Available online at http://scik.org
Adv. Fixed Point Theory, 8 (2018), No. 3, 313-340
https://doi.org/10.28919/afpt/3742
ISSN: 1927-6303

DEMICLODENESS AND FIXED POINTS OF G-ASYMPTOTICALLY NONEXPANSIVE MAPPING IN BANACH SPACES WITH GRAPH

MENGISTU GOA SANGAGO[1,*], TIBEBU WORRKU HUNDE[1], HABTU ZEGEYE HAILU[2]

[1]Department of Mathematics, College of Natural & Computational Sciences, Addis Ababa University, P.O. Box 31167, Addis Ababa, Ethiopia

[2]Department of Mathematics and Statistical Sciences, Botswana International University of Science and Technology, Private Bag 16, Palapye, Botswana

Abstract. Let C be a nonempty closed convex subset of a uniformly convex Banach space endowed with a transitive directed graph $G = (V(G), E(G))$, such that $V(G) = C$ and $E(G)$ is convex. We introduce the definition of G-asymptotically nonexpansive self-mapping on C. It is shown that such mappings are G-demiclosed. Finally, we prove the weak and strong convergence of a sequence generated by a modified Noor iterative process to a common fixed point of a finite family of G-asymptotically nonexpansive self-mappings defined on C with nonempty common fixed points set. Our results improve and generalize several recent results in the literature.

Keywords: weak and strong convergence; common fixed points; $G-$ asymptotically nonexpansive mapping; digraph; Property P.

2010 AMS Subject Classification: 47H09, 47H10.

1. Introduction

*Corresponding author
E-mail address: mengistu.goa@aau.edu.et
Received May 9, 2018

Let C be a nonempty subset of a real normed linear space X. A self-mapping $T : C \to C$ is called asymptotically nonexpansive (Goebel and Kirk [10]) if there exists a sequence $\{u_n\} \subset [0,\infty)$, $u_n \to 0$ as $n \to \infty$ such that $\forall x, y \in C$, the following inequality holds:

$$||T^n x - T^n y|| \leq \quad (1 + u_n)||x - y||, \ \forall n \geq 1.$$

T is called nonexpansive (Browder [5], Göhde [11], Kirk[16]) if

$$||Tx - Ty|| \leq \quad ||x - y||, \ \forall n \geq 1.$$

Recall that a Banach space X is said to satisfy ***Opial's condition*** (see [20]) if for each sequence $\{x_n\}$ weakly convergent to x and for $y \neq x$ we have

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||.$$

A point $x \in X$ is called a fixed point of a self-mapping T on X if $x = T(x)$. The fixed point set of a mapping T will be denoted by $F(T)$.

In 1972, Goebel and Kirk [10], proved the following fundamental theorem for existence of fixed point of asymptotically nonexpansive mappings:

Theorem 1.1. *If C is a nonempty bounded closed convex subset of a real uniformly convex Banach space X and if T is an asymptotically nonexpansive self-mapping on C, then T has at least one fixed point.*

In 1978, Bose [4] initiated the study of approximation of fixed points of asymptotically nonexpansive mapping and proved that, if C is a nonempty bounded closed convex subset of a uniformly convex Banach space X satisfying Opial's condition and $T : C \to C$ is an asymptotically nonexpansive mapping, then the sequence $\{T^n x\}$ converges weakly to a fixed point of T provided T is asymptotically regular at $x \in C$, i.e., $\lim_{n\to\infty} ||T^n x - T^{n+1}x|| = 0$. In 1982, Passty [22] proved that the requirement that X satisfies the Opial's condition can be replaced by the Frechet differentiable norm. In 1992, Tan and Xu [34] proved that the asymptotic regularity of T at x can be replaced by weak asymptotic regularity of T at x, i.e., $\omega - \lim_{n\to\infty}(T^n x - T^{n+1}x) = 0$.

In 1991, Schu [27] introduced the modified Mann iteration (see [17]) process:

$$x_{n+1} = \quad (1-\alpha_n)x_n + \alpha_n T^n x_n, \; n = 1,2,3,\cdots \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ which is bounded away from 0 and 1, i.e., $a \leq \alpha_n \leq b$ for all n for some $0 < a \leq b < 1$ to approximate fixed points of aymptotically nonexpansive self-mappings defined on nonempty bounded closed convex subsets of a Hilbert space. In parallel publication in 1991, Schu[28] also proved the same result in the setting of a uniformly convex Banach space which satisfies Opial's condition.

In 1993, Bruck et al.[6] constructed the following iterative scheme

$$x_{i+1} = \quad (1-\alpha_i)x_i + \alpha_i T^{n_i} x_i,$$

where $\{\alpha_i\}$ is a sequence in $(0,1)$ bounded away from 0 and 1 and $\{n_i\}$ a sequence of nonnegative integers and studied some convergence theorems for asymptotically nonexpansive mappings in the setting of Banach spaces with uniform $\tau-$Opail's property. In 1994, Tan and Xu [35] studied the modified Ishikawa iteration process and used the method to approximate fixed points for asymptotically nonexpansive mappings:

$$x_{n+1} = \quad (1-\alpha_n)x_n + \alpha_n T^n((1-\beta_n)x_n + \beta_n T^n x_n)), \; n = 1,2,3,\cdots$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ such that α_n is bounded away from 0 and 1 and β_n is bounded away from 1. Osilike and Aniagbosor [21] proved that the theorem of Schu remains true without the boundedness assumption on C provided that the fixed point set is nonempty. Furthermore, Chang et al.[7] proved convergence theorems for asymptotically nonexpansive mappings and nonexpansive mappings in Banach spaces without assuming any of the conditions (a) X satisfies Opial's condition; (b) T is weak-asymptotically regular; (c) C is bounded. Khan and Takahashi [15] have approximated common fixed points of two asymptotically nonexpansive self mappings by using the modified Ishikawa iteration (see [12]) process:

$$x_{n+1} = \quad (1-\alpha_n)x_n + \alpha_n T^n((1-\beta_n)x_n + \beta_n S^n x_n)), \; n = 1,2,3,\cdots$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ such that α_n is bounded away from 0 and 1 and β_n is bounded away from 1. Approximating fixed points of nonexpansive and asymptotically nonexpansive mappings has been extensively studied by several authors(see, e.g., [19, 21, 25, 30, 32]).

Fixed point theorems for monotone single valued mappings in a metric space endowed with partial orderings are first considered by Ran and Reurings [24] in 2004 and have been widely investigated (see, e.g., [2, 9, 18]). The theorem in [24] is a hybrid of the two independent fundamental theorems: Banach contraction principle [3] and Tarski's fixed point result [33]. Recently, Reich and Zaslavski in [23] obtained some fixed point results for different classes of contractive self-mappings in a partially ordered metric spaces.

On the other hand, Jachymski [13], investigated a new approach in metric fixed point theory by replacing an order structure with a graph structure on metric spaces. In this way, the results proved in ordered metric spaces are generalized (see for detail [13] and the reference therein).

Recall that a directed graph usually written as digraph is a pair $G = (V(G), E(G))$ where $V(G)$ is a nonempty set called vertices of the graph G and $E(G) = \{(u,v) : u,v \in V(G)\}$ is set of ordered pairs called edges of the graph G. Let C be a nonempty subset of a real Banach space X and Δ be the diagonal of $C \times C$. Let G be a digraph such that the set $V(G)$ of its vertices coincide with C and $\Delta \subseteq E(G)$, i.e., $E(G)$ contains all loops. Assume that G has no parallel Edges. If x and y are vertices of G, then a path in G from x to y of length $k \in \mathbb{N}$ is a finite sequence $\{x_i\}_{i=0}^{k}$ of vertices such that $x_0 = x,\ x_k = y$ and $(x_{i-1}, x_i) \in E(G)$, for $i = 1,2,3,\dots,k$. A directed graph G is said to be transitive if, for any $x,y,z \in V(G)$ such that (x,y) and (y,z) are in $E(G)$, we have $(x,z) \in E(G)$. For more detail of graph theory refer Diestel [8].

