theorem 3.1. Let (X,d) be a complete dislocated quasi metric spaces and $T, f : X \to X$ be self-maps satisfying the following conditions

i) $TX \subseteq fX$;

ii) T and f are weakly compatible and fX is closed subset of X;

iii) $d(Tx,Ty) \le a\varphi(d(fx, fy)) + b\varphi(\max\{d(fx, fy), d(fx,Tx)\}) +$

$$c\varphi\left(\frac{d(fx, fy)\left[1 + \sqrt{d(fx, fy)d(fx,Tx)}\right]^2}{(1 + d(fx, fy))^2} \right);$$

for all $x, y \in X$ and $a, b, c \ge 0$ with $a + b + c < 1$, and φ is a comparison function. Then T and f have a unique common fixed point if T and f commute at their coincidence points.

Proof

Let $x_0 \in X$, so that $y_0 = Tx_0 = fx_1$ and using condition (i) $Tx_1 \in fX$, then there exist $x_2 \in X$ such that $y_1 = Tx_1 = fx_2$. Continuing this process we construct a sequence $\{x_n\}$ and $\{y_n\}$ such that

$y_n = Tx_n = fx_{n+1}$ for $n \in \{0, 1, 2, \ldots\}$.

Now we consider two cases.

Case (i):

Suppose $y_n = y_{n+1}$ for some $n \in \{0, 1, 2, \ldots\}$, then $y_n = Tx_n = Tx_{n+1} = y_{n+1} = fx_{n+1}$.

So that x_{n+1} is coincidence point of T and f. Let $Tx_n = Tx_{n+1} = fx_{n+1} = u$, for some $u \in fX$.

Then by the weakly compatibility of T and f we obtain

(3.1) $\qquad\qquad Tu = Tfx_{n+1} = fTx_{n+1} = fu.$

YESHIMABET JIRA, KIDANE KOYAS, AYNALEM GIRMA

Therefore, $Tfu = fTu$ for $u \in X$. which shows that T and f commute at their coincidence point u.

Now we claim that $d(Tu,Tu) = 0.$

By using condition (iii) of Theorem 3.1

$$d(Tu,Tu) \le a\varphi\big(d\big(fu, fu\big)\big) + b\varphi\big(\max\{d\big(fu, fu\big), d(fu,Tu)\}\big) +$$
$$c\varphi\left(d\big(fu, fu\big)\frac{\big(1+\sqrt{d\big(fu, fu\big)d\big(fu,Tu\big)}\big)^2}{\big(1+\big(fu, fu\big)\big)^2}\right)$$
$$= a\varphi\big(d\big(Tu,Tu\big)\big) + b\varphi\big(\max\{d\big(Tu,Tu\big), d(Tu,Tu)\}\big) +$$
$$c\varphi\left(d\big(Tu,Tu\big)\frac{\big(1+\sqrt{d\big(Tu,Tu\big)d\big(Tu,Tu\big)}\big)^2}{\big(1+\big(Tu,Tu\big)\big)^2}\right);$$

Since $\varphi(t) \le t$ for all $t \ge 0,$ then we obtain

$$d(Tu,Tu) \le \ ad(Tu,Tu) + bd(Tu,Tu) + cd(Tu,Tu)$$
$$\le (a+b+c)d(Tu,Tu).$$

Since $0 \le a+b+c < 1,$ so the above inequality is possible if

(3.2) $$d(Tu,Tu) = 0.$$

We claim that $Tu = u.$

$$d(Tu,u) = d(Tu,Tx_{n+1})$$
$$\le a\varphi\big(d\big(fu, fx_{n+1}\big)\big) + b\varphi\big(max\{d\big(fu,Tu\big), d(fu, fx_{n+1})\}\big) +$$
$$c\varphi\left(d\big(fu, fx_{n+1}\big)\frac{\big(1+\sqrt{d\big(fu, fx_{n+1}\big)d\big(fu,Tu\big)}\big)^2}{\big(1+\big(fu, fx_{n+1}\big)\big)^2}\right)$$
$$= a\varphi\big(d\big(Tu,u\big)\big) + b\varphi\big(\max\{d\big(Tu,Tu\big), d(Tu,u)\}\big) +$$
$$c\varphi\left(d\big(Tu,u\big)\frac{\big(1+\sqrt{d\big(Tu,u\big)d\big(Tu,Tu\big)}\big)^2}{\big(1+\big(Tu,u\big)\big)^2}\right);$$

Since $d(Tu,Tu)=0,$ then $\max\{d(Tu,Tu),d(Tu,u)\}$ is $d(Tu,u)$ and

$$\left(d(Tu,u)\frac{\left(1+\sqrt{d(Tu,u)d(Tu,Tu)}\right)^2}{\left(1+(Tu,u)\right)^2} \right) \le d(Tu,u).$$

Thus,

$$d(Tu,u) \le a\varphi\big(d(Tu,u)\big)+b\varphi\big(d(Tu,u)\big)+c\varphi\big(d(Tu,u)\big).$$

