Available online at http://scik.org
Adv. Fixed Point Theory, 8 (2018), No. 3, 303-312
https://doi.org/10.28919/afpt/3741
ISSN: 1927-6303

STRONG CONVERGENCE THEOREM OF A NEW ITERATIVE METHOD FOR WEAK CONTRACTIONS AND COMPARISON OF THE RATE OF CONVERGENCE IN BANACH SPACE

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Abstract. In this paper, we first construct a new iteration method for approximating fixed points of a class of weak contractions in a Banach space and then prove strong convergence theorem of the proposed method under some control conditions. It is shown that our iteration method converges faster than Noor iteration. Moreover, we give some numerical example for comparing the rate of convergence between the Noor iteration and our iteration.

Keywords: strong convergence; rate of convergence; Noor iterations; weak contractions; Banach space.

2010 AMS Subject Classification: 47H10, 54H25.

1. Introduction and preliminaries

Fixed point theory plays very important role in nonlinear analysis and applications. It can be applied to study the existence of various equations. In 2003, Berinde [1] introduced a new type of contraction, called weak contraction, and proved a fixed point theorem for this type of mappings in a complete metric space by showing that the Picard sequences converge strongly

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Received May 7, 2018

to its fixed point. Recently, there are many iterative methods using to approximate fixed points of nonlinear mappings, such as Mann iteration, Ishikawa iteration and Noor iteration.

Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a mapping. A point $x \in C$ is a fixed point of T if $Tx = x$.

The **Mann** iteration (see [2]) is defined by $u_0 \in C$ and

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n T u_n \tag{1}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\}$ is a sequence in $[0,1]$.

For $\alpha_n = 1$, the iteration (1) called the **Picard** iteration.

The **Ishikawa** iteration (see [4]) is defined by $s_0 \in C$ and

$$\begin{cases} t_n &= (1-\beta_n)s_n + \beta_n T s_n \\ s_{n+1} &= (1-\alpha_n)s_n + \alpha_n T t_n \end{cases} \tag{2}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0,1]$

The **Noor** iteration (see [5]) is defined by $s_0 \in C$ and

$$\begin{cases} r_n &= (1-\gamma_n)w_n + \gamma_n T w_n, \\ q_n &= (1-\beta_n)w_n + \beta_n T r_n, \\ w_{n+1} &= (1-\alpha_n)w_n + \alpha_n T q_n \end{cases} \tag{3}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$.

It is easy to see that Mann iteration and Ishikawa iteration are special case of Noor iteration.

There are many papers have been concentrated on the iterative methods for approximating of fixed points of nonlinear mappings, (see [2], [3], [4], [5]), but there are a few papers pay attention on comparing the rate of convergence of those methods. In 2004, Berinde [6] provided the following concept to compare the rate of convergence of the iterative methods.

Definition 1.1 [6] Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real number that converge to a and b respectively, and assume that there exists

$$\ell = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} \tag{4}$$

(1) If $\ell = 0$, then it can be said that $\{a_n\}$ converge faster to a than $\{b_n\}$ to b.

(2) If $0 < \ell < \infty$, then it can be said that $\{a_n\}$ and $\{b_n\}$ have the same rate of convergence.

Suppose the two fixed point iterations $\{x_n\}$ and $\{y_n\}$ converge to the same fixed point z of a mapping T. By employing above concept of Berinde [6], we say that the sequence $\{x_n\}$ converge faster than the sequence $\{y_n\}$ if

$$\lim_{n\to\infty} \frac{|x_n - z|}{|y_n - z|} = 0 \tag{5}$$

Definition 1.2 Let C be a nonempty subset of a Banach space X. A mapping T is said to be weak contraction if there exists $L \geq 0$ and $\delta \in (0,1)$ such that

$$\|Tx - Ty\| \leq \delta\|x - y\| + L\|y - Tx\|, \quad \text{for all } x, y \in C. \tag{6}$$

Definition 1.3 Condition (*) Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is said to satisfy condition(*) if there exists $L' \geq 0$ and $\delta' \in (0,1)$ such that

$$\|Tx - Ty\| \leq \delta'\|x - y\| + L'\|x - Tx\|, \quad \text{for all } x, y \in C. \tag{7}$$

Theorem 1.4 [7] *Let C be a nonempty closed convex subset of a Banach space X, and $T : C \to C$ be a weak contraction. Then $F(T) \neq \varnothing$. Moreover, the Picard iteration $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$ converges to a fixed point of T. Moreover, if T satisfies the condition (*), then T has a unique fixed point.*