Definition 1.1. [13] A self map $T : C \to C$ is called G-contraction if there is a $\lambda \in [0,1)$ such that

(i) T preserves edges of G, i.e., $(x,y) \in E(G) \Rightarrow (Tx,Ty) \in E(G)$, and
(ii) $||Tx - Ty|| \leq \lambda ||x - y||$ for each $(x,y) \in E(G)$.

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Definition 1.2. [1] A self map $T : C \to C$ is called G-nonexpansive if it satisfies the conditions

(i) T preserves edges of G, and
(ii) $||Tx - Ty|| \leq ||x - y||$ for each $(x,y) \in E(G)$.

Definition 1.3. [1] Let C be a nonempty subset of a normed space X and let $G=(V(G),E(G))$ be a digraph such that $V(G)=C$. Then, C is said to have *Property P*, if for each sequence $\{x_n\}$ in C converging weakly to $x\in C$ and $(x_n,x_{n+1})\in E(G)$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k},x)\in E(G)$ for all $k\in\mathbb{N}$.

Remark 1.1. If G is transitive, then Property P is equivalent to the property: if $\{x_n\}$ is a sequence in C with $(x_n,x_{n+1})\in E(G)$ such that for any subsequence $\{x_{n_j}\}$ of the sequence $\{x_n\}$ converging weakly to x in X, then $(x_n,x)\in E(G)$ for all $n\in\mathbb{N}$.

Definition 1.4. Let C be a nonempty subset of a Banach space X endowed with a digraph $G=(V(G),E(G))$ such that $V(G)=C$ and let $T:C\to X$ be a mapping. Then T is said to be G-demiclosed at $y\in X$, if for any sequence $\{x_n\}$ in C with (x_n,x_{n+1}) and (x_n,Tx_n) are in $E(G)$ such that $\{x_n\}$ converges weakly to $x\in C$ and $\{Tx_n\}$ converges strongly to y imply $Tx=y$.

The concept of Monotone G-nonexpansive self-mappings in a Banach space with topology τ, which is weaker than the norm topology, is first introduced by Alfraidan [1] in 2015. In [1], the author studied the τ-convergence of Krasnoselskii sequence to fixed points of such class of mappings. Tiammee et al.[36] proved Browders theorem and the convergence of Halpern iteration for a G-nonexpansive mapping in a Hilbert space with a directed graph. In 2016, Tripak [37] proved weak and strong convergence of the Ishikawa iteration scheme to common fixed points of a couple of G-nonexpansive mappings in a Banach space with a directed graph. In [31], the author defined the concept of dominance in the following way.

Definition 1.5. [31] Let $x_1\in V(G)$ and A a subset of $V(G)$. We say that

(i) A is dominated by x_1 if $(x_1,x)\in E(G)$ for all $x\in A$.

(ii) A dominates x_1 if for each $x\in A$, $(x,x_1)\in E(G)$.

Using the concept of dominance assumptions, the author [37] proved the following convergence theorems.

Theorem 1.2. [37] *Let C be a nonempty closed convex subset of a uniformly convex Banach space endowed with a transitive directed graph $G=(V(G),E(G))$, such that $V(G)=C$ and $E(G)$ is convex. Let $T_i (i=1,2)$ be G-nonexpansive mappings from C to C with $F=$*

$F(T_1)\bigcap F(T_2)$ nonempty. Let $\{\alpha_n\}$, $\{\beta_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$. Let $\{x_n\}$ be a sequence generated from arbitrary $x_0 \in C$ given by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1 y_n \\ y_n = (1-\beta_n)x_n + \beta_n T_2 x_n \end{cases} \tag{1.2}$$

for $n = 0, 1, 2, \ldots$. Suppose that $T_i(i = 1, 2)$ satisfy the following conditions:

(1) There exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that, for all $x \in C$,

$$\max\{||x - T_1 x||, ||x - T_2 x||\} \geq f(d(x, F));$$

(2) F dominates x_0;

(3) F is dominated by x_0; and

(4) For each $z \in F$ and arbitrary $x_0 \in C$

$$(x_0, z), (y_0, z), (z, x_0), (z, y_0) \in E(G).$$

Then $\{x_n\}$ converges strongly to a common fixed point of T_i.

Definition 1.6. [29] Let C be a subset of a metric space (X, d). A mapping $T : C \to C$ is semi-compact if for a sequence $\{x_n\}$ in C with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in C$ as $j \to \infty$.

Theorem 1.3. [37] *Let C be a nonempty closed convex subset of a uniformly convex Banach space endowed with a transitive directed graph $G = (V(G), E(G))$, such that $V(G) = C$ and $E(G)$ is convex. Let $T_i(i = 1, 2)$ be G-nonexpansive mappings from C to C with $F = F(T_1)\bigcap F(T_2)$ nonempty. Let $\{\alpha_n\}$, $\{\beta_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$. Suppose that F dominates x_0, F is dominated by x_0 and $(x_0, z), (y_0, z), (z, x_0), (z, y_0) \in E(G)$ for each $z \in F$ and arbitrary $x_0 \in C$. Suppose that one of $T_i(i = 1, 2)$ is semi-compact. Then the sequence $\{x_n\}$ defined in (1.2) converges strongly to a common fixed point of T_i.*

Theorem 1.4. [37] *Let C be a nonempty closed convex subset of a uniformly convex Banach space X endowed with a transitive directed graph $G = (V(G), E(G))$, such that $V(G) = C$ and $E(G)$ is convex. Suppose X satisfies the Opial's property. Let $T_i(i = 1, 2)$ be G-nonexpansive* *mappings from C to C with $F = F(T_1)\bigcap F(T_2)$ nonempty. If $I - T_i$ is G-demiclosed at zero for each i, F dominates x_0, F is dominated by x_0 and $(x_0,z_0),(y_0,z_0),(z_0,x_0),\ (z_0,y_0) \in E(G)$ for $z_0 \in F$ and arbitrary $x_0 \in C$, then the sequence $\{x_n\}$ defined in (1.2) converges weakly to a common foxed point of T_i.*

In 2002, Xu and Noor [39] used a modified three step iterative method:

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n T^n x_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n z_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n,\ n = 1,2,3,,\cdots. \end{cases}$$

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ are sequences of real numbers in $[0,1]$ to approximate fixed points of asymptotically nonexpansive mappings in Banach spaces. In 2008, Khan et al.[14] extended the work of Xu and Noor [39], from one mapping to a finite family of mappings using the modified Noor iterative method:

$$\begin{cases} x_{n+1} = (1-\alpha_{k,n})x_n + \alpha_{k,n}T_k^n y_{k-1,n}, \\ y_{k-1,n} = (1-\alpha_{k-1,n})x_n + \alpha_{k-1,n}T_{k-1}^n y_{k-2,n}, \\ y_{k-2,n} = (1-\alpha_{k-2,n})x_n + \alpha_{k-2,n}T_{k-2}^n y_{k-3,n}, \\ \vdots \\ y_{2,n} = (1-\alpha_{2,n})x_n + \alpha_{2,n}T_2^n y_{1,n}, \\ y_{1,n} = (1-\alpha_{1,n})x_n + \alpha_{1,n}T_1^n x_n, \end{cases} \tag{1.3}$$

where $y_{0,n} = x_n$ for each $n \in \mathbb{N}$ and arbitrary $x_1 \in C$.

The purpose of this article is three fold:

(1) To introduce *G*-asymptotically nonexpansive self-mappings of a closed convex subset of a Banach space with digraph;

(2) To show that *G*-asymptotically nonexpansive self-mapping has *G*-demiclosedness property on a closed convex subset of a Banach space with digraph;

(3) To investigate approximations of fixed points of *G*-asymptotically nonexpansive self-mappings of a closed convex subset of a Banach space with digraph; in particular to

study some weak and strong convergence theorems for the sequence generated by the modified Noor iteration methods to common fixed points of a finite family of such mappings in real uniformly convex Banach spaces with digraph.

