Available online at http://scik.org
Adv. Fixed Point Theory, 8 (2018), No. 4, 463-474
https://doi.org/10.28919/afpt/3752
ISSN: 1927-6303

RESULTS IN PM SPACE WITH MENGER-HAUSDORFF METRIC

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Abstract: In this paper we obtain some common fixed point results with Menger-Hausdorff metric for occasionally weakly compatible mappings in PM space (Menger Space).

Keywords: occasionally weakly compatible mappings; common fixed point theorem; menger space.

1. Introduction

K. Menger [7] introduced the notion of probabilistic metric space, which is a generalization of the metric space. The study of this probabilistic metric space was done mainly with the pioneering works of Schweizer and Sklar [11, 12]. Generalization of such metric space appears to be well adapted for the investigation of physical quantities and many more. It has importance in probabilistic functional analysis and nonlinear analysis (see [3], [8], [9]). In 1972, Sehgal and Bharucha-Reid [13] initiated the study of contraction maps and obtained a generalization of Banach Contraction Principle on a complete Menger space or probabilistic metric space (shortly, PM-space) which is an important step in the development of fixed point theory and fixed point

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Received May 24, 2018

theorems in this space.

Fixed point theorems, involving four self-maps, began with the assumption that all of the maps are commuted. Sessa [14] weakened the condition of commutativity to that of pairwise weakly commuting. Jungck generalized the notion of weak commutativity to that of pairwise compatible [4] and then pairwise weakly compatible maps [5]. Jungck and Rhoades [6] introduced the concept of occasionally weakly compatible maps.

Abbas and Rhoades [1] generalized the concept of weak compatibility in the setting of single and multi-valued maps by introducing the notion of occasionally weakly compatible (owc).Also Abbas and Rhoades [2] extended the idea of owc maps to hybrid pairs of single-valued and multi-valued maps using a symmetric δ derived from an ordinary symmetric d.

The aim of this paper is to obtain some common fixed point results for owc maps with Menger-Hausdorff metric in PM space (Menger space).

2. Preliminaries

Definition 2.1[12]A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous $\mathrm{t-norm}$ if $*$ is satisfying conditions:

(i) $*$ is commutative and associative;

(ii) $*$ is continuous;

(iii) $a * 1 = a$ for all $a \in [0,1]$;

(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$.

Definition 2.2 [12] A mapping $F: \mathcal{R} \to \mathcal{R}^+$ is called a distribution function if it is non decreasing and left continuous with $\inf\{F(t): t \in \mathcal{R}\} = 0$ and $\sup\{F(t): t \in \mathcal{R}\} = 1$.

We shall denote by $\mathfrak{I}$ the set of all distribution functions defined on $[-\infty, \infty]$ while $H(t)$ will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & if\ t \leq 0; \\ 1, & if\ t > 1. \end{cases}$$

If X is a non-empty set, $\mathcal{F}: X \times X \to \mathfrak{I}$ is called a probabilistic distance on X and the value of $\mathcal{F}$ at $(x, y) \in X \times X$ is represented by $F_{x,y}$.

Definition 2.3[12] A PM-space is an ordered pair $(X,\mathcal{F})$, where X is a nonempty set of elements and $\mathcal{F}$ is a probabilistic distance satisfying the following conditions for all $x,y,z \in X$ and $t,s > 0$,

(1) $\mathrm{F_{x,y}(t) = H(t)}$ for all $t > 0$ if and only if $x = y$,

(2) $\mathrm{F_{x,y}(t) = F_{y,x}(t)}$,

(3) if $\mathrm{F_{x,y}(t) = 1}$ & $\mathrm{F_{y,z}(t) = 1}$, then $\mathrm{F_{x,z}(t+s) = 1}$.

The ordered triple $(X,\mathcal{F},*)$ is called a **Menger space** if $(X,\mathcal{F})$ is a PM-space $*$ is a t-norm and the following inequality holds:

$$\mathrm{F_{x,y}(t+s) \geq * \,(F_{x,z}(t), F_{z,y}(t))}$$

for all $x,y,z \in X$ and $t,s > 0$.