Since $\varphi(t)\le t$ for all $t\ge 0,$ then we obtain

$$d(Tu,u) \le \ ad(Tu,u)+bd(Tu,u)+cd(Tu,u)$$
$$\le (a+b+c)d(Tu,u).$$

Since $0\le a+b+c<1,$ so the above inequality is possible if

(3.3) $\qquad\qquad d(Tu,u)=0.$

Similarly,

$$d(u,Tu)=d(Tx_{n+1},Tu)$$
$$\le a\varphi\big(d(fx_{n+1},fu)\big)+b\varphi\big(\max\big\{d(fx_{n+1},fu),d(fx_{n+1},Tx_{n+1})\big\}\big)+$$
$$c\varphi\left(d(fx_{n+1},fu)\frac{\left(1+\sqrt{dfx_{n+1},fu)d(fx_{n+1},Tx_{n+1})}\right)^2}{\left(1+d(fx_{n+1},fu)\right)^2} \right)$$
$$= a\varphi(d(u,Tu))+b\varphi(\max\{d(u,Tu)d(u,u)\}+$$
$$c\varphi\left(d(u,Tu)\frac{\left(1+\sqrt{d(u,Tu)d(u,u)}\right)^2}{\left(1+(u,Tu)\right)^2} \right);$$

Since $d(u,u)=0,$ then $\max\{d(u,Tu),d(u,u)\}$ is $d(u,Tu)$ and

$$d(u,Tu)\frac{\left(1+\sqrt{d(u,Tu)d(u,u)}\right)^2}{\left(1+(u,Tu)\right)^2} \le d(u,Tu).$$

Thus,

YESHIMABET JIRA, KIDANE KOYAS, AYNALEM GIRMA

$$d(u,Tu) \leq a\varphi\big(d(u,Tu)\big) + b\varphi\big(d(u,Tu)\big) + c\varphi\big(d(u,Tu)\big).$$

Since $\varphi(t) \leq t$ for all $t \geq 0$, then we obtain

$$d(u,Tu) \leq ad(u,Tu) + bd(u,Tu) + cd(u,Tu)$$
$$\leq (a+b+c)d(u,Tu).$$

Since $0 \leq a+b+c < 1$, so the above inequality is possible if

(3.4) $\qquad\qquad d(u,Tu) = 0.$

From (3.3) and (3.4), we obtain $Tu = u.$

By (3.1) $Tu = fu = u.$

Therefore, u is a common fixed point of T and f.

Now we prove the uniqueness of the common fixed point.

Suppose u and v are two distinct fixed points of T and f.

That means $Tu = fu = u$ and $Tv = fv = v.$

By using condition (iii) of Theorem 3.1

$$d(u,v) = d(Tu,Tv)$$
$$\leq a\varphi\big(d(fu, fv)\big) + b\varphi\big(max\{d(fu, fv), d(fu,Tu)\}\big) +$$
$$c\varphi\left(d(fu, fv)\frac{\big(1+\sqrt{d(fu, fv)d(fu,Tu)}\big)^2}{\big(1+(fu, fv)\big)^2} \right)$$
$$= a\varphi\big(d(u,v)\big) + b\varphi\big(\max\{d(u,v), d(u,u)\}\big) +$$
$$c\varphi\left(d(u,v)\frac{\big(1+\sqrt{d(u,v)d(u,u)}\big)^2}{\big(1+d(u,v)\big)^2} \right);$$

Since $d(u,u) = 0,$ then $\max\{d(u,v), d(u,u)\} = d(u,v)$ and

$$d(u,v)\frac{\left(1+\sqrt{d(u,v)d(u,u)}\right)^2}{\left(1+d(u,v)\right)^2} \leq d(u,v).$$

Thus,

$$d(u,v) \leq a\varphi\big(d(u,v)\big) + b\varphi\big(d(u,v)\big) + c\varphi\big(d(u,v)\big).$$

Since $\varphi(t) \leq t$ for all $t \geq 0,$ then we obtain

$$d(u,v) \leq ad(u,v) + bd(u,v) + cd(u,v)$$
$$\leq (a+b+c)d(u,v).$$

Since $0 \leq a+b+c < 1,$ so the above inequality is possible if

(3.5) $\qquad\qquad d(u,v) = 0.$

Similarly,

$$d(v,u) = d(Tv,Tu)$$
$$\leq a\varphi(d(fv, fu)) + b\varphi(\max\{d(fv, fu), d(fv,Tv)\}) +$$
$$c\varphi\left(d(fv, fu)\frac{\left(1+\sqrt{d(fv, fu)d(fv,Tv)}\right)^2}{\left(1+d(fv, fu)\right)^2} \right)$$
$$\leq a\varphi(d(v,u)) + b\varphi(\max\{d(v,u), d(v,v)\}) +$$
$$c\varphi\left(d(v,u)\frac{\left(1+\sqrt{d(v,u)d(v,v)}\right)^2}{\left(1+d(v,u)\right)^2} \right);$$

Since $d(v,v) = 0,$ then $\max\{d(v,u), d(v,v)\} = d(v,u)$ and

$$d(v,u)\frac{\left(1+\sqrt{d(v,u)d(v,v)}\right)^2}{\left(1+d(v,u)\right)^2} \leq d(v,u).$$

Thus,

$$d(v,u) \leq a\varphi\big(d(v,u)\big) + b\varphi\big(d(v,u)\big) + c\varphi\big(d(v,u)\big).$$

Since $\varphi(t) \leq t$ for all $t \geq 0$, then we obtain

$$d(v,u) \leq ad(v,u) + bd(v,u) + cd(v,u)$$
$$\leq (a+b+c)d(v,u).$$

Since $0 \leq a+b+c < 1$, so the above inequality is possible if

$$(3.6) \qquad d(v,u) = 0.$$

From (3.5) and (3.6), we get $u = v$.

Hence, u is a unique common fixed point of T and f.

