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REMARKS ON COMMON FIXED POINT RESULTS FOR GENERALIZED α_*-ψ-CONTRACTION MULTIVALUED MAPPINGS IN b-METRIC SPACES

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Abstract. In this paper, we generalize, improve and complement several fixed point results for generalized α_*-ψ-contraction multivalued mappings in b-metric spaces. We provide some non-standard proof techniques which give shorter proofs of the obtained results.

Keywords: b-metric space; b-complete; b-Cauchy; b-continuous; Picard sequence.

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1. Introduction

For a mapping $f : X \to X$, $X \neq \emptyset$, a point $u \in X$ is called a fixed point if $f(u) = u$. The set of all fixed points of f is denoted with $F(f)$.

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Frechet introduced the concept of metric space in 1905, but the most basic fixed point theorem in analysis, known as the Banach Contraction Principle, was first stated and proved by Banach for the contraction maps in the setting of complete normed linear spaces. This remarkable theorem Banach established in his Ph.D. thesis (1920, published in 1922). Banach Contraction Principle is one of the most important results of analysis and considered as the main source of metric fixed point theory.

Theorem 1.1. [5] *Let (X,d) be a complete metric space and let a mapping $f : X \to X$ be a contraction, i.e. there exists a fixed constant $q \in [0,1)$ such that $d(f(x), f(y)) \leq qd(x,y)$, for all $x,y \in X$. Then f has a unique fixed point.*

There is vast amount of extensions of this important theorem. On the one side, the usual contractive condition is replaced by a weakly contractive condition (see for instance [11], [13], [22]), while, on the other side, the action space is replaced by some generalization of standard metric space ([4], [6], [10], [20]).

In recent years, various distances are introduced, and relations between these distances are established. Some significant generalizations are the following.

$$
\begin{array}{ccc}
\text{metric space} & \longrightarrow & b\text{-metric space} \\
\downarrow & & \downarrow \\
\text{rectangular metric space} & \longrightarrow & \text{rectangular } b\text{-metric space} \\
 & & \downarrow \\
 & & b_v(s)\text{-metric space}
\end{array}
$$

In [20], the notion of $b_v(s)$-metric space was introduced and some fixed point theorems for single-valued mappings in $b_v(s)$-metric spaces were proved.

This concept of b-metric spaces was independently introduced by Baktin (1989) and S. Czerwik (1993) replacing the triangle inequality with the next, more general, condition.

There is a nonnegative number $s \geq 1$ such that $d(x,z) \leq s(d(x,y) + d(y,z))$ holds for all $x,y,z \in X$.

The concept of a b-metric space is more general than that of a metric space, because each metric space is a b-metric space, but the contrary is not true ([1]-[4], [6]-[9], [10], [12], [13], [15]-[20], [23]-[26]).

It is worth noting that there is a significant difference between b-metric spaces and standard metric spaces. If $s > 1$, the triangle inequality is not satisfied. Also, b-metric is not continuous in general. Furthermore, the open (closed) balls generated by b-metric are not necessarily open (closed) sets. For further information, we refer the interested readers to the reference list (especially [13]).

There is vast amount of literature dealing with b-metric spaces with the coefficient $s \geq 1$. Some obtained results generalize those from metric space. However, there are some results where the cases $s > 1$ and $s = 1$ should be considered separately. Because of this, one recent work will be the main topic of this paper.

2. Preliminaries

We repeat some definitions and results, which will be needed in the sequel.

Definition 2.1. ([4],[6]) *Let X be a (non-empty) set and $s \geq 1$ a given real number. A function $d : X \times X \to [0,\infty)$ is said to be a b-metric on X if the following conditions are satisfied:*

(b_1) $d(x,y) = 0$ *if and only if* $x = y$;

(b_2) $d(x,y) = d(y,x)$ *for all* $x,y \in X$;

(b_3) $d(x,z) \leq s(d(x,y) + d(y,z))$ *for all* $x,y,z \in X$.

The triplet (X,d,s) is called a b-metric space with the coefficient s.

For more notions such as b-convergence, b-completeness, b-Cauchy sequence in the framework of b-metric spaces, the reader is referred to [1]-[3], [7], [9], [10], [12], [15]-[20], [23]-[26].

The following result is well known and important in the setting of b-metric spaces.

Theorem 2.1. [6] *Let (X,d,s) be a b-complete b-metric space and $T : X \to X$ a mapping satisfying $d(Tx,Ty) \leq \varphi(d(x,y))$, $x,y \in X$, where $\varphi : [0,\infty) \to [0,\infty)$ is an increasing function such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for each fixed $t > 0$. Then T has exactly one fixed point z and $\lim_{n\to\infty} d(T^n x,z) = 0$ for each $x \in X$.*

Very recently, R. Miculescu and A. Mihail [19, Lemma 2.2] proved the next result.