2. Preliminaries

The following technical Lemmas are crucial in proving our main results of the article.

Lemma 2.1. [7] *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers with $\sum_{n=1}^{\infty} b_n < \infty$. If one of the following conditions is satisfied:*

(i) $a_{n+1} \leq a_n + b_n,\ n \geq 1,$

(ii) $a_{n+1} \leq (1+b_n)a_n,\ n \geq 1,$

then $\lim_{n\to\infty} a_n$ *exists.*

Lemma 2.2. [40] *Let X be a Banach space, and $R > 1$ be a fixed number. Then X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$||\lambda x + (1-\lambda)y||^2 \leq \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)g(||x-y||)$$

for all $x, y \in B_R(0) = \{x \in X : ||x|| \leq R\}$ and $\lambda \in [0,1]$.

Lemma 2.3. [27] *Let X be a uniformly convex Banach space and $\{\alpha_n\}$ a sequence in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. Suppose that sequences $\{x_n\}$ and $\{y_n\}$ in X are such that $\limsup_{n\to\infty} ||x_n|| \leq c$, $\limsup_{n\to\infty} ||y_n|| \leq c$ and $\lim_{n\to\infty} ||\alpha_n x_n + (1-\alpha_n)y_n|| = c$ for some $c \geq 0$ then $\lim_{n\to\infty} ||x_n - y_n|| = 0$.*

Lemma 2.4. [7] *Let X be a uniformly convex Banach space, C be a nonempty bounded convex subset of X. Then there exists a strictly increasing continuous convex function $\gamma : [0,\infty) \to [0,\infty)$ with $\gamma(0) = 0$ such that, for any Lipschitzian mapping $T : C \to X$ with the Lipschitz constant $L \geq 1$, any finite many elements $\{x_i\}_{i=1}^n$ in C and any finite many nonnegative numbers $\{t_i\}_{i=1}^n$ with $\sum_{i=1}^{n} t_i = 1$, the following inequality holds:*

$$||T(\sum_{i=1}^{n} t_i x_i) - \sum_{i=1}^{n} t_i T x_i|| \leq L\gamma^{-1} \max_{1\leq i,j\leq n} (||x_i - x_j|| - L^{-1}||Tx_i - Tx_j||).$$

Lemma 2.5. [26] *Let $\{x_n\}$ be a bounded sequence in a reflexive Banach space X. If for any weakly convergent subsequence $\{x_{n_j}\}$ of $\{x_n\}$, both $\{x_{n_j}\}$ and $\{x_{n_j+1}\}$ converge weakly to the same point in X, then the sequence $\{x_n\}$ is weakly convergent.*

3. Main results

Throughout this section C denotes a nonempty closed convex subset of a real uniformly convex Banach space X endowed with a directed graph $G = (V(G), E(G))$ such that $V(G) = C$ and $E(G)$ is convex. We also suppose that the graph G is transitive.

Definition 3.1. A self map $T : C \to C$ is said to be G-asymptotically nonexpansive if it satisfies the conditions:

(i) T preserves edges of G, and

(ii) there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\sum_{n=1}^{\infty}[k_n - 1] < \infty$ and for each $(x,y) \in E(G)$ and $n \in \mathbb{N}$

$$||T^n x - T^n y|| \leq k_n ||x - y||.$$

Proposition 3.1. *Let $\{T_i\}_{i=1}^{k}$ be a family of G-asymptotically nonexpansive mappings on C such that $F = \bigcap_{i=1}^{k} F(T_i)$ nonempty. Let $z \in F$ be such that (x_1, z) and (z, x_1) are in $E(G)$ for arbitrary $x_1 \in C$. Then, for a sequence $\{x_n\}$ generated by x_1 with iterative scheme defined by (1.3), we have $(x_n, z), (z, x_n), (x_n, y_{i,n}), (y_{i,n}, x_n), (z, y_{i,n})$, $(y_{i,n}, z)$ and (x_n, x_{n+1}) are in $E(G)$ for each $i = 1, 2, 3, \cdots, k$ and $n = 1, 2, 3, \cdots$.*

Proof. We proceed by induction. First we let $(x_1, z) \in E(G)$. Since T_1 is edge-preserving, we have $(T_1 x_1, z) \in E(G)$. By the convexity of $E(G)$, we have

$$(1-\alpha_{1,1})(x_1, z) + \alpha_{1,1}(T_1 x_1, z) = ((1-\alpha_{1,1})x_1 + \alpha_{1,1} T_1 x_1, z) = (y_{1,1}, z),$$

so that $(y_{1,1}, z) \in E(G)$. Since T_2 is edge-preserving, $(T_2 y_{1,1}, z) \in E(G)$ and again by the convexity of $E(G)$ we have

$$(1-\alpha_{2,1})(x_1, z) + \alpha_{2,1}(T_2 y_{1,1}, z) = ((1-\alpha_{2,1})x_1 + \alpha_{2,1} T_2 y_{1,1}, z) = (y_{2,1}, z),$$

so that $(y_{2,1},z)\in E(G)$. Assume that $(y_{l,1},z)\in E(G)$, for some $l\in\{1,2,3,\cdots,k-2\}$. As T_{l+1} is edge-preserving, $(T_{l+1}y_{l,1},z)\in E(G)$ and by using the convexity of $E(G)$, we get

$$(1-\alpha_{l+1,1})(x_1,z)+\alpha_{l+1,1}(T_{l+1}y_{l,1},z)= \quad ((1-\alpha_{l+1,1})x_1+\alpha_{l+1,1}T_{l+1}y_{l,1},z)= \quad (y_{l+1,1},z),$$

so that $(y_{l+1,1},z)\in E(G)$. Thus $(y_{i,1},z)\in E(G)$ for each $i=1,2,3,\cdots,k-1$.

In particular, for $i=k-1$

$$(y_{k-1,1},z)\in E(G).$$

Since T_k is edge-preserving, we have

$$(T_k y_{k-1,1},z)\in E(G).$$

Using the convexity of $E(G)$, we have

$$(1-\alpha_{k,1})(x_1,z)+\alpha_{k,1}(T_k y_{k-1,1},z)= \quad ((1-\alpha_{k,1})x_1+\alpha_{k,1}T_k y_{k-1,1},z)= \quad (x_2,z),$$

so that $(x_2,z)\in E(G)$. Thus, we obtain $(y_{i,1},z)\in E(G)$ for $i=1,2,3,\cdots,k-1$ and (x_2,z) is also in $E(G)$.

Since $\{T_i\}_{i=1}^k$ are edge preserving, $\{T_i^2\}_{i=1}^k$ are also edge preserving. Thus, repeating the previous process for (x_2,z) in place of (x_1,z) and using the operators T_i^2 in place of T_i, we obtain $(y_{i,2},z)\in E(G)$ for $i=1,2,3,\cdots,k-1$ so that (x_3,z) is also in $E(G)$.

Assume that $(x_m,z)\in E(G)$ for some $m\in\mathbb{N}$. Since T_i is edge-preserving, we have T_i^m are also edge preserving and hence, we have $(T_1^m x_m,z)\in E(G)$ and by using the convexity of $E(G)$, we get

$$(1-\alpha_{1,m})(x_m,z)+\alpha_{1,m}(T_1^m x_m,z)= \quad ((1-\alpha_{1,m})x_m+\alpha_{1,m}T_1^m x_m,z)= \quad (y_{1,m},z),$$

so that $(y_{1,m},z)\in E(G)$. As T_2^m is edge-preserving, $(T_2^m y_{1,m},z)\in E(G)$, as $E(G)$ is convex, we have

$$(1-\alpha_{2,m})(x_m,z)+\alpha_{2,m}(T_2^m y_{1,m},z)= \quad ((1-\alpha_{2,m})x_m+\alpha_{2,m}T_2^m y_{1,m},z)= \quad (y_{2,m},z),$$

so that $(y_{2,m},z)\in E(G)$. By repeating the process, we conclude that $(y_{i,m},z)$ and (x_{m+1},z) are in $E(G)$ for all $i=1,2,3,\cdots,k-1$.