Let (X,d) be a metric space, $CB(X)$ be the family of all nonempty bounded closed subsets of X and δ be the *Hausdorff metric* induced by d, that is,

$$\delta(A,B) = \max\left\{sup_{x\in A}d(x,B), sup_{y\in B}d(y,A)\right\},$$

for any $A,B \in CB(X)$, where $d(x,A) = inf_{y\in A}d(x,y)$.

Let $(X,\mathcal{F},*)$ be a Menger space and Ω be the family of all nonempty probabilistically bounded τ-closed subsets of X. For any $A,B \in \Omega$, define the distribution functions as follows:

$\tilde{\mathcal{F}}(A,B)(t) = \tilde{F}_{A,B}(t) = sup_{s<t} * (inf_{x\in A}sup_{y\in B}F_{x,y}(s), inf_{y\in B}sup_{x\in A}F_{x,y}(s)), \qquad s,t \in \mathbb{R}$,

$\mathcal{F}(x,A)(t) = F_{x,A}(t) = sup_{s<t}sup_{y\in A}F_{x,y}(s), \qquad s,t \in \mathbb{R}$,

where $\tilde{\mathcal{F}}$ is called the Menger-Hausdorff metric induced by $\mathcal{F}$.

Lemma 2.4[10] If a Menger space $(X,\mathcal{F},*)$ satisfies the condition $\mathrm{F_{x,y}(t) = C}$ for all $\mathrm{t} > 0$ with fixed $x,y \in X$. Then we have $C = 1$ and $x = y$.

Lemma 2.5[16] Let $(X,\mathcal{F},*)$ be a Menger space. Then for any $A,B \in \Omega$ and any $x \in A$, $F_{x,B}(t) \geq \tilde{\mathcal{F}}_{A,B}(t)$ for all $t \geq 0$.

Definition 2.6[1] Maps $f: X \longrightarrow X$ and $T: X \longrightarrow CB(X)$ are said to be weakly compatible if they commute at their coincidence points, that is $fx \in Tx$ for some $x \in X$ then $fTx = Tfx$.

Definition 2.7[1] Maps $f: X \longrightarrow X$ and $T: X \longrightarrow CB(X)$ are said to be occasionally weakly compatible (owc) if and only if there exist some point x in X such that $fx \in Tx$ and $fTx \subseteq Tfx$.

Example 2.8 Let $(X,\mathcal{F},*)$ be a Menger space, where $X=[0,\infty)$ and

$$F_{x,y}(t)=\begin{cases}\dfrac{t}{t+|x-y|}, & \text{if } t>0;\\ 0, & \text{if } t=0.\end{cases}$$

Let $A:X\to X$ & $B:X\to CB(X)$ be single valued and set-valued maps defined by

$$A(x)=\begin{cases}0, & if\ x=0;\\ x^2, & if\ x\in(0,\infty).\end{cases}\qquad B(X)=\begin{cases}\{0\}, & if\ x=0;\\ \{3x\}, & if\ x\in(0,\infty).\end{cases}$$

Here, 0 and 3 are two coincidence points of A and B. That is $A0=\{0\}\in B(0)$, $A(3)=\{9\}\in B(3)$, $but\ AB(0)=\{0\}=BA(0)$, $AB(3)\neq BA(3)$. Thus A and B are owc but not weakly compatible.

3. Main Results

Theorem 3.1 Let $(X,\mathcal{F},*)$ be a menger space. Let $A,B:X\to X$ and $S,T:X\to\Omega$ such that the pairs $\{A,S\}$ & $\{B,T\}$ are owc. If

$$\tilde{F}_{Sx,Ty}\geq \min\{F_{Ax,By}\,.\,F_{Ax,Sx},\ F_{Ax,By}\,.\,F_{By,Ty},\ F_{Ax,Sx}\,.\,F_{By,Ty},\ F_{Ax,Ty},\ F_{By,Sx}\} \qquad (3.1)$$

for all $x,y\in X$ & $t>0$. Then A,B,S & T have a unique common fixed point.