Lemma 2.1. [19] *Let (X,d,s) be a b-metric space and $\{x_n\}$ a sequence in X. If there exists $\gamma \in [0,1)$ such that $d(x_{n+1},x_n) \leq \gamma d(x_n,x_{n-1})$ for all $n \in \mathbb{N}$, then $\{x_n\}$ is a b-Cauchy sequence.*

Remark 2.1. *In the several papers based on b-metric concept, the authors assume that* $\gamma \in [0, \frac{1}{s})$ *instead of* $\gamma \in [0,1)$, *which is obviously stronger condition. Under this stronger condition they show that the corresponding Picard sequence,* $\{x_n = Tx_{n-1}\}_{n\in\mathbb{N}}$ *with the initial point* $x_0 \in X$, *is a b-Cauchy. To prove this they use the following inequality:*

$$d(x_m,x_n) \leq sd(x_m,x_{m+1}) + s^2 d(x_{m+1},x_{m+2}) + \cdots + s^{n-m-1} d(x_{n-2},x_{n-1}) + s^{n-m-1} d(x_{n-1},x_n),$$

where $n,m \in \mathbb{N}$ *and* $n > m$.

The next lemma play an important role in many papers in the context of b-metric spaces. In [2], this lemma is also an essential tool for proving that the defined sequence $\{x_n\}$ is a b-Cauchy sequence.

Lemma 2.2. [1] *Let* (X,d) *be a b-metric space with* $s \geq 1$, *and suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are b-convergent with the limits x and y, respectively. Then we have*

$$\frac{1}{s^2} d(x,y) \leq \liminf_{n\to\infty} d(x_n,y_n) \leq \limsup_{n\to\infty} d(x_n,y_n) \leq s^2 d(x,y).$$

In particular, if $x = y$, *then we have* $\lim_{n\to\infty} d(x_n,y_n) = 0$. *Moreover, for each* $z \in X$, *we have*

$$\frac{1}{s} d(x,z) \leq \liminf_{n\to\infty} d(x_n,z) \leq \limsup_{n\to\infty} d(x_n,z) \leq s\, d(x,z).$$

Definition 2.2. *Let* $T : X \to X$ *be a mapping and* $\alpha : X \times X \to [0,\infty)$ *a function. The mapping* T *is said to be triangular* α*-admissible if the following conditions are satisfied:*

(T_1) T *is* α*-admissible;*

(T_2) $\alpha(x,u) \geq 1$ *and* $\alpha(u,y) \geq 1$ *imply* $\alpha(x,y) \geq 1$.

Definition 2.3. *Let* $T : X \to X$ *be a mapping and* $\alpha : X \times X \to [0,\infty)$ *a function. The mapiing* T *is said to be* α*-orbital admissible if*

(T_3) $\alpha(x,Tx) \geq 1$ *implies* $\alpha\left(Tx,T^2x\right) \geq 1$.

Definition 2.4. *Let* $T : X \to X$ *be a mapping and* $\alpha : X \times X \to [0,\infty)$ *a function. The mapping* T *is said to be triangular* α*-orbital admissible if* T *is* α*-orbital admissible an*

(T_4) $\alpha(x,y) \geq 1$ *and* $\alpha(y,Ty) \geq 1$ *imply* $\alpha(x,Ty) \geq 1$.

Let (X,d) be a b-metric space. We will denote by $CB(X)$ the family of all bounded and closed subsets of X. For $x\in X$ and $A,B\in CB(X)$, we define

$$D(x,A)=\inf_{a\in A}d(x,a)\quad\text{ and }\quad D(A,B)=\sup_{a\in A}D(a,B).$$

The mapping $H:CB(X)\times CB(X)\to[0,\infty)$ defined by

$$H(A,B)=\max\left\{\sup_{x\in A}D(x,B),\sup_{y\in B}D(y,A)\right\},\quad A,B\in CB(X),$$

is called a Hausdorff-Pompeiu b-metric induced by the b-metric space (X,d).

For the convenience of the reader, we now repeat some well known results in the context of b-metric spaces, thus making our exposition self-contained (see [2] and references therein).

Lemma 2.3. *Let (X,d) be a b-metric space. The following properties are satisfied.*

1) $D(x,B)\le d(x,b)$ *for all* $x\in X$, $b\in B$ *and* $B\in CB(X)$.
2) $D(x,B)\le H(A,B)$ *for all* $x\in X$ *and* $A,B\in CB(X)$.
3) $D(x,A)\le s(d(x,y)+D(y,B))$ *for all* $x,y\in X$ *and* $A,B\in CB(X)$.

The next result is well known in the standard metric spaces [21], but for the case of b-metric, we provide the proof.