Continuing the process once again for (x_{m+1}, z), we have $(y_{i,m+1}, z) \in E(G)$ for all $i = 1,2,3,\cdots,k-1$. Therefore, by induction, we conclude that $(x_n, z), (y_{i,n}, z) \in E(G)$ for all $i = 1,2,3,\cdots,k-1$ and $n = 1,2,3,\cdots$.

Using a similar argument, we can show that $(z, x_n), (z, y_{i,n}) \in E(G)$ for all $i = 1,2,3,\cdots,k-1$ and $n = 1,2,3,\cdots$, under the assumption that $(z, x_1) \in E(G)$. The transitivity property of G implies that $(x_n, x_{n+1}), (x_n, y_{i,n}), (y_{i,n}, x_n)$ are in $E(G)$ for all $i = 1,2,3,\cdots,k-1$ and $n = 1,2,3,\cdots$. This completes the proof.

Lemma 3.2. *Let $\{T_i\}_{i=1}^k$ be a finite family of G-asymptotically nonexpansive mappings on C such that $F = \bigcap_{i=1}^k F(T_i)$ nonempty. Suppose that for all $(x,y) \in E(G)$,*

$$||T_i^n x - T_i^n y|| \leq (1 + u_{i,n})||x - y||$$

where $\{u_{i,n}\} \subset [0,\infty)$ with $\sum_{n=1}^{\infty} u_{i,n} < \infty$ for each $i \in \{1,2,3,\cdots,k\}$. Suppose that (x_1, z) and (z, x_1) are in $E(G)$ for arbitrary $x_1 \in C$ and $z \in F$. If $\{x_n\}$ is the sequence generated by (1.3) with $\{\alpha_{i,n}\} \subset [\delta, 1-\delta]$ for some δ in $(0,1)$, then

(i) $\lim_{n\to\infty} ||x_n - z||$ *exists;*

(ii) $\lim_{n\to\infty} ||x_n - T_i^n y_{i-1,n}|| = 0,$ *for each* $i = 2,3,4,\cdots,k$;

(iii) $\lim_{n\to\infty} ||x_n - T_i^n x_n|| = 0,$ *for each* $i = 1,2,3,\cdots,k$;

(iv) $\lim_{n\to\infty} ||x_n - T_i x_n|| = 0,$ *for each* $i = 1,2,3,\cdots,k$.

Proof. First we prove (i). Let $x_1 \in C$ and $z \in F$ be as in the hypothesis and let $\{x_n\}$ be a sequence generated by (1.3). By Proposition 3.1., $(x_n, z), (z, x_n), (x_n, y_{i,n}), (y_{i,n}, x_n)$ and (x_n, x_{n+1}) are in $E(G)$. Set $v_n = \max_{1\leq i\leq k} u_{i,n}$, for all n. Since $\sum_{n=1}^{\infty} u_{i,n} < \infty$, for each i, we must have

$$\sum_{n=1}^{\infty} v_n < \infty. \tag{3.1}$$

Now by the G-asymptotically nonexpansiveness of T_1 and (1.3), we have

$$\begin{aligned} ||y_{1,n} - z|| &\leq (1 - \alpha_{1,n})||x_n - z|| + \alpha_{1,n}||T_1^n x_n - z|| \\ &\leq (1 - \alpha_{1,n})||x_n - z|| + \alpha_{1,n}(1 + u_{1,n})||x_n - z|| \\ &= (1 + \alpha_{1,n} u_{1,n})||x_n - z_0|| \\ &\leq (1 + v_n)||x_n - z||. \end{aligned}$$

Thus

$$||y_{1,n}-z|| \leq \quad (1+v_n)||x_n-z||. \tag{3.2}$$

Assume that, for some $m \in \{1,2,3,\cdots,k-2\}$,

$$||y_{m,n}-z|| \leq (1+v_n)^m||x_n-z||. \tag{3.3}$$

By G-asymptotically nonexpansiveness of T_{m+1} and using (1.3) and (3.3), we have

$$\begin{aligned}
||y_{m+1,n}-z|| &\leq (1-\alpha_{m+1,n})||x_n-z||+\alpha_{m+1,n}||T^n_{m+1}y_{m,n}-z|| \\
&\leq (1-\alpha_{m+1,n})||x_n-z||+\alpha_{m+1,n}(1+u_{m+1,n})||y_{m,n}-z|| \\
&\leq (1-\alpha_{m+1,n})||x_n-z|| \\
&\quad +\alpha_{m+1,n}(1+u_{m+1,n})(1+v_n)^m||x_n-z|| \\
&\leq (1-\alpha_{m+1,n})||x_n-z||+\alpha_{m+1,n}(1+v_{m,n})^{m+1}||x_n-z|| \\
&= [1-\alpha_{m+1,n}+\alpha_{m+1,n}(1+\sum_{j=1}^{m+1}\binom{m+1}{j}v_n^j)]||x_n-z|| \\
&= (1+\alpha_{m+1,n}\sum_{j=1}^{m+1}\binom{m+1}{j}v_n^j)||x_n-z|| \\
&\leq (1+\sum_{j=1}^{m+1}\binom{m+1}{j}v_n^j)||x_n-z|| \\
&= (1+v_n)^{m+1}||x_n-z||,
\end{aligned}$$

where

$$\binom{r}{s}=\frac{r!}{(r-s)!s!}.$$

Thus, for each $i=1,2,3,\cdots,k-1$, we have

$$||y_{i,n}-z|| \leq \quad (1+v_n)^i||x_n-z||. \tag{3.4}$$

In particular for $i = k-1$, we have

$$\begin{aligned} ||x_{n+1} - z|| &= ||(1-\alpha_{k,n})(x_n - z) + \alpha_{k,n}(T_k^n y_{k-1,n} - z)|| \\ &\leq (1-\alpha_{k,n})||x_n - z|| + \alpha_{k,n}(1+u_{k,n})||y_{k-1,n} - z|| \\ &\leq (1-\alpha_{k,n})||x_n - z|| + \alpha_{k,n}(1+u_{k,n})(1+v_n)^{k-1}||x_n - z|| \\ &\leq (1-\alpha_{k,n})||x_n - z|| + \alpha_{k,n}(1+v_n)^{k}||x_n - z|| \\ &= [1-\alpha_{k,n} + \alpha_{k,n}(1+\sum_{j=1}^{k}\binom{k}{j}v_n^j\ )]||x_n - z|| \\ &\leq (1+\sum_{j=1}^{k}\binom{k}{j}v_n^j\ )||x_n - z||. \end{aligned}$$

Therefore, for each $n = 1, 2, 3, \cdots$, we have

$$||x_{n+1} - z|| \leq \ (1+\sum_{j=1}^{k}\binom{k}{j}v_n^j\ )||x_n - z||. \tag{3.5}$$

If we set $b_n = \sum_{j=1}^{k}\binom{k}{j}v_n^j$, we have

$$b_n = \sum_{j=1}^{n}\binom{k}{j}v_n^j \leq \sum_{j=1}^{n}\binom{k}{j}v_n = v_n(2^k - 1). \tag{3.6}$$

Using (3.1) and (3.6), we obtain that

$$\sum_{n=1}^{\infty} b_n \leq (2^k - 1)\sum_{n=1}^{\infty} v_n < \infty. \tag{3.7}$$

Using (3.5), (3.7) to apply (ii) of Lemma 2.1 with $a_n = ||x_n - z||$, we conclude that $\lim_{n\to\infty} ||x_n - z||$ exists.