Theorem 1.5 [8] *Let C be a nonempty closed convex subset of a Banach space X, and $T : C \to C$ be a weak contraction satisfying the condition (*) and let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ with $\sum_{n=0}^{\infty} \alpha_n = \infty$. Suppose $\{u_n\}, \{s_n\}$ and $\{w_n\}$ are sequences generated by Mann iteration* (1)*, Ishikawa iteration* (2) *and Noor iteration* (3)*, respectively. Then $\{u_n\}, \{s_n\}$ and $\{w_n\}$ converge strongly to a unique fixed point of T.*

In this paper, we propose a new iteration method as the following :

Let C be a nonempty convex subset of a Banach space X and $T : C \to C$, be a mapping . The iteration is defined by $x_0 \in C$ and

$$(8)\qquad \begin{cases} z_n &= (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n &= (1-\beta_n)T x_n + \beta_n T z_n, \\ x_{n+1} &= (1-\alpha_n)T z_n + \alpha_n T y_n \end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$. Then we prove the convergence theorem of the proposed method for fixed point of weak contraction in a Banach space, and also compare the rate of convergence between Noor iterations and our iterations.

2. Main results

Firstly, we prove the following convergence theorem of our iteration method defined in (8)

Theorem 2.1 *Let C be a nonempty closed convex subset of a Banach space X, and $T : C \to C$ be a weak contraction satisfying the condition (*). Let $\{x_n\}$ be a sequence generated by* (8)*. Suppose that $\sum_{n=0}^{\infty} \beta_n \gamma_n = \infty$. Then $\{x_n\}$ converges strongly to a unique fixed point of T.*

Proof. By Theorem 1.4, T has a unique fixed point, say p. By condition (*), we have

$$\begin{aligned} \|T x_n - p\| &= \|T x_n - T p\| \\ &= \|T p - T x_n\| \\ &\le \delta' \|p - x_n\| + L' \|p - T p\| \\ &\le \delta' \|x_n - p\|. \end{aligned}$$

Similarly, we also have $\|T y_n - p\| \le \delta' \|y_n - p\|$ and $\|T z_n - p\| \le \delta' \|z_n - p\|$ for all $n \in \mathbb{N} \cup \{0\}$. These imply

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n)T z_n + \alpha_n T y_n - p\| \\ &= \|(1-\alpha_n)(T z_n - p) + \alpha_n (T y_n - p)\| \\ &\le (1-\alpha_n)\|T z_n - p\| + \alpha_n \|T y_n - p\| \\ &\le (1-\alpha_n)\delta' \|z_n - p\| + \alpha_n \delta' \|y_n - p\|, \qquad (9) \end{aligned}$$

$$\begin{aligned}
\|z_n - p\| &= \|(1-\gamma_n)x_n + \gamma_n T x_n - p\| \\
&= \|(1-\gamma_n)(x_n - p) + \gamma_n(T x_n - p)\| \\
&\leq (1-\gamma_n)\|(x_n - p)\| + \gamma_n\|T x_n - p\| \\
&\leq (1-\gamma_n)\|(x_n - p)\| + \gamma_n\delta'\|T x_n - p\| \\
&\leq (1-\gamma_n + \gamma_n\delta')\|x_n - p\| \qquad (10)
\end{aligned}$$

and

$$\begin{aligned}
\|y_n - p\| &= \|(1-\beta_n)T x_n + \beta_n T z_n - p\| \\
&= \|(1-\beta_n)(T x_n - p) + \beta_n(T z_n - p)\| \\
&\leq (1-\beta_n)\|(T x_n - p)\| + \beta_n\|T z_n - p\| \\
&\leq (1-\beta_n)\delta'\|(x_n - p)\| + \beta_n\delta'\|z_n - p)\| \\
&\leq (1-\beta_n)\delta'\|(x_n - p)\| + \beta_n\delta'(1-\gamma_n + \gamma_n\delta')\|x_n - p\| \\
&= [(1-\beta_n)\delta' + \beta_n\delta'(1-\gamma_n + \gamma_n\delta')]\|x_n - p\| \\
&= \delta'(1-\beta_n\gamma_n + \beta_n\gamma_n\delta')\|x_n - p\| \qquad (11)
\end{aligned}$$