Lemma 2.4. *Let A and B be nonempty, closed, bounded subsets of a b-metric space (X,d,s) and $q<1$. Then, for $a\in A$, there exists $b\in B$ such that*

$$q\,d(a,b)\le H(A,B). \tag{2.1}$$

Proof. If $H(A,B)=0$, then $a\in B$ and so (2.1) holds for $b=a$.

Suppose that $H(A,B)>0$. By the definitions of $D(a,B)$ and $H(A,B)$, for any $\varepsilon>0$ there exists $b\in B$ such that $d(a,b)\le D(a,B)+\varepsilon\le H(A,B)+\varepsilon$. For $\varepsilon=\left(\frac{1}{q}-1\right)H(A,B)>0$, which we may assume, we obtain (2.1). This completes the proof.

Let us note that the proof does not depend on s.

Definition 2.5. [2] *Let $T:X\to CB(X)$ be a multi-valued mapping and $\alpha:X\times X\to[0,\infty)$ a given function. Then T is said to be α_*-admissible if $\alpha(x,y)\ge1$ implies $\alpha_*(Tx,Ty)\ge1$, where* $\alpha_*(A,B)=\inf\{\alpha(x,y)\mid x\in A,y\in B\}$.

With Ω will be denoted the class of all functions $\beta:[0,\infty)\to[0,1)$ such that for any bounded sequence $\{t_n\}$ of positive real numbers, $\beta(t_n)\to1$ implies $t_n\to0$.

Theorem 2.2. [8] *Let (X,d) be a metric space and T a self mapping of X. Suppose that there exists $\beta \in \Omega$ such that for all $x,y \in X$, holds $d(Tx,Ty) \leq \beta(d(x,y))d(x,y)$. Then T has a unique fixed point $x^* \in X$ and $\{T^n x\}$ converges to x^* for each $x \in X$.*

Definition 2.6. [2] *Let (X,d) be a b-metric space and $\alpha : X \times X \to [0,\infty)$ a function. Then X is said to be α-complete if every b-Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ converges in X.*

Definition 2.7. [2] *Let (X,d) be a metric space, $T : X \to X$ a mapping and $\alpha,\eta : X \times X \to [0,\infty)$ two functions. We say that T is α-η-continuous mapping on (X,d) if for given $x \in X$ and a sequence $\{x_n\}$ in X with the properties $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, we have $Tx_n \to Tx$ as $n \to \infty$.*

If $\eta(x_n,x_{n+1}) = 1$, then T is called an $\alpha-$continuous mapping.

We follow the notation used in [2] [14] and denote by Ψ the class of the functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the conditions: ψ is nondecreasing, continuous and $\psi(t) = 0$ if and only if $t = 0$.

To facilitate access to our main results, we repeat some definitions and results from [2].

Definition 2.8. [2] *Let $S,T : X \to CB(X)$ be two multi-valued mappings and $\alpha : X \times X \to [0,\infty)$ a function. The pair (S,T) is said to be triangular α_*-admissible if the following conditions hold:*

1) *(S,T) is α_*-admissible, that is, $\alpha(x,y) \geq 1$ implies $\alpha_*(Sx,Ty) \geq 1$ and $\alpha_*(Tx,Sy) \geq 1$, where $\alpha_*(A,B) = \inf\{\alpha(x,y) \mid x \in A, y \in B\}$,*

2) *$\alpha(x,u) \geq 1$ and $\alpha(u,y) \geq 1$ imply $\alpha(x,y) \geq 1$.*

Definition 2.9. [2] *Let $S,T : X \to CB(X)$ be two multi-valued mappings and $\alpha : X \times X \to [0,\infty)$ a function. The pair (S,T) is said to be α_*-orbital admissible if the conditions $\alpha_*(x,Sx) \geq 1$ and $\alpha_*(x,Tx) \geq 1$ imply $\alpha_*\left(Sx,T^2x\right) \geq 1$ and $\alpha_*\left(Tx,S^2x\right) \geq 1$.*

Definition 2.10. [2] *Let $S,T : X \to CB(X)$ be two multi-valued mappings and $\alpha : X \times X \to [0,\infty)$ a function. The pair (S,T) is said to be triangular α_*-orbital admissible, if the following conditions are satisfied:*

(i) (S,T) is α_*-orbital admissible,

(ii) $\alpha(x,y) \geq 1, \alpha_*(y,Sy) \geq 1$ and $\alpha_*(y,Ty) \geq 1$ imply $\alpha_*(x,Sy) \geq 1$ and $\alpha_*(x,Ty) \geq 1$.