Next, we prove (ii). From (i), we have $\lim_{n\to\infty} ||x_n - z||$ exists and hence $\{x_n\}$ is a bounded sequence. Let

$$\lim_{n\to\infty} ||x_n - z|| = \ c \text{ for some } c \geq 0. \tag{3.8}$$

From (3.4), for each $m \in \{1, 2, 3, \cdots, k-1\}$, we have

$$||y_{m,n} - z|| \leq \ (1+v_n)^m ||x_n - z||$$

and using (3.8), we get

$$\limsup_{n\to\infty} ||y_{m,n} - z|| \leq c. \tag{3.9}$$

On the other hand, from (1.3) we have

$$\begin{aligned}
||x_{n+1}-z|| &\leq (1-\alpha_{k,n})||x_n-z||+\alpha_{k,n}(1+v_n)||y_{k-1,n}-z)|| \\
&\leq (1-\alpha_{k,n})||x_n-z||+\alpha_{k,n}(1+v_n)[(1-\alpha_{k-1,n})||x_n-z|| \\
&\quad +\alpha_{k-1,n}(1+v_n)||y_{k-2,n}-z||] \\
&= [1-\alpha_{k,n}+\alpha_{k,n}(1+v_n)(1-\alpha_{k-1,n})]||x_n-z|| \\
&\quad +\alpha_{k,n}\alpha_{k-1,n}(1+v_n)^2||y_{k-2,n}-z|| \\
&= (1-\alpha_{k,n}\alpha_{k-1,n}+\alpha_{k,n}v_n-\alpha_{k,n}\alpha_{k-1,n}v_n)||x_n-z||+ \\
&\quad +\alpha_{k,n}\alpha_{k-1,n}(1+v_n)^2||y_{k-2,n}-z|| \\
&\leq (1+v_n)[(1-\alpha_{k,n}\alpha_{k-1,n}||x_n-z|| \\
&\quad +\alpha_{k,n}\alpha_{k-1,n})(1+v_n)(1-\alpha_{k-2,n})||x_n-z|| \\
&\quad +\alpha_{k-2,n}(1+v_n)||y_{k-3,n}-z||] \\
&\leq (1-\alpha_{k,n}\alpha_{k-1,n}\alpha_{k-2,n})(1+v_n)^2||x_n-z||^2 \\
&\quad +\alpha_{k,n}\alpha_{k-1,n}\alpha_{k-2,n}(1+v_n)^3||y_{k-3,n}-z||.
\end{aligned}$$

Continuing the process, we obtain

$$\begin{aligned}
||x_{n+1}-z|| &\leq (1-\alpha_{k,n}\alpha_{k-1,n}\cdots\alpha_{j+1,n})(1+v_n)^{k-j-1}||x_n-z|| \\
&\quad +\alpha_{k,n}\alpha_{k-1,n}\cdots\alpha_{j+1,n}(1+v_n)^{k-j}||y_{j,n}-z||
\end{aligned}$$

for each $j=1,2,3,\cdots,k-1$.

By rearranging, we get

$$\begin{aligned}
\frac{||x_{n+1}-z||}{(1+v_n)^{k-j-1}} &\leq (1-\alpha_{k,n}\alpha_{k-1,n}\cdots\alpha_{j+1,n})||x_n-z||+ \\
&\quad \alpha_{k,n}\alpha_{k-1,n}\cdots\alpha_{j+1,n}(1+v_n)||y_{j,n}-z||.
\end{aligned}$$

Which was simplified to

$$\Big(\frac{||x_{n+1}-z||}{(1+v_n)^{k-j-1}}-||x_n-z||\Big)\frac{1}{\alpha_{k,n}\alpha_{k-1,n}\cdots\alpha_{j+1,n}}+||x_n-z||\leq(1+v_n)||y_{j,n}-z||.$$

Since $\alpha_{i,n}\in[\delta,1-\delta]$, the above inequality was further simplified to

$$\begin{aligned}(\frac{||x_{n+1}-z||}{(1+v_n)^{k-j-1}}-||x_n-z||)\frac{1}{(1-\delta)^{k-j}}+||x_n-z|| \\ \leq (1+v_n)||y_{j,n}-z||.\end{aligned} \tag{3.10}$$

Taking limit inferior of (3.10) and using (3.8), we get

$$c\leq \liminf_{n\to\infty}||y_{j,n}-z||. \tag{3.11}$$

Thus, from (3.9) and (3.11), we conclude that for each $j=1,2,3,\cdots,k-1$

$$\lim_{n\to\infty}||y_{j,n}-z|| = c. \tag{3.12}$$

Thus, for $j=2,3,4,\cdots,k-1$, we have

$$\lim_{n\to\infty}||(1-\alpha_{j,n})(x_n-z)+\alpha_{j,n}(T_j^n y_{j-1,n}-z)|| = c. \tag{3.13}$$

For $j=2,3,4,\cdots,k-1$, we have

$$||T_j^n y_{j-1,n}-z|| \leq (1+u_{j,n})||y_{j-1,n}-z||$$

and hence

$$\limsup_{n\to\infty}||T_j^n y_{j-1,n}-z|| \leq c. \tag{3.14}$$

Using (3.8), (3.13), (3.14) and apply Lemma 2.3, we have

$$\lim_{n\to\infty}||x_n-T_j^n y_{j-1,n}|| = 0 \text{ for } j=2,3,4,\cdots,k-1. \tag{3.15}$$

For the case $j=k$, using (3.4) and G-asymptotically nonexpansiveness of T_k, we have

$$||T_k^n y_{k-1,n}-z|| \leq (1+u_{k,n})||y_{k-1,n}-z|| \leq (1+v_n)^k||x_n-z||. \tag{3.16}$$

Taking limit superior of (3.16) and using (3.8), we obtain

$$\limsup_{n\to\infty}||T_k^n y_{k-1,n}-z|| \leq c.$$

Since

$$\lim_{n\to\infty}\|x_{n+1}-z\| = \lim_{n\to\infty}\|(1-\alpha_{k,n})(x_n-z)-\alpha_{k,n}(T_k^n y_{k-1,n}-z)\| = c.$$

Applying Lemma 2.3 once again, we get

$$\lim_{n\to\infty}||x_n-T_k^n y_{k-1,n}|| = 0. \tag{3.17}$$

Therefore, from (3.15) and (3.17), for each $j = 2,3,4,\cdots,k$, we obtain that

$$\lim_{n\to\infty} ||x_n - T_j^n y_{j-1,n}|| = 0.$$

Next, we show (iii). Since for each $i = 1,2,\cdots,k-1$,

$$\lim_{n\to\infty} ||x_n - z|| = \lim_{n\to\infty} ||y_{i,n} - z|| = c,$$

we have $\{x_n\}$ and $\{y_{i,n} - z\}$ are bounded sequences and hence $\{T_i^n y_{i,n} - z\}$ is also bounded. Therefore, there is $R > 0$ such that

$$\bigcup_{i=1}^{k-1} \{x_n\} \bigcup \{y_{i,n}\} \bigcup \{T_i^n y_{i,n}\} \subset \overline{B(z,R)}.$$

By Lemma 2.2, there is continuous and strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ such that

$$\begin{aligned} ||y_{1,n} - z||^2 &\leq (1-\alpha_{1,n})||x_n - z||^2 + \alpha_{1,n}||T_1^n x_n - z||^2 \\ &\quad -\alpha_{1,n}(1-\alpha_{1,n})g(||x_n - T_1^n x_n||) \\ &\leq (1-\alpha_{1,n})||x_n - z||^2 + \alpha_{1,n}(1+u_{1,n})^2||x_n - z||^2 \\ &\quad -\delta^2 g(||x_n - T_1^n x_n||) \\ &\leq (1+u_{1,n})^2||x_n - z||^2 - \delta^2 g(||x_n - T_1^n x_n||) \end{aligned}$$