Proof. Since the pairs $\{A,S\}$ & $\{B,T\}$ are owc, therefore, there exist two elements $u,v\in X$ such that $Au\in Su, ASu\subseteq SAu$ and $Bv\in Tv, BTv\subseteq TBv$. As $Au\in Su$ so $AAu\subset ASu\subset SAu, Bv\in Tv\ so\ BBv\subset BTv\subset TBv$

First we prove that $Au=Bv$.

We have $F_{A^2u,B^2v}\geq\tilde{F}_{SAu,TBv}$.

Suppose that $\tilde{F}_{SAu,TBv}<1$. Then by (3.1)

Put $x=Au, y=Bv$

$$\tilde{F}_{SAu,TBv}\geq\min\{F_{A^2u,B^2v}\,.\,F_{AAu,SAu},\quad F_{AAu,BBv}\,.\,F_{BBv,TBv},\quad F_{AAu,SAu}\,.\,F_{BBv,TBv},\quad F_{AAu,TBv},\quad F_{BBv,SAu}\}$$

$$\geq\min\{\tilde{F}_{SAu,TBv},\ \tilde{F}_{SAu,TBv},\ 1,\ \tilde{F}_{SAu,TBv},\ \tilde{F}_{SAu,TBv}\}$$

$$=\ \tilde{F}_{SAu,TBv}$$, a contradiction.

Hence $Au=Bv$.

Also, $$F_{A^2u,Au}=F_{A^2u,Bv}\geq\tilde{F}_{SAu,Tv}$$

Now we claim that $A^2u=Au$. If not, then $\tilde{F}_{SAu,Tv}<1$.

Considering (3.1) for $Au=x, y=v$

$\tilde{F}_{SAu,Tv} \geq \min\{F_{A^2u,Bv} \,.\, F_{AAu,SAu}, \quad F_{AAu,Bv} \,.\, F_{Bv,Tv}, \quad F_{AAu,SAu} \,.\, F_{Bv,Tv}, \quad F_{AAu,Tv}, \quad F_{Bv,SAu}\}$

$\geq \min\{\tilde{F}_{SAu,Tv}, \; \tilde{F}_{SAu,Tv}, \; 1, \; \tilde{F}_{SAu,Tv}, \; \tilde{F}_{SAu,Tv}\}$

$= \tilde{F}_{SAu,Tv}$, which is again a contradiction and hence $A^2u = Au$.

Similarly, we can get $B^2v = Bv$. If $Au = Bv = z$ then $Az = z = Bz$, $z \in Sz$ & $z \in Tz$.

Therefore z is the common fixed point of A, B, S & T . Now suppose that A, B, S & T have another common fixed point $z' \neq z$. Then $F_{z,z'} = F_{Az,Bz'} \geq \tilde{F}_{Sz,Tz'}$.

Assume that $\tilde{F}_{Sz,Tz'} < 1$. Then by (3.1) for $x = z$ & $y = z'$.

$\tilde{F}_{Sz,Ty} \geq \min\{F_{Az,Bz'} \,.\, F_{Az,Sz}, \; F_{Az,Bz'} \,.\, F_{Bz',Tz'}, \; F_{Az,Sz} \,.\, F_{Bz',Tz'}, \; F_{Az,Tz'}, \; F_{Bz',Sz}\}$

$\geq \min\{F_{Az,Bz'}, \; F_{Az,Bz'}, \; 1, \; F_{Az,Bz'}, \; F_{Az,Bz'}\}$

$= F_{Az,Bz'}$, a contradiction.

Hence $z = z'$. Thus, A, B, S & T have a unique common fixed point.