Lemma 2.5. [2] *Let $S,T : X \to CB(X)$ be two multi-valued mappings such that the pair (S,T) is triangular α_*-orbital admissible. Assume that there exists $x_0 \in X$ such that $\alpha_*(x_0, Sx_0) \geq 1$. Define the sequence $\{x_n\}$ in X by $x_{2n+1} \in Sx_{2n}$ and $x_{2n+2} \in Tx_{2n+1}$, where $n \in \mathbb{N} \cup \{0\}$. Then for $n,m \in \mathbb{N} \cup \{0\}$ with $m > n$, we have $\alpha(x_n, x_m) \geq 1$.*

Definition 2.11. [2] *Let (X,d) be a b-metric space, $S : X \to CB(X)$ a multi-valued mapping and $\alpha : X \times X \to [0,\infty)$ a function. We say that S is an α-continuous multi-valued mapping on $(CB(X), H)$ if whenever $\{x_n\}$ is a sequence in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = 0$, then $\lim_{n\to\infty} H(Sx_n, Sx) = 0$.*

The next definition from [2] gives a completely new notion in the setting of b-metric spaces.

Definition 2.12. [2] *Let (X,d) be a b-metric space, $\alpha : X \times X \to [0,\infty)$ a function and $S,T : X \to CB(X)$ two multi-valued mappings. The pair (S,T) is called a generalized α_*-ψ-Geraghty contraction type multi-valued mapping if there exist $\beta \in \Omega$ and $\psi \in \Psi$ such that for $x,y \in X$, with $\alpha(x,y) \geq 1$, the pair (S,T) satisfies the following inequality:*

$$\psi\left(s^3 H(Sx,Ty)\right) \leq \beta\left(\psi(M(x,y))\right) \cdot \psi(M(x,y)),$$

where

$$M(x,y) = \max\left\{d(x,y), D(x,Sx), D(y,Ty), \frac{D(x,Ty)+D(y,Sx)}{2s}\right\}.$$

In [2], the authors also proved the following results, Theorem 2.1 and 2.2. In Theorem 2.2, the continuity of the mappings S and T (property (v_1)) is replaced by the suitable new condition (property (v_2)).

Theorem 2.3. [2] *Let (X,d) be a b-metric space and $\alpha : X \times X \to [0,\infty)$ a function. Suppose that $S,T : X \to CB(X)$ are two multi-valued mappings satisfying the following conditions.*

(i) *(X,d) is an α-complete b-metric space;*

(ii) *(S,T) is a generalized α_*-ψ-Geraghty contraction type multi-valued mapping;*

(iii) *(S,T) is triangular α_*-orbital admissible;*

(iv) *There exists $x_0 \in X$ such that $\alpha_*(x_0, Sx_0) \geq 1$;*

(v) (v_1) *S and T are α-continuous multi-valued mappings;*

or

(v_2) *If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x^* \in X$*

as $n \to \infty$, *then there exists a subsequence* $\left\{x_{n(k)}\right\}$ *of* $\{x_n\}$ *such that* $\alpha\left(x_{n(k)}, x^*\right) \geq 1$ *for all* $k \in \mathbb{N} \cup \{0\}$.

Then S *and* T *have a common fixed point.*

3. Main results

In this section we give a genuine generalization of the results obtained in [2]. We provide the much shorter proofs than ones in the recent paper of E. Ammer et al. Essential to our proofs are the properties of the functions β and ψ as well as Lemma 2.1. Also, we shall use the definitions of the distances $D(a,B)$, $D(b,A)$, $H(A,B)$. From our proofs, we conclude that the functions β and ψ in the results obtained in [2] are superfluous. Also, it is sufficient to assume that $\varepsilon > 1$ instead of $\varepsilon = 3$.

Our first result is a generalization of [2, Theorem 2.1.].

Theorem 3.1. *Let* $(X, d, s > 1)$ *be a b-metric space,* $\alpha : X \times X \to [0, \infty)$ *a function and* $\varepsilon > 1$. *Let* $S, T : X \to CB(X)$ *be two multi-valued mappings such that for* $x, y \in X$, *with* $\alpha(x,y) \geq 1$, *the pair* (S,T) *satisfies the inequality*

$$H(Sx, Ty) \leq \frac{1}{s^{\varepsilon}} M(x,y), \tag{3.1}$$

where $M(x,y) = \max\left\{d(x,y), D(x,Sx), D(y,Ty), \frac{D(x,Ty)+D(y,Sx)}{2s}\right\}$.

Suppose that the following conditions are satisfied.

(i) (X,d) *is an* α*-complete b-metric space.*

(ii) (S,T) *is triangular* α_**-orbital admissible.*

(iii) *There exists* $x_0 \in X$ *such that* $\alpha_*(x_0, Sx_0) \geq 1$.