On rearranging, we obtain

$$\delta^2 g(||x_n - T_1^n x_n||) \leq (1+u_{1,n})^2||x_n - z||^2 - ||y_{1,n} - z||^2. \tag{3.18}$$

Taking the superior limit of (3.18) and using (3.8) and (3.12) to get

$$\delta^2 \limsup_{n\to\infty} g(||x_n - T_1^n x_n) \leq 0.$$

Which gives that

$$\lim_{n\to\infty} g(||x_n - T_1^n x_n||) = 0.$$

Since g is continuous and monotonically increasing we conclude that

$$\lim_{n\to\infty} ||x_n - T_1^n x_n|| = 0. \tag{3.19}$$

Again,

$$\begin{aligned} ||x_n - T_2^n x_n|| &\leq ||x_n - T_2^n y_{1,n}|| + ||T_2^n y_{1,n} - T_2^n x_n|| \\ &\leq ||x_n - T_2^n y_{1,n}|| + (1+u_{2,n})||y_{1,n} - x_n|| \\ &= ||x_n - T_2^n y_{1,n}|| + \alpha_{2,n}(1+u_{2,n})||x_n - T_1^n x_n||. \end{aligned}$$

Which implies that

$$||x_n - T_2^n x_n|| \leq ||x_n - T_2^n y_{1,n}|| + \alpha_{2,n}(1+u_{2,n})||x_n - T_1^n x_n||.$$

Applying (ii) of Lemma 3.2, and applying (3.19) and the fact that the sequence $\{\alpha_{2,n}(1+u_{2,n})\}$ is bounded, we have

$$\lim_{n\to\infty} ||x_n - T_2^n x_n|| = 0. \tag{3.20}$$

Repeatedly we apply Lemma 2.2, for $i = 3,4,5,\cdots,k-1$ and get

$$\begin{aligned} ||y_{i,n} - z||^2 &\leq (1-\alpha_{i,n})||x_n - z||^2 + \alpha_{i,n}||T_i^n y_{i-1,n} - z||^2 \\ &\quad -\alpha_{i,n}(1-\alpha_{i,n})g(||x_n - T_i^n y_{i-1,n}||) \\ &\leq (1-\alpha_{i,n})||x_n - z||^2 + \alpha_{i,n}||T_i^n y_{i-1,n} - z||^2 \\ &\quad -\delta^2 g(||x_n - T_i^n y_{i-1,n}||) \\ &\leq (1-\alpha_{i,n})||x_n - z||^2 + \alpha_{i,n}(1+u_{i,n})^2||y_{i-1,n} - z||^2 \\ &\quad -\delta^2 g(||x_n - T_i^n y_{i-1,n}||). \end{aligned}$$

On rearranging, we obtain

$$\begin{aligned} &\delta^2 g(||x_n - T_i^n y_{i-1,n}||) \\ &\leq (1-\alpha_{i,n})||x_n - z||^2 - ||y_{i-1,n} - z||^2 - \alpha_{i,n}(1+u_{i,n})^2||y_{i-1,n} - z||^2 \\ &= (||x_n - z||^2 - ||y_{i,n} - z||^2) + \alpha_{i,n}[(1+u_{i,n})^2||y_{i-1,n} - z||^2 - ||x_n - z||^2 \\ &\leq 2R(||x_n - z|| - ||y_{i,n} - z||)[1 - \alpha_{i,n}(1+u_{i,n})^2]. \end{aligned} \tag{3.21}$$

Taking limit superior of (3.21) and using (3.8) and (3.12), we get

$$\delta^2 \limsup_{n\to\infty} g(||x_n - T_i^n y_{i-1,n}||) \leq 0.$$

Using property of g, we conclude that for each $i = 3,4,5,\cdots,k-1$,

$$\lim_{n\to\infty} ||x_n - T_i^n y_{i-1,n}|| = 0. \tag{3.22}$$

On the other hand, for $i = 3,4,5,\cdots,k$, we have

$$\begin{aligned} ||x_n - T_i^n x_n|| &\leq ||x_n - T_i^n y_{i-1,n}|| + ||T_i^n y_{i-1,n} - T_i^n x_n|| \\ &\leq ||x_n - T_i^n y_{i-1,n}|| + (1+u_{i,n})||y_{i-1,n} - x_n|| \\ &= ||x_n - T_i^n y_{i-1,n}|| + (1+u_{i,n})\alpha_{i-1,n}||T_{i-1}^n y_{i-2,n} - x_n||. \end{aligned}$$

This implies that for each $i = 3,4,5,\cdots,k$

$$||x_n - T_i^n x_n|| \leq ||x_n - T_i^n y_{i-1,n}|| + (1+u_{i,n})\alpha_{i-1,n}||T_{i-1}^n y_{i-2,n} - x_n|| \tag{3.23}$$

Taking limit superior of (3.23) and using (3.22), we get that

$$\limsup_{n\to\infty} ||x_n - T_i^n x_n|| \leq 0. \tag{3.24}$$

Which implies that, for each $i = 3,4,5,\cdots,k$, we have

$$\lim_{n\to\infty} ||x_n - T_i^n x_n|| = 0. \tag{3.25}$$

Thus, from (3.19), (3.20) and (3.25), we conclude that for each $i = 1,2,3,\cdots,k$,

$$\lim_{n\to\infty} ||x_n - T_i^n x_n|| = 0. \tag{3.26}$$

Finally, we prove (iv). Fix $m \in \{1,2,3,\cdots,k\}$ but arbitrary. By the G-asymptotically nonexpansiveness of T_m we have that

$$\begin{aligned} ||x_n - T_m x_n|| &\leq ||x_n - x_{n+1}|| + ||x_{n+1} - T_m^{n+1} x_{n+1}|| \\ &\quad + ||T_m^{n+1} x_{n+1} - T_m^{n+1} x_n|| + ||T_m^{n+1} x_n - T_m x_n|| \\ &\leq ||x_n - x_{n+1}|| + ||x_{n+1} - T_m^{n+1} x_{n+1}|| \\ &\quad + k_{n+1}||x_n - x_{n+1}|| + k_1||x_n - T_m^n x_n|| \\ &= (1+k_{n+1})||x_n - x_{n+1}|| + k_1||x_n - T_m^n x_n|| \\ &\quad + ||x_{n+1} - T_m^{n+1} x_{n+1}||. \end{aligned}$$

where $k_n = \max_{1\le i\le k}(1+u_{i,n})$. This implies that

$$||x_n - T_m x_n|| \le \quad (1+k_{n+1})||x_n - x_{n+1}|| + k_1||x_n - T_m^n x_n|| + ||x_{n+1} - T_m^{n+1}x_{n+1}||. \qquad (3.27)$$

From (1.3), we have that

$$||x_{n+1} - x_n|| = \alpha_{k,n}||x_n - T_k^n y_{k-1,n}||. \qquad (3.28)$$

Since $\alpha_{k,n} \in [\delta, 1-\delta]$, using (iii) and (3.28), we obtain

$$\lim_{n\to\infty} ||x_n - x_{n+1}|| = \quad 0. \qquad (3.29)$$

Taking the superior limit of (3.27) and using (3.26) and (3.29), we conclude that

$$\lim_{n\to\infty} ||x_n - T_m x_n|| = \quad 0. \qquad (3.30)$$

Since m was arbitrary, we obtain the required result. This completes the proof.

In the next theorem, we proved the G-Demiclosedness principle without assuming the Opial's property of the Banach space X.

Theorem 3.3. *Suppose that C has* **Property P** : $x_n \rightharpoonup x$ *and* $(x_n, x_{n+1}) \in E(G)$, *there exists a subsequence* $\{x_{n_k}\}$ *such that for each* k, $(x_{n_k}, x) \in E(G)$. *Let T be a G-asymptotically nonexpansive mapping on C with asymptotic coefficient* $\{k_n\}$ *such that*

$$\sum_{n=1}^{\infty}(k_n - 1) < \quad \infty.$$

Then $I - T$ is G-demiclosed at 0.