Case (ii): Suppose $y_n \neq y_{n+1}$ for each $n \in \{0,1,2,...\}$.

$$
\begin{aligned}
d(y_n, y_{n+1}) &= d(Tx_n, Tx_{n+1}) \\
&\leq a\varphi\big(d(fx_n, fx_{n+1})\big) + b\varphi\big(\max\{d(fx_n, fx_{n+1}), d(fx_n, Tx_n)\}\big) + \\
&\quad c\varphi\left(d(fx_n, fx_{n+1}) \frac{\left(1 + \sqrt{d(fx_n, fx_{n+1})d(fx_n, Tx_n)}\right)^2}{\left(1 + d(fx_n, fx_{n+1})\right)^2}\right) \\
&= a\varphi(d(y_{n-1}, y_n)) + b\varphi(\max\{d(y_{n-1}, y_n), d(y_{n-1}, y_n)\}) + \\
&\quad c\varphi\left(d(y_{n-1}, y_n) \frac{\left(1 + \sqrt{d(y_{n-1}, y_n)d(y_{n-1}, y_n)}\right)^2}{\left(1 + d(y_{n-1}, y_n)\right)^2}\right);
\end{aligned}
$$

Since $\varphi(t) \leq t$ for all $t \geq 0$, then we obtain

$$d(y_n, y_{n+1}) \leq ad(y_{n-1}, y_n) + bd(y_{n-1}, y_n) + cd(y_{n-1}, y_n)$$
$$\leq (a+b+c)d(y_{n-1}, y_n);$$

Let $h = a+b+c$.

$$(3.7) \qquad d(y_n, y_{n+1}) \leq hd(y_{n-1}, y_n).$$

Since, $0 \le h < 1.$ we obtain

$$d\left(y_{n-1}, y_n\right) \le hd\left(y_{n-2}, y_{n-1}\right).$$

Similarly,

$$d(y_n, y_{n+1}) \le h^2 d(y_{n-2}, y_{n-1}).$$

Now, continuing this process we get

$$d(y_n, y_{n+1}) \le h^n d\left(y_0, y_1\right).$$

Since, $0 \le h < 1$ we have

$$\lim_{n \to \infty} h^n d\left(y_0, y_1\right) = 0.$$

Thus,

(3.8) $$\lim_{n \to \infty} d\left(y_n, y_{n+1}\right) = 0.$$

Similarly, we can show

(3.9) $$\lim_{n \to \infty} d\left(y_{n+1}, y_n\right) = 0.$$

Now we can show $\{y_n\}$ is a Cauchy sequence in X.

Let $m, n \in \mathbb{N}$ with $m > n.$

Applying triangular inequality

$$
\begin{aligned}
d(y_n, y_m) &\le d(y_n, y_{n+1}) + d\left(y_{n+1}, y_m\right) \\
&\le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + ... + d\left(y_{m-1}, y_m\right) \\
&\le h^n d(y_0, y_1) + h^{n+1} d\left(y_0, y_1\right) + ... + h^{m-1} d\left(y_0, y_1\right) \\
&\le h^n \left(1 + h + ... + h^{m-n-1}\right) d\left(y_0, y_1\right) \\
&= h^n \left(\sum_{i=0}^{m-n-1} h^i\right) d\left(y_0, y_1\right) \\
&\le h^n \left(\sum_{i=0}^{\infty} h^i\right) d\left(y_0, y_1\right) \\
&= \frac{h^n}{1-h} d\left(y_0, y_1\right);
\end{aligned}
$$

YESHIMABET JIRA, KIDANE KOYAS, AYNALEM GIRMA

Thus,

$$d(y_n, y_m) \le \frac{h^n}{1-h} d(y_0, y_1).$$

Since $0 \le h < 1$, then $\lim_{h \to 0} \frac{h^n}{1-h} d(y_0, y_1) = 0.$

This implies,

(3.10) $$\lim_{n,m \to \infty} d(y_n, y_m) = 0.$$

Let $m, n \in \mathbb{N}$ with $n > m$.

By similar line of proof we obtain.

(3.11) $$\lim_{n,m \to \infty} d(y_m, y_n) = 0.$$

Hence from (3.10) and (3.11) we have

$$\lim_{m,n \to \infty} d(y_n, y_m) = \lim_{m,n \to \infty} d(y_m, y_n) = 0.$$

Thus, the sequence $\{y_n\}$ is a Cauchy sequence in X for $n \in \{0, 1, 2, \ldots\}$.

Since, X is complete there exists $z \in X$, such that $\lim_{n \to \infty} y_n = z$.

Thus, $\lim_{n \to \infty} Tx_n = \lim_{n \to \infty} fx_{n+1} = z.$

Since fX is a closed subset of X, there exist $u \in fX$ such that $z = fu$

Now we show that $Tu = z$.

$$d(Tu, z) = d(Tu, Tx_n)$$
$$\le a\varphi(d(fu, fx_n)) + b\varphi(\max d(fu, fx_n), (fu, Tu)\})$$
$$+ c\varphi\left(d(fu, fx_n)\left[\frac{\left(1 + \sqrt{d(fu, fx_n)d(fu, Tu)}\right)^2}{\left(1 + d(fu, fx_n)\right)^2} \right] \right);$$

Since $Tu = fu$, then

$$d(Tu, z) \leq a\varphi(d(Tu, fx_n)) + b\varphi(\max\{d(Tu, fx_n), d(Tu, Tu)\}) +$$

$$c\varphi\left(d(Tu, fx_n)\left(\frac{\left(1 + \sqrt{d(Tu, fx_n)d(Tu, Tu)}\right)^2}{\left(1 + d(Tu, fx_n)\right)^2}\right)\right);$$