(iv) S *and* T *are* α*-continuous multi-valued mappings.*

Then S *and* T *have a common fixed point.*

Proof. From (iii), there exists $x_1 \in Sx_0$ such that $\alpha(x_0, x_1) \geq 1$ and $x_1 \neq x_0$. By the inequality (3.1) and Lemma 2.3, we have

$$0 < D(x_1, Tx_1) \leq H(Sx_0, Tx_1) \leq \frac{1}{s^{\varepsilon}} M(x_0, x_1). \tag{3.2}$$

Using Lemma 2.4 for $q=\frac{1}{s}<1$, there exists $x_2\in Tx_1$ such that

$$\frac{1}{s}d(x_1,x_2)\le H(Sx_0,Tx_1)\le \frac{1}{s^{\varepsilon}}M(x_0,x_1), \tag{3.3}$$

where

$$\begin{aligned} M(x_0,x_1) &= \max\left\{d(x_0,x_1),D(x_0,Sx_0),D(x_1,Tx_1),\frac{D(x_0,Tx_1)+D(x_1,Sx_0)}{2s}\right\} \\ &= \max\left\{d(x_0,x_1),D(x_1,Tx_1),\frac{D(x_0,Tx_1)+0}{2s}\right\} \\ &= \max\left\{d(x_0,x_1),D(x_1,Tx_1),\frac{D(x_0,Tx_1)}{2s}\right\}. \end{aligned}$$

According to Lemma 2.3, we have

$$\begin{aligned} \frac{D(x_0,Tx_1)}{2s} &\le \frac{s(d(x_0,x_1)+D(x_1,Tx_1))}{2s}=\frac{d(x_0,x_1)+D(x_1,Tx_1)}{2} \\ &\le \max\{d(x_0,x_1),D(x_1,Tx_1)\}, \end{aligned}$$

and, thus, it follows that $M(x_0,x_1)=\max\{d(x_0,x_1),D(x_1,Tx_1)\}$.

If $M(x_0,x_1)=D(x_1,Tx_1)$, then from (3.2), we obtain $0<D(x_1,Tx_1)\le\frac{1}{s^{\varepsilon}}D(x_1,Tx_1)$, a contradiction. Hence, we conclude that $\max\{d(x_0,x_1),D(x_1,Tx_1)\}=d(x_0,x_1)$. According to (3.3), we have $d(x_1,x_2)\le\frac{1}{s^{\varepsilon-1}}d(x_0,x_1)$.

Similarly, for $x_2\in Tx_1$, Lemma 2.4 gives $x_3\in Sx_2$, such that

$$\frac{1}{s}d(x_2,x_3)\le H(Tx_1,Sx_2)\le\frac{1}{s^{\varepsilon}}M(x_1,x_2), \tag{3.4}$$

where

$$\begin{aligned} M(x_1,x_2) &= \max\left\{d(x_1,x_2),D(x_1,Tx_1),D(x_2,Sx_2),\frac{D(x_1,Sx_2)+D(x_2,Tx_1)}{2s}\right\} \\ &= \max\left\{d(x_1,x_2),D(x_2,Sx_2),\frac{D(x_1,Sx_2)}{2s}\right\} \\ &= \max\{d(x_1,x_2),D(x_2,Sx_2)\}, \end{aligned}$$

since

$$\begin{aligned} \frac{D(x_1,Sx_2)}{2s} &\le \frac{s(d(x_1,x_2)+D(x_2,Sx_2)}{2s}=\frac{d(x_1,x_2)+D(x_2,Sx_2)}{2} \\ &\le \max\{d(x_1,x_2),D(x_2,S_x2)\}. \end{aligned}$$

By the inequality (3.1) and Lemma 2.3, we have

$$0<D(x_2,Sx_2)\le H(Tx_1,Sx_2)\le\frac{1}{s^{\varepsilon}}M(x_1,x_2). \tag{3.5}$$

If $M(x_1,x_2)=D(x_2,Sx_2)$, then from (3.5) we obtain $0<D(x_2,Sx_2)\leq \frac{1}{s^{\varepsilon}}D(x_2,Sx_2)$, which is impossible. This clearly forces $\max\{d(x_1,x_2),D(x_2,Sx_2)\}=d(x_1,x_2)$ and by (3.4) we obtain

$$d(x_2,x_3)\leq \frac{1}{s^{\varepsilon-1}}d(x_1,x_2).$$

We continue in this manner. In general, $x_{2n+1}\in X$ is chosen such that $x_{2n+1}\in Sx_{2n}$ and $x_{2n+2}\in Tx_{2n+1}$ for all $n\in\mathbb{N}\cup\{0\}$.

Since $\alpha_*(x_0,Sx_0)\geq 1$ and (S,T) is triangular α_*-orbital admissible, by Lemma 2.5, we have $\alpha(x_n,x_{n+1})\geq 1$ for all $n\in\mathbb{N}\cup\{0\}$.