Example 3.1.1 Let $X = [0,4]$ with the metric d defined by $d(x,y) = |x-y|$ and for each $t \in [0,1]$, define

$$M(x,y,t) = \begin{cases} \dfrac{t}{t + |x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0 \end{cases}$$

for all $x, y \in X$. Let $(X, \mathcal{F}, *)$ be a menger space. Let $A, B : X \to X$ and $S, T: X \to \Omega$ such that the pairs $\{A, S\}$ & $\{B, T\}$ are owc defined by

$$S(x) = \begin{cases} \{2\}, & \text{if } 0 \leq x \leq 2; \\ \{0\}, & \text{if } 2 \leq x \leq 4. \end{cases} \qquad A(X) = \begin{cases} x, & \text{if } 0 \leq x \leq 2; \\ 3, & \text{if } 2 \leq x \leq 4. \end{cases}$$

$$T(x) = \begin{cases} \{2\}, & \text{if } 0 \leq x \leq 2; \\ \{4\}, & \text{if } 2 \leq x \leq 4. \end{cases} \qquad B(X) = \begin{cases} 2, & \text{if } 0 \leq x \leq 2; \\ \dfrac{x}{4}, & \text{if } 2 \leq x \leq 4. \end{cases}$$

Clearly all the conditions of the above theorem are satisfied. That is,

$$A(2) = \{2\} \in S(2) \text{ and } SA(2) = \{2\} = AS(2),$$

$$B(2) = \{2\} \in T(2) \text{ and } TB(2) = \{2\} = BT(2),$$

So, A and S as well as B and T are owc maps. Also 2 is the unique common fixed point of A, B, S and T. On the other hand, it is clear to see that the maps are discontinuous at 2.

Further, we have

$$S(X) = \{0,2\} \text{ is not a subset of } B(X) = \left(\tfrac{1}{2}, 1\right] \cup \{2\}$$

and

$$T(X) = \{2,4\} \text{ is not a subset of } A(X) = [0,2] \cup \{3\},$$

which generalizes our result.

Theorem 3.2 Let $(X, \mathcal{F}, *)$ be a menger space. Let $A, B : X \to X$ and $S, T: X \to \Omega$ such that the pairs $\{A, S\}$ & $\{B, T\}$ are owc. If

$$\tilde{F}_{Sx,Ty} \geq \min\{F_{Ax,By},\ F_{Ax,Sx},\ F_{By,Ty},\ F_{Ax,Ty},\ F_{By,Sx}\} \qquad (3.2)$$

for all $x, y \in X$ & $t > 0$. Then A, B, S & T have a unique common fixed point.

Proof. Since the pairs $\{A, S\}$ & $\{B, T\}$ are owc, therefore, there exist two elements $u, v \in X$ such that $Au \in Su, ASu \subseteq SAu$ and $Bv \in Tv, BTv \subseteq TBv$. As $Au \in Su$ so $AAu \subset ASu \subset SAu, Bv \in Tv$ so $BBv \subset BTv \subset TBv$

First we prove that $Au = Bv$.

We have $F_{A^2u,B^2v} \geq \tilde{F}_{SAu,TBv}$.

Suppose that $\tilde{F}_{SAu,TBv} < 1$. Then by (3.2)

Put $x = Au, y = Bv$

$$\begin{aligned} \tilde{F}_{SAu,TBv} &\geq \min\{F_{A^2u,B^2v},\ F_{AAu,SAu},\ F_{BBv,TBv},\ F_{AAu,TBv},\ F_{BBv,SAu}\} \\ &\geq \min\{\tilde{F}_{SAu,TBv},\ 1,\ 1,\ \tilde{F}_{SAu,TBv},\ \tilde{F}_{SAu,TBv}\} \\ &= \tilde{F}_{SAu,TBv}, \text{ a contradiction.} \end{aligned}$$

Hence $Au = Bv$.

Also, $\qquad F_{A^2u,Au} = F_{A^2u,Bv} \geq \tilde{F}_{SAu,Tv}$

Now we claim that $A^2u = Au$. If not, then $\tilde{F}_{SAu,Tv} < 1$.