Letting $n \to \infty$, then $\lim_{n\to\infty} fx_n = z$.

$$d(Tu, z) \leq a\varphi(d(Tu, z)) + b\varphi(\max\{d(Tu, z), d(Tu, Tu)\}) +$$

$$c\varphi\left(d(Tu, z)\left(\frac{\left(1 + \sqrt{d(Tu, z)d(Tu, Tu)}\right)^2}{\left(1 + d(Tu, z)\right)^2}\right)\right).$$

Since by (3.2) $d(Tu, Tu) = 0$, then $\max\{d(Tu, z), d(Tu, Tu)\} = d(Tu, z)$ and

$$\left(d(Tu, z)\frac{\left(1 + \sqrt{d(Tu, z)d(Tu, Tu)}\right)^2}{\left(1 + d(Tu, z)\right)^2}\right) \leq d(Tu, z).$$

Thus,

$$d(Tu, z) \leq a\varphi(d(Tu, z)) + b\varphi(d(Tu, z)) + c\varphi(d(Tu, z)).$$

Since $\varphi(t) \leq t$ for all $t \geq 0$, then we obtain

$$d(Tu, z) \leq ad(Tu, z) + bd(Tu, z) + cd(Tu, z)$$
$$\leq (a + b + c)d(Tu, z).$$

Since $0 \leq a + b + c < 1$, so the above inequality is possible if

(3.12) $\qquad\qquad d(Tu, z) = 0.$

Similarly,

$$d(z,Tu) = d(Tx_n,Tu)$$

$$\leq a\varphi(d(fx_{n,}fu)) + b\varphi(\max\{d(fx_{n,}fu), d(fx_{n,}Tx_n)\}) +$$

$$+ c\varphi\left(d(fx_n,fu)\frac{\left(1+\sqrt{d(fx_{n,}fu)d(fx_{n,}Tx_n)}\right)^2}{\left(1+d(fx_n,fu)\right)^2}\right);$$

Since $Tu = fu$, then

$$d(z,Tu) \leq a\varphi(d(fx_n,Tu)) + b\varphi(\max\{d(fx_n,Tu), d(fx_n,Tx_n)\}) +$$

$$+ c\varphi\left(d(fx_n,Tu)\frac{\left(1+\sqrt{d(fx_n,Tu)d(fx_n,Tx_n)}\right)^2}{\left(1+d(fx_n,Tu)\right)^2}\right);$$

Letting $n \to \infty$, then $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} fx_n = z.$

$$d(z,Tu) \leq a\varphi(d(z,Tu)) + b\varphi(\max\{d(z,Tu), d(z,z)\}) +$$

$$+ c\varphi\left(d(z,Tu)\frac{\left(1+\sqrt{d(z,Tu)d(z,z)}\right)^2}{\left(1+d(z,Tu)\right)^2}\right);$$

Since $d(z,z) = 0,$ then $\max\{d(z,Tu), d(z,z)\}$ is $d(z,Tu)$ and

$$\left(d(z,Tu)\frac{\left(1+\sqrt{d(z,Tu)d(z,z)}\right)^2}{\left(1+d(z,Tu)\right)^2}\right) \leq d(z,Tu).$$

Thus,

$$d(z,Tu) \leq a\varphi(d(z,Tu)) + b\varphi(d(z,Tu)) + c\varphi(d(z,Tu)).$$

Since $\varphi(t) \leq t$ for all $t \geq 0,$ then we obtain

$$d(z,Tu) \leq ad(z,Tu) + bd(z,Tu) + cd(z,Tu)$$
$$\leq (a+b+c)d(z,Tu).$$

Since $0 \le a+b+c < 1,$ so the above inequality is possible if

(3.13) $\qquad\qquad d\left(z,Tu\right)=0.$

Using (3.12) and (3.13) we have $z=Tu.$

Then, $z=Tu=fu.$

By the weakly compatibility of T and f, we have $Tfu = fTu.$

Then $Tz = Tfu = fTu = fz.$

Which implies that z is a coincidence point of T and f.

Consider

$$d\left(Tz,z\right)=d(Tz,Tu)$$
$$\le a\varphi(d\left(fz,fu\right))+b\varphi(\max\left\{d\left(fz,fu\right),d\left(fz,Tz\right)\right\})+$$
$$c\varphi\left(d\left(fz,fu\right)\left(\frac{\left(1+\sqrt{d\left(fz,fu\right)d(fz,Tz)}\right)^2}{\left(1+d\left(fz,fu\right)\right)^2}\right)\right)$$
$$= a\varphi(d\left(Tz,z\right))+b\varphi(\max\left\{d\left(Tz,z\right),d\left(Tz,Tz\right)\right\})+$$
$$c\varphi\left(d\left(Tz,z\right)\left(\frac{\left(1+\sqrt{d\left(Tz,z\right)d(Tz,Tz)}\right)^2}{\left(1+d\left(Tz,z\right)\right)^2}\right)\right).$$

Since $\varphi(t)\le t,$ $\max\{d\left(Tz,z\right),d\left(Tz,Tz\right)\}$ is $d\left(Tz,z\right)$ and

$$d\left(Tz,z\right)\left(\frac{\left(1+\sqrt{d\left(Tz,z\right)d(Tz,Tz)}\right)^2}{\left(1+d\left(Tz,z\right)\right)^2}\right)\le d\left(Tz,z\right).$$