Proof. Let $\{x_n\}$ be a sequence in C with (x_n, x_{n+1}) and (x_n, Tx_n) are in $E(G)$ such that $x_n \rightharpoonup q \in C$ as $n \to \infty$ and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. By Property P, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $(x_{n_j}, q) \in E(G)$ for all $j \in \mathbb{N}$. By Remark 1.1, $(x_n, q) \in E(G)$ for all $n \in \mathbb{N}$.
We claim that, as $n \to \infty$

$$T^n q \quad \to \quad q.$$

Note that, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} ||x_k - T^n x_k|| &\leq (1+\sum_{r=1}^{n-1} k_r)||x_k - Tx_k|| \\ &= (1+\sum_{r=1}^{n-1}(1+u_r))||x_k - Tx_k|| \\ &= (n+\sum_{r=1}^{n-1} u_r)||x_k - Tx_k|| \\ &\leq (n+\sum_{r=1}^{\infty} u_r)||x_k - Tx_k|| \\ &\leq (n+M)||x_k - Tx_k|| \end{aligned}$$

where, $M = \sum_{r=1}^{\infty} u_r = \sum_{r=1}^{\infty}(k_r - 1) < \infty$.
Thus, we have

$$||x_k - T^n x_k|| \leq (n+M)||x_k - Tx_k||. \tag{3.31}$$

Since $\lim_{k\to\infty} ||x_k - Tx_k|| = 0$, we have that, for fixed $n \in \mathbb{N}$, there is a positive integer $N = N(n)$, such that

$$k \geq N \Rightarrow ||x_k - Tx_k|| < \frac{1}{(n+M)^2}. \tag{3.32}$$

Hence, from (3.31) and (3.32), we obtain

$$k \geq N \Rightarrow ||x_k - T^n x_k|| < \frac{1}{n+M}.$$

Therefore,

$$\overline{\lim_k} ||x_k - T^n x_k|| \leq \frac{1}{n+M}. \tag{3.33}$$

This implies that

$$\overline{\lim_n}\, \overline{\lim_k} ||x_k - T^n x_k|| = 0. \tag{3.34}$$

Therefore, for an arbitrary $\varepsilon > 0$, we can choose n_0 such that

$$\limsup_{k\to\infty} ||x_k - T^n x_k|| < \varepsilon, \ \forall n \geq n_0. \tag{3.35}$$

Since $x_n \rightharpoonup q$, by Mazur's theorem (Cf. [38]), for each positive integer k, there exists a convex combination $y_k = \sum_{i=1}^{p(k)} t_i^{(k)} x_{i+k}$ with $t_i^{(k)} \geq 0$ and $\sum_{i=1}^{p(k)} t_i^{(k)} = 1$ such that

$$||y_k - q|| < \frac{1}{k}. \tag{3.36}$$

Since $\{x_n\}$ weakly converges in a uniformly convex Banach space X, it is bounded and hence there exists $r > 0$ such that $\{x_n\} \subset D =: C \cap \overline{B(q,r)}$. Then D is nonempty closed convex subset of C. Thus, $T : D \to C$ is G-asymptotically nonexpansive mapping. Therefore, $T^n : D \to C$ is a Lipschitzian mapping with Lipschitz constant $k_n \geq 1$.

Again, we have

$$
\begin{aligned}
||T^n y_k - y_k|| &= ||T^n y_k - \sum_{i=1}^{p(k)} t_i^{(k)} T^n x_{i+k} + \sum_{i=1}^{p(k)} t_i^{(k)} T^n x_{i+k} - \sum_{i=1}^{p(k)} t_i^{(k)} x_{i+k}|| \\
&\leq ||T^n y_k - \sum_{i=1}^{p(k)} t_i T^n x_{i+k}|| + ||\sum_{i=1}^{p(k)} t_i^{(k)} T^n x_{i+k} - \sum_{i=1}^{p(k)} t_i^{(k)} x_{i+k}|| \qquad (3.37) \\
&\leq ||T^n y_k - \sum_{i=1}^{p(k)} t_i^{(k)} T^n x_{i+k}|| + \sum_{i=1}^{p(k)} t_i^{(k)} ||T^n x_{i+k} - x_{i+k}||.
\end{aligned}
$$

From (3.35), we get

$$
\sum_{i=1}^{p(k)} t_i^{(k)} ||T^n x_{i+k} - x_{i+k}|| < \varepsilon, \ \forall\, n \geq n_0. \qquad (3.38)
$$

Using Lemma 2.4, we obtain

$$
\begin{aligned}
||T^n y_k - \sum_{i=1}^{p(k)} t_i^{(k)} T^n x_{i+k}|| &\leq k_n \gamma^{-1} \{\max(||x_{i+k} - x_{i+p}|| - k_n^{-1} ||T^n x_{i+k} - T^n x_{i+p}||)\} \\
&\leq k_n \gamma^{-1} \{\max(2\varepsilon + (1 - k_n^{-1}) k_n ||x_{i+k} - x_{i+p}||)\}.
\end{aligned}
$$

Since $\{x_k\} \subset D$, we must have

$$
||x_{i+k} - x_{i+p}|| \leq 2r,
$$

so that

$$
||T^n y_k - \sum_{i=1}^{p(k)} t_i^{(k)} T^n x_{i+k}|| \leq k_n \gamma^{-1} (2\varepsilon + 2r(k_n - 1)). \qquad (3.39)
$$

Substituting (3.38) and (3.39) in (3.37), for each $k \in \mathbb{N}$ and $n \geq n_0$ we obtain

$$
||T^n y_k - y_k|| \leq k_n \gamma^{-1} (2\varepsilon + 2r(k_n - 1)) + \varepsilon \qquad (3.40)
$$

Thus we have

$$
\limsup_{k \to \infty} ||T^n y_k - y_k|| \leq k_n \gamma^{-1} (2\varepsilon + 2r(k_n - 1)) + \varepsilon. \qquad (3.41)
$$

On the other hand, for each $n \in \mathbb{N}$, we have

$$
||q - T^n q|| \leq ||q - y_k|| + ||y_k - T^n y_k|| + ||T^n y_k - T^n q||. \qquad (3.42)
$$

Since $y_k \to q$ as $k \to \infty$, which gives $y_k \rightharpoonup q$ and thus, by Property P and Remark 1.1, we have $(y_k, q) \in E(G)$, for all $k \in \mathbb{N}$. This in turn gives $(T^n y_k, T^n q) \in E(G)$, So that

$$||T^n y_k - T^n q|| \leq \quad k_n ||y_k - q||. \tag{3.43}$$

Substituting (3.43) in (3.42), we obtain

$$||q - T^n q|| \leq \quad (1 + k_n)||y_k - q|| + ||y_k - T^n y_k||. \tag{3.44}$$

From (3.36) and (3.44), we get

$$||q - T^n q|| \leq \quad \frac{1 + k_n}{k} + ||y_k - T^n y_k||. \tag{3.45}$$

Taking limit superior of (3.44) as $k \to \infty$, we have

$$||T^n q - q|| \leq \quad \limsup_{k \to \infty} ||y_k - T^n y_k||. \tag{3.46}$$

Combining (3.41) and (3.46), we infer that for all $n \geq n_0$

$$||T^n q - q|| \leq \quad k_n \gamma^{-1}(2\varepsilon + 2r(k_n - 1)) + \varepsilon. \tag{3.47}$$

Taking limit superior of (3.47) as $n \to \infty$ and using the arbitrariness of ε, we get

$$\limsup_{n \to \infty} ||T^n q - q|| \leq \gamma^{-1}(0) \quad = \quad 0.$$

Therefore,

$$\lim_{n \to \infty} ||q - T^n q|| = \quad 0. \tag{3.48}$$

But,

$$||q - Tq|| \leq ||q - T^{n+1} q|| + ||T^{n+1} q - Tq||. \tag{3.49}$$

Since $T^n q \to q$ as $n \to \infty$ and the fact that strong convergence implies weak convergence, we can obtain that $(T^n q, q) \in E(G)$, $\forall n \in \mathbb{N}$. As T is edge preserving, we have $(T^{n+1} q, Tq) \in E(G)$, so that

$$||T^{n+1} q - Tq|| \leq k_1 ||T^n q - q||. \tag{3.50}$$

From (3.49) and (3.50), we get that

$$||q - Tq|| \leq ||q - T^{n+1} q|| + k_1 ||q - T^n q||. \tag{3.51}$$

Taking limit superior of (3.51) and using (3.48), we obtain that

$$\|q - Tq\| \leq (1+k_1)\limsup_{n\to\infty}\|q - T^n q\| = 0.$$

This shows that

$$q = Tq.$$

This completes the proof.