For all $k\in\mathbb{N}\cup\{0\}$, we have

$$0<D(x_{2k+1},Tx_{2k+1})\leq H(Sx_{2k},Tx_{2k+1})\leq \frac{1}{s^{\varepsilon}}M(x_{2k},x_{2k+1}), \tag{3.6}$$

and

$$\frac{1}{s}d(x_{2k+2},x_{2k+1})\leq H(Sx_{2k},Tx_{2k+1})\leq \frac{1}{s^{\varepsilon}}M(x_{2k},x_{2k+1}), \tag{3.7}$$

where

$$M(x_{2k},x_{2k+1})=\max\Big\{d(x_{2k},x_{2k+1}),D(x_{2k},Sx_{2k}),D(x_{2k+1},Tx_{2k+1}),$$
$$\frac{D(x_{2k},Tx_{2k+1})+D(x_{2k+1},Sx_{2k})}{2s}\Big\}.$$

According to Lemma 2.3, we have

$$\begin{aligned}&\frac{D(x_{2k},Tx_{2k+1})+D(x_{2k+1},Sx_{2k})}{2s}=\frac{D(x_{2k},Tx_{2k+1})+0}{2s}=\frac{D(x_{2k},Tx_{2k+1})}{2s}\\&\leq\frac{s\left(d(x_{2k},x_{2k+1})+D(x_{2k+1},Tx_{2k+1})\right)}{2s}=\frac{d(x_{2k},x_{2k+1})+D(x_{2k+1},Tx_{2k+1})}{2}\\&\leq\max\{d(x_{2k},x_{2k+1}),D(x_{2k+1},Tx_{2k+1})\}.\end{aligned}$$

From what has already been proved and the inequality $D(x_{2k},Sx_{2k})\leq d(x_{2k},x_{2k+1})$, it follows that $M(x_{2k},x_{2k+1})=\max\{d(x_{2k},x_{2k+1}),D(x_{2k+1},Tx_{2k+1})\}$.

If $\max\{d(x_{2k},x_{2k+1}),D(x_{2k+1},Tx_{2k+1})\}=D(x_{2k+1},Tx_{2k+1})$, then from (3.6), we obtain

$$0<D(x_{2k+1},Tx_{2k+1})\leq\frac{1}{s^{\varepsilon}}D(x_{2k+1},Tx_{2k+1}),$$

which contradicts the fact $\varepsilon>1$. Hence, $M(x_{2k},x_{2k+1})=d(x_{2k},x_{2k+1})$. Further, by (3.7), we get $d(x_{2k+2},x_{2k+1})\leq\frac{1}{s^{\varepsilon-1}}d(x_{2k+1},x_{2k})$. It follows that for all $n\in\mathbb{N}\cup\{0\}$ the inequality

$d(x_{n+1},x_{n+2}) \leq \frac{1}{s^{\varepsilon-1}} d(x_n,x_{n+1})$ holds. Lemma 2.1 now shows that the sequence $\{x_n\}$ is a b-Cauchy sequence.

Since (X,d) is an α-complete b-metric space and $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, there exists $x^* \in X$ such that $\lim_{n\to\infty} d(x_n,x^*) = 0$ implies that $\lim_{k\to\infty} d(x_{2k+1},x^*) = 0$ and $\lim_{k\to\infty} d(x_{2k+2},x^*) = 0$. The α-continuity of T implies $\lim_{k\to\infty} H(Tx_{2k+1},Tx^*) = 0$. Thus,

$$\begin{aligned} D(x^*,Tx^*) &\leq s\left(d(x^*,x_{2k+1}) + D(x_{2k+1},Tx^*)\right) \\ &\leq s\left(d(x^*,x_{2k+1}) + H(Tx_{2k+1},Tx^*)\right) \\ &\to s\cdot(0+0) = 0, \end{aligned}$$

and so, $x^* \in Tx^*$. Similarly, we obtain $x^* \in Sx^*$. Hence, S and T have a common fixed point $x^* \in X$.

This completes the proof.

Remark 3.1. *It is clear that for* $1 < \varepsilon \leq 3$ *and* $s > 1$ *we have*

$$\begin{aligned} \psi(sH(Sx,Ty)) \leq \psi(s^{\varepsilon}H(Sx,Ty)) &\leq \psi\left(s^3H(Sx,Ty)\right) \\ &\leq \beta(\psi((M(x,y)))) \cdot \psi(M(x,y)), \end{aligned}$$

where

$$M(x,y) = \max\left\{d(x,y), D(x,Sx), D(y,Ty), \frac{D(x,Ty)+D(y,Sx)}{2s}\right\},$$

and consequently the condition (2.1) *from [2] implies the condition* (3.1)*. We can conclude that Theorem* 3.1 *extends the main result, Theorem* 2.1*, from [2]. It is worth notice that our proof is much shorter and also all redundant properties are avoided.*

In the next result we show that the α-continuity of the mappings S and T can be replaced with a new suitable condition.