We get

$$d\left(Tz,z\right)\le\left(a+b+c\right)d\left(Tz,z\right).$$

Since $0 \leq a+b+c < 1$, so the above inequality is possible if

$$(3.14) \qquad d(Tz, z) = 0.$$

Similarly,

$$d(z,Tz) = d(Tu,Tz)$$
$$\leq a\varphi(d(fu, fz)) + b\varphi(\max\{d(fu, fz), d(fu,Tu)\}) +$$
$$c\varphi\left(d(fu, fz)\left(\frac{\left(1+\sqrt{d(fu, fz)d(fu,Tu))}\right)^2}{\left(1+d(fu, fz)\right)^2}\right)\right)$$
$$= a\varphi(d(z,Tz)) + b\varphi(\max\{d(z,Tz), d(z, z)\}) +$$
$$c\varphi\left(d(z,Tz)\left(\frac{\left(1+\sqrt{d(z,Tz)d(z, z))}\right)^2}{\left(1+d(z,Tz)\right)^2}\right)\right);$$

Since, $\max\{d(z,Tz)d(z, z)\}$ is $d(z,Tz)$ and $\varphi(t) \leq t$.

Then we have

$$d(z,Tz) \leq ad(z,Tz) + bd(z,Tz) + cd(z,Tz)$$
$$\leq (a+b+c)d(z,Tz).$$

Since $0 \leq a+b+c < 1$, so the above inequality is possible if

$$(3.15) \qquad d(z,Tz) = 0.$$

So, from (3.14) and (3.15) we have $Tz = z$.

Thus, $Tz = fz = z$.

Therefore, z is a common fixed point of T and f.

Now we show the uniqueness of common fixed point of T and f in X.

Let w be another common fixed point of T and f in X. That is $Tw = fw = w$.

Consider

$$d(z,w) = d(Tz,Tw)$$
$$\leq a\varphi(d(fz, fw)) + b\varphi(\max\{d(fz, fw), d(fz,Tz)\}) +$$
$$c\varphi\left[d(fz, fw)\left(d(fz, fw)\frac{\left(1+\sqrt{d(fz, fw)d(fz,Tz)}\right)^2}{\left(1+d(fz, fw)\right)^2}\right)\right]$$
$$= a\varphi(d(z, w)) + b\varphi(\max\{d(z, w), d(z, z)\}) +$$
$$c\varphi\left[d(z,w)\left(d(z,w)\frac{\left(1+\sqrt{d(z,w)d(z,z)}\right)^2}{\left(1+d(z,w)\right)^2}\right)\right];$$

Since $max\{d(z,w), d(z,z)\}$ is $d(z,w)$ and $\varphi(t) \leq t$, then the above inequality becomes

$$d(z,w) \leq (a+b+c)d(z,w).$$

Since $0 \leq a+b+c < 1$, so the above inequality is possible if

(3.16) $\qquad d(z,w) = 0.$

Using similar line of proof

$$d(w,z) = d(Tw,Tz)$$
$$\leq a\varphi(d(fw, fz)) + b\varphi(\max\{d(fw, fz), d(fw,Tw)\}) +$$
$$c\varphi\left[d(fw, fz)\left(\frac{\left(1+\sqrt{d(fw, fz)d(fw,Tw)}\right)^2}{\left(1+d(fw, fz)\right)^2}\right)\right]$$
$$= a\varphi(d(w,z)) + b\varphi(\max\{d(w,z), d(w,w)\}) +$$
$$c\varphi\left[d(w,z)\left(\frac{\left(1+\sqrt{d(w,z)d(w,w)}\right)^2}{\left(1+d(w,z)\right)^2}\right)\right]$$

Since $\varphi(t) \leq t$ and $max\{d(w,z), d(w,w)\}$ is $d(w,z)$, then the above inequality becomes

YESHIMABET JIRA, KIDANE KOYAS, AYNALEM GIRMA

$$d(w,z) \leq ad(w,z) + bd(w,z) + cd(w,z)$$
$$\leq (a+b+c)d(w,z).$$

Since $0 \leq a+b+c < 1$, so the above inequality is possible if

(3.17) $\qquad\qquad d(w,z) = 0.$

From (3.16) and (3.17), we conclude that $w = z$.

So, z is a unique common fixed point of T and f in X.

The following is an example in support of Theorem 3.1.

Example 4.2: Let $X = [0,4]$, define $d : X \times X \to [0,\infty)$ by

$d(x,y) = x^2 + y$ and $T, f : X \to X$ by

$$T(x) = \begin{cases} 0, & if\ 0 \leq x \leq 3 \\ \dfrac{1}{4}, & if\ 3 < x \leq 4 \end{cases} \text{and } f(x) = \begin{cases} \dfrac{x}{3}, & if\ 0 \leq x \leq 3 \\ 4, & if\ 3 < x \leq 4 \end{cases}$$

If $a = \dfrac{1}{2}, b = \dfrac{1}{5}, c = \dfrac{1}{5}$ and $\varphi(t) = \dfrac{3}{4}t$, for all $t \geq 0$.

In fact both properties of Definition 2.2 holds true.

i)$d(x,y) = x^2 + y = y^2 + x = d(y,x) = 0$ implies $x = y = 0$.

ii)$d(x,y) = x^2 + y \leq x^2 + z + z^2 + y = d(x,z) + d(z,y)$.

We observe that (X,d) is a dq-metric space and also $T0 = f0 = 0$.