Theorem 3.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and suppose that C has Property P. Let $\{T_i\}_{i=1}^k$ be a finite family of G-asymptotically nonexpansive mappings on C with the nonempty common fixed points set $F = \bigcap_{i=1}^k F(T_i)$. Let $x_1 \in C$ be fixed so that (x_1, z) and (z, x_1) are in $E(G)$ for some $z \in F$. If $\{x_n\}$ is a sequence generated by x_1 with iterative scheme (1.3) such that $\{\alpha_{i,n}\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$ and $\sum_{n=1}^\infty u_{i,n} < \infty$ for each $i = 1,2,3,\cdots,k$, then $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i\}_{i=1}^k$.*

Proof. Let $z \in F$ such that (x_1, z) and (z, x_1) are in $E(G)$. By Lemma 3.2, we have

$$\begin{aligned}
&1.\ \lim_{n\to\infty}||x_n - z|| \text{ exists.}\\
&2.\ \lim_{n\to\infty}||x_n - T_i^n x_n|| = 0, \text{ for } i = 1,2,3,\cdots,k.\\
&3.\ \lim_{n\to\infty}||x_n - T_i^n y_{i-1,n}|| = 0, \text{ for } i = 2,3,4,\cdots,k.\\
&4.\ \lim_{n\to\infty}||x_n - T_i x_n|| = 0, \text{ for } i = 1,2,3,\cdots,k.
\end{aligned} \tag{3.52}$$

From (3.52(1)), we see that $\{x_n\}$ is a bounded sequence in C. Since C is nonempty closed convex subset of a uniformly convex Banach space X, by the weak compactness of bounded sets there exists a subsequence $\{x_{n_h}\}$ of the sequence $\{x_n\}$ such that $\{x_{n_h}\}$ converges weakly to some point $p \in C$. It follows from (3.52(2)) that for each $i = 1,2,3,\cdots,k$,

$$\lim_{h\to\infty}||T_i^{n_h} x_{n_h} - x_{n_h}|| = 0.$$

By Proposition 3.1, we have (x_n, x_{n+1}) and $(x_n, T_i x_n)$ are in $E(G)$ for all $n \in \mathbb{N}$ and hence (x_{n_h}, x_{n_h+1}) and $(x_{n_h}, T_i x_{n_h})$ are also in $E(G)$ for each $i = 1,2,3,\cdots,k$. Thus, by Theorem 3.3 we conclude that $p \in F$.

To complete the proof it suffices to show that $\{x_n\}$ converges weakly to p. To this end we need to show that $\{x_n\}$ satisfies the hypothesis of Lemma 2.5.

Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ which converges weakly to some point $q \in C$. By similar arguments as above q is in F.

Now for each $j \geq 1$, we have

$$x_{n_j+1} = x_{n_j} + \alpha_{k,n_j}(T_k^{n_j} y_{k-1,n_j} - x_{n_j}). \tag{3.53}$$

It follows from (3.52(3)) that

$$\lim_{j\to\infty} ||T_k^{n_j} y_{k-1,n_j} - x_{n_j}|| = 0.$$

Since $\alpha_{k,n_j} \in [\delta, 1-\delta]$ for each $j \in \mathbb{N}$ and $\delta \in (0, \frac{1}{2})$, we have

$$\lim_{j\to\infty} \alpha_{k,n_j} ||T_k^{n_j} y_{k-1,n_j} - x_{n_j}|| = 0. \tag{3.54}$$

Thus, from (3.53) and (3.54), we conclude that

$$weak - \lim_{j\to\infty} x_{n_j+1} = q.$$

Therefore, the sequence $\{x_n\}$ satisfies the hypothesis of Lemma 2.5 which in turn implies that $\{x_n\}$ weakly converges to q so that $p = q$. This completes the proof.

Theorem 3.5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and suppose that C has Property P. Let $\{T_i\}_{i=1}^k$ be a family of G-asymptotically nonexpansive mappings on C with the nonempty common fixed points set $F = \bigcap_{i=1}^k F(T_i)$ and $\sum_{n=1}^\infty u_{i,n} < \infty$ for each $i = 1,2,3,\cdots,k$. Let $x_1 \in C$ be fixed so that (x_1, p) and (p, x_1) are in $E(G)$ for some $p \in F$. If for some $l \in \{1,2,3,\cdots,k\}$, T_l^m is semi-compact for some positive integer m, then the iteration $\{x_n\}$ generated by x_1 with iterative scheme (1.3) such that $\{\alpha_{i,n}\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$ converges strongly to a common fixed point of the family $\{T_i\}_{i=1}^k$.*

Proof. Fix $m \in \{1,2,3,\cdots,k\}$ and assume that T_m^s is semi-compact for some $s \in \mathbb{N}$. Let $p \in F$ such that $(x_1, p), (p, x_1)$ are in $E(G)$. It follows from (3.52(2)) and (3.52(3)) that $\{x_n\}$ is a bounded sequence in C and

$$\lim_{n\to\infty} ||x_n - T_m x_n|| = 0. \tag{3.55}$$

As $\{x_n\}$ is bounded, by the definition of semi-compactness, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that for some $z \in C$,

$$\lim_{j\to\infty} ||x_{n_j} - z|| = 0. \tag{3.56}$$

Since strong convergence implies weak convergence and using Remark 1.1, we have $(x_{n_j}, z) \in E(G)$. Now it is obvious that z is a fixed point of T_m. By the G-asymptotically nonexpansiveness of T_i for each $i \in \{1,2,3,\cdots,k\}$, and using (3.53) and (3.56), we have

$$\begin{aligned} ||T_iz - z|| &\leq ||T_iz - T_ix_{n_j}|| + ||T_ix_{n_j} - x_{n_j}|| + ||x_{n_j} - z|| \\ &\leq (1+k_1)||z - x_{n_j}|| + ||T_ix_{n_j} - x_{n_j}|| \to 0 \text{ as } j \to \infty. \end{aligned} \tag{3.57}$$

Thus, z is a common fixed point of the family $\{T_i\}_{i=1}^{k}$ so that $\lim_{n\to\infty} ||x_n - z||$ exists. Hence it must be the case that

$$\lim_{n\to\infty} ||x_n - z|| = 0.$$

This completes the proof.

Our results generalize the results in the corresponding literature in two ways: first, Banach spaces satisfying Opial's property are very limited so that relaxing this property substantially generalizes the related results on the space with this condition. secondly, the class of G-asymptotically nonexpansive mappings are more general than G-nonexpansive mappings as well as asymptotically nonexpansive mappings.

4. Acknowledgements

This work was supported by Simmons Foundation based at Botswana International University of Science and Technology. The first and second authors were also supported by the International Science Program(ISP)- Sweden, based in the Department of Mathematics, Addis Ababa University, Ethiopia

Conflict of Interests

The authors declare that there is no conflict of interests.

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