Theorem 3.2. *Let* $(X,d,s>1)$ *be a b-metric space,* $\alpha : X\times X \to [0,\infty)$ *a function and* $\varepsilon > 1$*. Let* $S,T : X \to CB(X)$ *be two multi-valued mappings such that for* $x,y \in X$*, with* $\alpha(x,y) \geq 1$*, the pair* (S,T) *satisfies the inequality*

$$H(Sx,Ty) \leq \frac{1}{s^{\varepsilon}} M(x,y), \tag{3.8}$$

where $M(x,y) = \max\left\{d(x,y), D(x,Sx), D(y,Ty), \frac{D(x,Ty)+D(y,Sx)}{2s}\right\}$.

Suppose, further, that $S,T : X \to CB(X)$ *satisfy the following conditions:*

(i) *(X,d) is an α-complete b-metric space,*

(ii) *(S,T) is triangular α_*-orbital admissible,*

(iii) *there exists $x_0 \in X$ such that $\alpha_*(x_0, Sx_0) \geq 1$,*

(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq 1$ for all $k \in \mathbb{N} \cup \{0\}$.*

Then S and T have a common fixed point $x^* \in X$.

Proof. In the same way as in the proof of Theorem 3.1, we construct the sequence $\{x_n\}$ in X defined by $x_{2n+1} \in Sx_{2n}$, $x_{2n+2} \in Tx_{2n+1}$, $n \in \mathbb{N} \cup \{0\}$, with the properties $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{x_n\}$ converges to $x^* \in X$. By condition (iv), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq 1$ for all $k \in \mathbb{N} \cup \{0\}$. According to Lemma 2.3, we have

$$\begin{aligned}\frac{1}{s} D(x^*, Tx^*) &\leq d\left(x^*, x_{2n(k)+1}\right) + D\left(x_{2n(k)+1}, Tx\right) \\ &\leq d\left(x^*, x_{2n(k)+1}\right) + H\left(Sx_{2n(k)}, Tx^*\right) \\ &\leq d\left(x^*, x_{2n(k)+1}\right) + \frac{1}{s^{\varepsilon}} M\left(x_{2n(k)}, x^*\right),\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}M\left(x_{2n(k)}, x^*\right) = \max\Big\{ & d\left(x_{2n(k)}, x^*\right), D\left(x_{2n(k)}, Sx_{2n(k)}\right), D(x^*, Tx^*), \\ & \frac{D\left(x_{2n(k)}, Tx^*\right) + D\left(x^*, Sx_{2n(k)}\right)}{2s}\Big\}.\end{aligned} \tag{3.10}$$

Since $\frac{1}{s} D\left(x_{2n(k)}, Tx^*\right) \leq d\left(x_{2n(k)}, x^*\right) + D(x^*, Tx^*)$ and $D\left(x^*, Sx_{2n(k)}\right) \leq d\left(x^*, x_{2n(k)}\right)$, we can conclude that

$$\limsup_{k \to \infty} \frac{D\left(x_{2n(k)}, Tx^*\right) + D\left(x^*, Sx_{2n(k)}\right)}{2s} \leq \frac{D(x^*, Tx^*)}{2s}.$$

Letting $k \to \infty$ in (3.10), we obtain $\lim_{k \to \infty} M\left(x_{2n(k)}, x^*\right) = D(x^*, Tx^*)$.

If we assume that x^* is not the fixed point of T, i.e. $D(x^*, Tx^*) > 0$, we obtain a contradiction. Indeed, letting $k \to \infty$ in (3.9), we get $\frac{1}{s} D(x^*, Tx^*) \leq \frac{1}{s^{\varepsilon}} D(x^*, Tx^*)$, which contradicts the fact that $\varepsilon > 1$. Therefore, $x^* \in Tx^*$, i.e., x^* is the fixed point of T. Similarly, we can show that $x^* \in Sx^*$. Consequently, $x^* \in X$ is the common fixed point of S and T.

This completes the proof.

Remark 3.2. *It is easy to notice that the previous result is a proper generalization of Theorem* 2.2 *from [2]. Namely, we do not use the Geraghty condition to prove the existence of the fixed point.*

Assuming that $S = T$ and $M(x,y) = d(x,y)$ in Theorem 3.1 and Theorem 3.2, we obtain the new results which are genuine generalizations of those in [2].

4. Improvement results and remarks on a resent paper

Now, we give some remarks on the results obtained in [2].

1) In Corollary 2.1 and Corollary 2.2, it is not necessary to assume the completeness of X, since the condition (i) is then redundant.

2) Considering the result of Theorem 2.3, we can conclude that a very similar approach would be more illuminating. This result can be generalized with little effort to the cases $\varepsilon \in (1,3]$ without functions ψ and β. Also, it is clear that a new result can be established if we replace (v) with the α-continuity of multi-valued mappings S and T. Extending the theorem in those ways would have a much greater impact. In the both cases, the multivalued mappings S and T satisfy the contractive condition of the form $d(Sx,Ty) \le \frac{1}{s^{\varepsilon}} M(x,y)$, where $s > 1$ and $M(x,y) = \max\left\{d(x,y), D(x,Sx), D(y,Ty), \frac{D(x,Ty)+D(y,Sx)}{2s}\right\}$.