This implies 0 is a coincidence point of T and f. Moreover $fX = \{4\} \cup [0,1]$ and $TX = \left\{0, \dfrac{1}{4}\right\}$.

Hence, $TX \subseteq fX$ and also $T0 = Tf0 = fT0 = f0 = 0$, then f and T are weakly compatible.

To see contractive condition (iii) of Theorem 3.1 we consider the following cases.

<u>Case</u> (i): Suppose $x, y \in [0,3]$. Then,

$$d(Tx,Ty) = d(0,0) = 0.$$

$$d(fx, fy) = d\left(\frac{x}{3}, \frac{y}{3}\right) = \left(\frac{x}{3}\right)^2 + \frac{y}{3} = \frac{x^2}{9} + \frac{y}{3} = \frac{x^2 + 3y}{9}.$$

$$d(fx, Tx) = d\left(\frac{x}{3}, 0\right) = \frac{x^2}{9}.$$

Then by contractive condition (iii) of Theorem 3.1 we have

$$d(Tx,Ty) \le a\varphi(d(fx, fy)) + b\varphi(\max\{d(fx, fy), d(fx,Tx)\})$$

$$+ c\varphi\left[d(fx, fy) \frac{\left(1 + \sqrt{d(fx, fy)d(fx,Tx)}\right)^2}{\left(1 + d(fx, fy)\right)^2} \right].$$

This implies $0 \le \dfrac{3}{8}\left(\dfrac{x^2 + 3y}{9}\right) + \dfrac{3}{20}\left(\max\left\{\dfrac{x^2 + 3y}{9}, \dfrac{x^2}{9}\right\}\right) + \dfrac{3}{20}\left(\dfrac{x^2 + 3y}{9}\right).$

Always true for all $x, y \in [0,3)$.

<u>Case</u> (ii): Suppose $x, y \in (3,4]$. In this case,

$$d(Tx,Ty) = d\left(\frac{1}{4}, \frac{1}{4}\right) = \left(\frac{1}{4}\right)^2 + \frac{1}{4} = \frac{5}{16}$$

$$d(fx, fy) = d(4,4) = (4)^2 + 4 = 20$$

$$d(fx,Tx) = d\left(4, \frac{1}{4}\right) = \frac{65}{4}$$

$$d\left(Tx,Ty\right) \le a\varphi(d\left(fx,fy\right)) + b\varphi(\max\left\{d\left(fx,fy\right),d\left(fx,Tx\right)\right\}) +$$

$$c\varphi\left(d\left(fx,fy\right)\frac{\left(1+\sqrt{d\left(fx,fy\right)d\left(fx,Tx\right)}\right)^2}{\left(1+d\left(fx,fy\right)\right)^2}\right).$$

$$\frac{5}{16} \le \frac{3}{8}(20) + \frac{3}{20}\left(\max\left\{20,\frac{65}{4}\right\}\right) + \frac{3}{20}\left(\left(\frac{65}{4}\right)\frac{\left(1+\sqrt{20\left(\frac{65}{4}\right)}\right)^2}{\left(1+\frac{65}{4}\right)^2}\right). \quad \text{True for all } x, y \in (3,4].$$

Case (iii): If $x \in [0,3]$ and $y \in (3,4]$, then

$$d\left(Tx,Ty\right) = d\left(0,\frac{1}{4}\right) = \frac{1}{4}.$$

$$d\left(fx,fy\right) = d\left(\frac{x}{3},4\right) = \left(\frac{x}{3}\right)^2 + 4 = \frac{x^2}{9} + 4 = \frac{x^2+36}{9}.$$

$$d\left(fx,Tx\right) = d\left(\frac{x}{3},0\right) = \left(\frac{x}{3}\right)^2 = \frac{x^2}{9}.$$

$$d\left(Tx,Ty\right) \le a\varphi\left(d\left(fx,fy\right)\right) + b\varphi(\max\left\{d\left(fx,fy\right),d\left(fx,Tx\right)\right\}) +$$

$$c\varphi\left(d\left(fx,fy\right)\frac{\left(1+\sqrt{d\left(fx,fy\right)d\left(fx,Tx\right)}\right)^2}{\left(1+d\left(fx,fy\right)\right)^2}\right).$$

$$\frac{1}{4} \le \frac{3}{8}(\frac{x^2+36}{9}) + \frac{3}{20}\left(\frac{x^2+36}{9}\right) + \frac{3}{20}\left(\left(\frac{x^2+36}{9}\right)\frac{\left(1+\sqrt{\left(\frac{x^2}{9}\right)\frac{x^2+36}{9}}\right)^2}{\left(1+\frac{x^2+36}{9}\right)^2}\right).$$

True for all $x \in [0,3]$ and $y \in (3,4]$.

Case (iv): Suppose $x \in (3,4]$ and $y \in [0,3]$, then we have

$$d(Tx,Ty) = d\left(\frac{1}{4},0\right) = \frac{1}{16}.$$

$$d(fx, fy) = d\left(4, \frac{y}{3}\right) = 16 + \frac{y}{3}.$$

$$d(fx,Tx) = d\left(4, \frac{1}{4}\right) = 4^2 + \frac{1}{4} = \frac{65}{4}.$$

$$d(Tx,Ty) \le a(\varphi d(fx, fy)) + b\varphi(\max\{d(fx, fy), d(fx,Tx)\}) +$$

$$c\varphi\left[d(fx, fy)\frac{\left(1 + \sqrt{d(fx, fy)d(fx,Tx)}\right)^2}{\left(1 + d(fx, fy)\right)^2}\right].$$

$$\frac{1}{16} \le \frac{3}{8}(16 + \frac{1}{3}y) + \frac{3}{20}\left(16 + \frac{1}{3}y\right) + \frac{3}{20}\left[\left(16 + \frac{1}{3}y\right)\frac{\left(1 + \sqrt{\left(16 + \frac{1}{3}y\right)\frac{65}{4}}\right)^2}{\left(1 + 16 + \frac{1}{3}y\right)^2}\right].$$

True for all $x \in (3,4]$ and $y \in [0,3]$.