By (9), (10) and (11) , we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq (1-\alpha_n)\delta'\|z_n - p\| + \alpha_n\delta'\|y_n - p\| \\
&\leq (1-\alpha_n)\delta'(1-\gamma_n + \gamma_n\delta')\|x_n - p\| \\
&\quad + \alpha_n\delta'(\delta'(1-\beta_n\gamma_n + \beta_n\gamma_n\delta'))\|x_n - p\| \\
&= [(1-\alpha_n)\delta'(1-\gamma_n + \gamma_n\delta') \\
&\quad + \alpha_n\delta'\delta'(1-\beta_n\gamma_n + \beta_n\gamma_n\delta')]\|x_n - p\| \\
&= [(1-\alpha_n)(1-\gamma_n(1-\delta'))\delta' \\
&\quad + \alpha_n\delta'\delta'(1-\beta_n\gamma_n(1-\delta'))]\|x_n - p\|
\end{aligned}$$

$$\begin{aligned}
&\leq [(1-\alpha_n)(1-\beta_n\gamma_n(1-\delta'))\delta' \\
&\quad +\alpha_n\delta'\delta'(1-\beta_n\gamma_n(1-\delta'))]\|x_n-p\| \\
&\leq [(1-\alpha_n)(1-\beta_n\gamma_n(1-\delta')) \\
&\quad +\alpha_n(1-\beta_n\gamma_n(1-\delta'))]\|x_n-p\| \\
&= [(1-\beta_n\gamma_n(1-\delta'))(1-\alpha_n+\alpha_n)]\|x_n-p\| \\
&= (1-\beta_n\gamma_n(1-\delta'))\|x_n-p\|. \qquad (12)
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|x_{n+1}-p\| &\leq (1-\beta_n\gamma_n(1-\delta'))\|x_n-p\| \\
&\ \ \vdots \qquad (13) \\
&\leq \prod_{k=1}^{n}(1-\beta_k\gamma_k(1-\delta'))\|x_1-p\|.
\end{aligned}$$

It implies that $\|x_{n+1}-p\| \to 0$. Hence, $\{x_n\}$ converge to $p \in F(T)$ as $n \to \infty$

Theorem 2.2 *Assume X, C, T are as in Theorem 2.1. Let $\{w_n\}$ and $\{x_n\}$ be sequence generated by Noor iteration* (3) *and the iteration* (8)*, respectively, with $w_1 = x_1$. Suppose that $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ satisfying the following conditions:*

(C1) $\sum_{n=0}^{\infty}\alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $\sum_{n=0}^{\infty}\beta_n\gamma_n = \infty$ *and* $\lim_{n\to\infty}\beta_n\gamma_n = a \in (0,1)$.

Then $\{x_n\}$ and $\{w_n\}$ converge strongly to a unique fixed point of T and moreover $\{x_n\}$ converges faster than Noor iteration (3).

Proof. By Theorem 1.5, and Theorem 2.1, we obtain that $\{w_n\}$ and $\{x_n\}$ converge strongly to a unique fixed point of T. By (13), we have

$$\|x_{n+1}-p\| \leq \prod_{k=1}^{n}(1-\beta_k\gamma_k(1-\delta'))\|x_1-p\|, \quad \text{for all } n \in \mathbb{N}\cup\{0\}. \qquad (14)$$

By the inequality (2.4) of [6], we also have

$$\|w_{n+1}-p\| \geq \prod_{k=1}^{n}(1-\alpha_k(1+\delta'))\|w_1-p\|, \quad \text{for all } n \in \mathbb{N}\cup\{0\} \qquad (15)$$

It follows by (14) and (15)

$$\begin{aligned}\frac{\|x_{n+1}-p\|}{\|w_{n+1}-p\|} &\leq \frac{\prod_{k=1}^{n}(1-\beta_k\gamma_k(1-\delta'))\|x_1-p\|}{\prod_{k=1}^{n}(1-\alpha_k(1+\delta'))\|w_1-p\|} \\ &\leq a_n\end{aligned}$$

where $a_n = \dfrac{\prod_{k=1}^{n}(1-\beta_k\gamma_k(1-\delta'))}{\prod_{k=1}^{n}(1-\alpha_k(1+\delta'))}$.

Note that

$$\frac{a_{n+1}}{a_n} = \frac{1-\beta_{n+1}\gamma_{n+1}(1-\delta')}{1-\alpha_{n+1}(1+\delta')}.$$

By condition (C1) and (C2), we get

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 1-a(1-\delta') < 1.$$

This implies by the ratio test that $\lim_{n\to\infty} a_n = 0$.

Hence , $\lim_{n\to\infty}\frac{\|x_{n+1}-p\|}{\|w_{n+1}-p\|} = 0$, so we can conclude that the sequence $\{x_n\}$ converges faster than $\{w_n\}$.