3) The authors provide an example on page 14. They say that $\left([0,\frac{1}{2}),d\right)$ is a complete b-metric space. This statement is not correct. Indeed, considering the sequence $x_n = \frac{1}{2} - \frac{1}{n+1}$, $n \in \mathbb{N}$, we have $x_n \to \frac{1}{2}$ as $n \to \infty$ in the b-metric space $\left([0,\frac{1}{2}),d\right)$ but the limit $\frac{1}{2}$ does not belong to $[0,\frac{1}{2})$. However, it is easy to check that for certain values of $\varepsilon > 1$ and $s > 1$ and some ψ and β, the inequality $d(Sx,Ty) \le \frac{1}{s^{\varepsilon}} M(x,y)$ is satisfied but the following condition $\psi\left(s^3 d(Sx,Ty)\right) \le \beta(\psi(M(x,y)))M(x,y)$ does not hold.

We think that a different approach would be more useful. The authors should consider $Tx = \begin{cases} \left\{\frac{2}{245}x\right\}, & \text{if } x \in [0,\frac{1}{2}) \\ \{1\}, & \text{if } x \in [\frac{1}{2},1] \end{cases}$ and $Sx = \{0\}$ and use Theorem 2.1.

4) It is easy to verify that Section 3 is a direct consequence of the previous section. Indeed, X can be identify with a proper subset of $CB(X)$ considering $\{x\}$ instead of x. Then Definition 3.1 becomes a special case of Definition 2.5. Also, Theorem 3.1 and 3.2 can be obtained by

Theorem 2.1 and 2.2. The same case is with Corollary 3.1. The only conclusion in Section 3 different from Section 2 is the uniqueness of fixed point of S and T.

5) In Section 4, the results are used to establish the existence of a solution for a pair of ordinary differential equations. We will show that it is not essential to use the b-metric

$$(x,y) \mapsto \sup_{t\in[a,b]} (x(t)-y(t))^2$$

in the space of continuous functions $C[a,b]$ for solving ordinary differential or integral equations since we can use the standard metric

$$(x,y) \mapsto \sup_{t\in[a,b]} |x(t)-y(t)|.$$

As a mater of fact, if we denote with d a b-metric and with D a metric on $C[a,b]$, then we have $D(x,y)=\sqrt{d(x,y)}$. Since $(C[a,b],d)$ is a b-metric space with the coefficient $s=2$, then the condition $\psi\left(s^3 d(Sx,Ty)\right) \leq \beta\left(\psi(M(x,y))\right)\cdot M(x,y)$ implies

$$d(Sx,Ty) \leq \frac{1}{8} M(x,y), \tag{4.1}$$

where

$$\begin{aligned} M(x,y) &= \max\left\{d(x,y), d(x,Sx), d(y,Ty), \frac{d(x,Ty)+d(Sx,y)}{4}\right\} \\ &= \max\left\{|x-y|^2, |x-Sx|^2, |y-Ty|^2, \frac{|x-Ty|^2+|Sx-y|^2}{4}\right\}. \end{aligned}$$

Since $\frac{|x-Ty|^2+|Sx-y|^2}{4} \leq \frac{1}{2}\max\left\{|x-Ty|^2, |Sx-y|^2\right\}$, from (4.1), we have

$$|Sx-Ty|^2 \leq \frac{1}{8}\max\left\{|x-y|^2, |x-Sx|^2, |y-Ty|^2, \frac{|x-Ty|^2}{2}, \frac{|Sx-y|^2}{2}\right\}.$$

By taking square roots, we obtain

$$\begin{aligned} D(Sx,Ty) &\leq \frac{1}{2\sqrt{2}}\max\left\{|x-y|, |x-Sx|, |y-Ty|, \frac{|x-Ty|}{\sqrt{2}}, \frac{|Sx-y|}{\sqrt{2}}\right\} \\ &= \frac{1}{2\sqrt{2}}\max\left\{D(x,y), D(x,Sx), D(y,Ty), \frac{D(x,Ty)}{\sqrt{2}}, \frac{D(Sx,y)}{\sqrt{2}}\right\}. \end{aligned}$$

Now, we derive the following assertion.

Let (X,D) be a complete metric space and $S,T: X \to X$ mappings such that the condition

$$D(Sx,Ty) \leq \frac{1}{2\sqrt{2}}\max\left\{D(x,y), D(x,Sx), D(y,Ty), \frac{D(x,Ty)}{\sqrt{2}}, \frac{D(Sx,y)}{\sqrt{2}}\right\}$$

is satisfied. Then S and T have a unique fixed point.

As a consequence, we can conclude that Theorem 4.1 is an application of the standard metric D on the space $C[a,b]$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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