From cases $(i) - (iv)$ all the conditions of Theorem 3.1 are satisfied and 0 is the unique common fixed point of f and T.

Theorem 3.2. Let (X,d) be a complete dq-metric space. $T, f : X \to X$ be continuous self-mappings satisfying the contractive condition of Theorem 3.1. Then f and T have a unique common fixed point.

Proof

YESHIMABET JIRA, KIDANE KOYAS, AYNALEM GIRMA

Following as in the proof of Theorem 3.1 we construct a sequence $\{y_n\}$. Let $\{x_{2n}\}$ and $\{x_{2n+1}\}$ be subsequences of the sequence $\{y_n\}$. As in the Theorem 3.1 we define $x_{2n+1} = Tx_{2n}$ and

$x_{2n} = fx_{2n-1}$.

Similarly, as shown in the proof of Theorem 3.1 we can show that the sequence $\{y_n\}$ is a Cauchy sequence.

By the completeness of X one can show that $\lim_{n\to\infty} y_n = u$ for some $u \in X$.

Since $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are subsequences of $\{y_n\}$, then $\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n+1} = u$.

Next since, f and T are continuous we arrive at

$$Tu = T\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} x_{2n+1} = u.$$

Then

(3.18) $\qquad\qquad Tu = u.$

Similarly,

$$fu = f\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} fx_{2n-1} = \lim_{n\to\infty} x_{2n} = u.$$

Then

(3.19) $\qquad\qquad fu = u.$

So, from (3.18) and (3.19) we get $Tu = u = fu$.

Therefore u is common fixed point of f and T.

Now we want to show uniqueness of the common fixed point.

Let q be another common fixed point of f and T. That is $fq = Tq = q$.

Then by the contractive condition (iii) of Theorem 3.1 we have

$$d(u,q) = d(Tu,Tq)$$

$$\leq a\varphi(d(fu, fq)) + b\varphi(\max\{d(fu, fq), d(fu,Tu)\}) +$$

$$c\varphi\left[d(fu, fq)\frac{\left(1+\sqrt{d(fu, fq)d(fu,Tu)}\right)^2}{\left(1+d(fu, fq)\right)^2}\right]$$

$$\leq a\varphi(d(u,q)) + b\varphi(\max\{d(u,q), d(u,u)\}) +$$

$$c\varphi\left[d(u,q)\frac{\left(1+\sqrt{d(u,q)d(u,u)}\right)^2}{\left(1+d(u,q)\right)^2}\right].$$

Since, $\max\{d(u,q), d(u,u)\}$ is $d(u,q)$ and $\varphi(t) \leq t$ for all $t \geq 0$.

Then the above inequality becomes

$$d(u,q) \leq ad(u,q) + bd(u,q) + cd(u,q)$$
$$\leq (a+b+c)d(u,q).$$

Since $0 \leq a+b+c < 1$, so the above inequality is possible if

(3.20) $\qquad d(u,q) = 0.$

Similarly,

$$d(q,u) = d(Tq,Tu)$$

$$\leq a\varphi(d(fq, fu)) + b\varphi(\max\{d(fq, fu), d(fq,Tq)\} +$$

$$c\varphi\left[d(fq, fu)\frac{\left(1+\sqrt{d(fq, fu)d(fq,Tq)}\right)^2}{\left(1+d(fq, fu)\right)^2}\right]$$

$$\leq a\varphi(d(q,u)) + b\varphi(\max\{d(q,u), d(q,q)\}) +$$

$$c\varphi\left[d(q,u)\frac{\left(1+\sqrt{d(q,u)d(q,q)}\right)^2}{\left(1+d(q,u)\right)^2}\right].$$

Since, $\max\{d(q,u), d(q,q)\}$ is $d(q,u)$ and $\varphi(t) \leq t$ for all $t \geq 0$.

Then the above inequality becomes

$$(q,u) \le ad(q,u) + bd(q,u) + cd(q,u)$$
$$\le (a+b+c)d(q,u).$$

Since $0 \le a+b+c < 1,$ so the above inequality is possible if

(3.21) $\qquad\qquad (q,u) = 0.$

So, from (3.20) and (3.21) we obtain $u = q.$

Thus u is a unique common fixed point of f and $T.$

The following is an example in support of Theorem 3.2.