Theorem 2.3 *Assume X, C, T are as in Theorem 2.1. Let $\{x_n\}$ and $\{w_n\}$ be sequences in Theorem 2.2. Suppose that $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ satisfying the following conditions:*

(C3) $\sum_{n=0}^{\infty}\alpha_n = \infty$ and $\sum_{n=0}^{\infty}\beta_n\gamma_n = \infty$,

(C4) $\alpha_n < \frac{1}{1+\delta'}$, for all $n \in \mathbb{N}\cup\{0\}$ and $\sup_n\left(\dfrac{1-\beta_n\gamma_n(1-\delta')}{1-\alpha_n(1+\delta')}\right) < 1$.

Then the sequence $\{x_n\}$ and $\{w_n\}$ converge strongly to a unique fixed point of T and moreover $\{x_n\}$ converges faster than $\{w_n\}$.

Proof. By Theorem 2.1 and Theorem 1.5, under the condition (C3), we know that $\{x_n\}$ and $\{w_n\}$ converge strongly to a unique fixed point of T.

Using the same proof as in Theorem 2.2 together with the condition (C3) and (C4), we have

$$\begin{aligned}\frac{\|x_{n+1}-p\|}{\|w_{n+1}-p\|} &\leq \frac{\prod_{k=1}^{n}(1-\beta_k\gamma_k(1-\delta'))}{\prod_{k=1}^{n}(1-\alpha_k(1+\delta'))} \\ &= \prod_{k=1}^{n}\left(\frac{1-\beta_k\gamma_k(1-\delta')}{1-\alpha_k(1+\delta')}\right) \\ &\leq \prod_{k=1}^{n}\eta = \eta^n,\end{aligned}$$

where $\sup_n \left(\frac{1-\beta_n\gamma_n(1-\delta')}{1-\alpha_n(1+\delta')} \right) < \eta < 1$.

This implies that $\lim_{n\to\infty} \frac{\|x_{n+1}-p\|}{\|w_{n+1}-p\|} = 0$. Thus the sequence $\{x_n\}$ converges faster than $\{w_n\}$.

The following example shows that our iteration converges faster than Noor iteration.

Example 2.4 Let $T : [0,1] \to [0,1]$ be defined by

$$Tx = \begin{cases} \frac{x^2}{3}, & \text{if } x \in \left[0, \frac{2}{5}\right), \\ \frac{2x}{5} + \frac{1}{2}, & \text{if } x \in \left[\frac{2}{5}, 1\right]. \end{cases}$$

It is easy to see that T is a weak contraction with $\delta = \frac{1}{2}, L = 3$ and T also satisfying the condition (*). The following table shows numerical experiment of Noor iteration and our iteration (8) when $\gamma_n = \frac{1}{2}, \beta_n = \alpha_n = \frac{1}{n}$ and the initial point $x_0 = 0.3$.

TABLE 1. Numerical experiment of Noor iteration and our iteration when the initial point $w_0 = x_0 = 0.3$.

	Noor iteration		**Our iteration**	
n	w_n	$\|w_{n+1} - w_n\|$	x_n	$\|x_{n+1} - x_n\|$
1	1.5398030648E-01	1.4601969352E-01	4.6011189844E-03	2.9539888102E-01
2	1.1660716637E-01	3.7373140112E-02	1.3272079746E-06	4.5997917764E-03
3	9.7699426091E-02	1.8907740282E-02	1.2232517822E-13	1.3272078522E-06
4	8.5792254624E-02	1.1907171467E-02	1.0910848394E-27	1.2232517822E-13
⋮	⋮	⋮	⋮	⋮

TABLE 2. Comparing the rate of convergence using Berinde idea with the initial point $w_0 = x_0 = 0.3$.

	Noor iteration	**Our iteration**	**Convergence Rate**
n	w_n	x_n	$\frac{\|x_n-0\|}{\|w_n-0\|}$
1	1.5398030648E-01	4.6011189844E-03	2.9881217212E-02
2	1.1660716637E-01	1.3272079746E-06	1.1381873137E-05
3	9.7699426091E-02	1.2232517822E-13	1.2520562619E-12
4	8.5792254624E-02	1.0910848394E-27	1.2717754583E-26
⋮	⋮	⋮	⋮

From Table 1 and Table 2, we can conclude the sequence $\{x_n\}$ generated by our iteration converge to a fixed point of T faster than the sequence $\{w_n\}$ generated by Noor iteration.

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgements

The author would like to thank Faculty of Science, Lampang Rajabhat university, for financial support during the preparation of this this paper.

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