Example 2.2: Let $X = [0,1]$ define $d : X \times X \to [0,\infty)$ by

$d(x, y) = x$ for all $x, y \in X.$

And also define $f, T : X \to X$ by

$$Tx = \frac{x}{9}, \; fx = \frac{8x}{9}, \; \text{and} \; \varphi(t) = \frac{4}{5}t.$$

Let $a = \frac{3}{4}, b = \frac{1}{6}$ and $c = \frac{1}{16}.$

$$d(Tx,Ty) = d(\frac{x}{9}, \frac{y}{9}) = \frac{x}{9}.$$

$$d(fx, fy) = d\left(\frac{8x}{9}, \frac{8y}{9}\right) = \frac{8x}{9}.$$

$$d(fx,Tx) = d(\frac{8x}{9}, \frac{8x}{9}) = \frac{8x}{9}.$$

From the contractive condition (iii) of the Theorem 3.1 we have

$$d\left(Tx,Ty\right) \le a\varphi(d\left(fx,fy\right)) + b\varphi(\max\left\{d\left(fx,fy\right),d\left(fx,Tx\right)\right\}) +$$

$$c\varphi\left[d\left(fx,fy\right)\frac{\left(1+\sqrt{d\left(fx,fy\right)d\left(fx,Tx\right)}\right)^2}{\left(1+d\left(fx,fy\right)\right)^2}\right];$$

$$\frac{x}{9} \le \frac{12}{20}\left(\frac{8x}{9}\right) + \frac{4}{30}\left(\max\{\frac{8x}{9},\frac{8x}{9}\}\right) + \frac{4}{80}\left[\left(\frac{8x}{9}\right)\frac{\left(1+\sqrt{(\frac{8x}{9})(\frac{8x}{9})}\right)^2}{\left(1+\frac{8x}{9}\right)^2}\right].$$

$$\frac{x}{9} \le \frac{24x}{45} + \frac{16x}{135} + \frac{2x}{45}.$$

Which, is true for all $x \in [0,1]$.

f and T satisfies all the conditions of the Theorem 3.2 and f and T have a unique fixed point.

$T0 = f0 = 0.$

Therefore 0 is the unique common fixed point of f and T.

Remark 1: For $f = I$ (I identity map on X) from contractive condition of Theorem 3.1 we get

$$d\left(Tx,Ty\right) \le a\varphi\left(d\left(x,y\right)\right) + b\varphi\left(\max\left\{d\left(x,y\right),d\left(x,Tx\right)\right\}\right) +$$

$$c\varphi\left[d\left(x,y\right)\frac{\left(1+\sqrt{d\left(x,y\right)d\left(x,Tx\right)}\right)^2}{\left(1+d\left(x,y\right)\right)^2}\right].$$

Whenever, $f = I$ contractive condition of Theorem 3.1 is simplified to Theorem 2.11.

Hence, Theorem 2.11 follows as a corollary to Theorem 3.1.

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgement

YESHIMABET JIRA, KIDANE KOYAS, AYNALEM GIRMA

REFERENCES

[1] I.Y. Alber and S. Guerre-Delabriere, Principle of weakly contractive maps in Hilbert space, New Results in Operator, in: I. Gohberg and Y. Lyubich, (Eds.): Advances and Applications, Birkhauser, Basel, Switzerland 98 (1997), 7-22.

[2] Al-Thagafi, N. Shazad, Generalized I-non expansive self-maps and invariant approximations, Acta. Math. Sin. (Engl. Ser.)24 (2008), 867-876.

[3] F. M. Zeyada, G. H. Hassan, M. A. Ahmed, A generalization of a fixed point theorem due to Hitzler and Seda in dislocated quasi-metric spaces, Arab. J. Sci. Eng. Sect. A Sci. 31 (2006), 111–114.

[4] G. Jungck, Commuting mappings and fixed points, Amer. Math. Monthly 83 (1976), 261-263.

[5] G. Jungck, Compatible mappings and common fixed points, Internat. J. Math. Math. Sci. 9 (1986), 771-779.

[6] G. Jungck and B. E. Rhoades, Fixed point for set valued functions without continuity, Indian J. Pure Appl. Math. 29(3)(1998), 227-238.

[7] V.Berinde,on the approximation of fixed points of weak contractive mappings,Carpathian Journal of Math.19 (1)(2003), 7-22.

[8] S.K. Tiwariand P.Vishuw, Dislocated quasi b - metric spaces and new common fixed point results, Dr.C.V. Raman University, Kota Bilaspur 7(3) (2017), 60 - 64.

[9] S. Reich, Some fixed point problems, Atti. Accad. Nz. Lincei. (1974), 51-57.

[10] B. E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Anal., Theory Method Appl.47 (4) (2001), 2683-2693.

[11] D. W. Boyd and J. S. W. Wong, on nonlinear Contractions, Proc. Amer. Math. Soc. 20(2) (1969), 458-464.

[12] S.Sessa, on a weakly commutativity condition in a fixed point considerations, Publ. Inst. Math. 32(46) (1986), 149-153.

[13] K. P. R. Sastry, I. R. S. Krishna Murthy, Common fixed points of two partially commuting tangential self-maps on a metric space, J. Math. Anal. Appl.250 (2000), 731-734.

[14] Y. Song, Coincidence Points for None commuting f -Weakly Contractive Mappings, Int. J. Comput. Appl. Math. 2(1) (2007), 51-57.

[15] Q. Zhang, Y. Song, Fixed point theory for generalized j -Weak Contractions, Appl. Math. Lett. 22 (2009), 75-78.

[16] M.U. Rahman, M. Sarwar, Fixed point results in dislocated quasi metric spaces, Pal. Journal of Math. 5(2016), 171-176.

[17] B.K.Dass, S.Gupta, An extension of Banach Contraction principles through rational expression, Indian J. Pure Appl. Math. 6(1975) ,1455-1458.

[18] C.T.Aage, J. N.Salunke, The results on fixed points in dislocated and dislocated quasi-metric space, Appl. Math. Sci.2 (59) (2008), 2941-2948.

[19] P.Hitzler, A.K.Seda, Dislocated topologies, J. Electr.Engin. 51(12) (2000), 3-7.