Considering (3.2) for $Au = x, y = v$

$$\begin{aligned} \tilde{F}_{SAu,Tv} &\geq \min\{F_{A^2u,Bv},\ F_{AAu,SAu},\ F_{Bv,Tv},\ F_{AAu,Tv},\ F_{Bv,SAu}\} \\ &\geq \min\{\tilde{F}_{SAu,Tv},\ 1,\ 1,\ \tilde{F}_{SAu,Tv},\ \tilde{F}_{SAu,Tv}\} \\ &= \tilde{F}_{SAu,Tv} \text{ , which is again a contradiction and hence } A^2u = Au. \end{aligned}$$

Similarly, we can get $B^2v = Bv$. If $Au = Bv = z$ then $Az = z = Bz$, $z \in Sz$ & $z \in Tz$. Therefore z is the common fixed point of A, B, S & T . Now suppose that A, B, S & T have another common fixed point $z' \neq z$. Then $F_{z,z'} = F_{Az,Bz'} \geq \tilde{F}_{Sz,Tz'}$.

Assume that $\tilde{F}_{Sz,Tz'} < 1$. Then by (3.2) for $x = z$ & $y = z'$.

$\tilde{F}_{Sz,Ty} \geq \min\{F_{Az,Bz'}, F_{Az,Sz}, F_{Bz',Tz'}, F_{Az,Tz'}, F_{Bz',Sz}\}$

$\geq \min\{F_{Az,Bz'}, 1, 1, F_{Az,Bz'}, F_{Az,Bz'}\}$

$= F_{Az,Bz'}$, a contradiction.

Hence $z = z'$. Thus, A, B, S & T have a unique common fixed point.

Theorem 3.3 Let $(X, \mathcal{F}, *)$ be a menger space. Let $A, B : X \to X$ and $S, T: X \to \Omega$ such that the pairs $\{A, S\}$ & $\{B, T\}$ are owc. If

$$\tilde{F}_{Sx,Ty} \geq \min\{F_{Ax,By}, F_{Ax,Sx}, F_{By,Ty}, \frac{F_{Ax,Ty} + F_{By,Sx}}{2}\} \qquad (3.3)$$

for all $x, y \in X$ & $t > 0$. Then A, B, S & T have a unique common fixed point.

Proof. Since the pairs $\{A, S\}$ & $\{B, T\}$ are owc, therefore, there exist two elements $u, v \in X$ such that $Au \in Su, ASu \subseteq SAu$ and $Bv \in Tv, BTv \subseteq TBv$. As $Au \in Su$ so $AAu \subset ASu \subset SAu, Bv \in Tv$ so $BBv \subset BTv \subset TBv$

First we prove that $Au = Bv$.

We have $F_{A^2u,B^2v} \geq \tilde{F}_{SAu,TBv}$.

Suppose that $\tilde{F}_{SAu,TBv} < 1$. Then by (3.3)

Put $x = Au, y = Bv$

$$\tilde{F}_{SAu,TBv} \geq \min\{F_{A^2u,B^2v}, \quad F_{AAu,SAu}, \quad F_{BBv,TBv}, \quad \frac{F_{AAu,TBv} + F_{BBv,SAu}}{2}\}$$

$$\geq \min\{\tilde{F}_{SAu,TBv}, 1, 1, \tilde{F}_{SAu,TBv}\}$$

$$= \tilde{F}_{SAu,TBv}$$, a contradiction.

Hence $Au = Bv$.

Also, $\quad F_{A^2u,Au} = F_{A^2u,Bv} \geq \tilde{F}_{SAu,Tv}$

Now we claim that $A^2u = Au$. If not, then $\tilde{F}_{SAu,Tv} < 1$.

Considering (3.3) for $Au = x, y = v$

$$\tilde{F}_{SAu,Tv} \geq \min\{F_{A^2u,Bv}, \quad F_{AAu,SAu}, \quad F_{Bv,Tv}, \quad \frac{F_{AAu,Tv} + F_{Bv,SAu}}{2}\}$$

$$\geq \min\{\tilde{F}_{SAu,Tv}, 1, 1, \tilde{F}_{SAu,Tv}\}$$

$= \tilde{F}_{SAu,Tv}$, which is again a contradiction and hence $A^2u = Au$.

Similarly, we can get $B^2v = Bv$. If $Au = Bv = z$ then $Az = z = Bz$, $z \in Sz$ & $z \in Tz$.

Therefore z is the common fixed point of A, B, S & T. Now suppose that A, B, S & T have another common fixed point $z' \neq z$. Then $F_{z,z'} = F_{Az,Bz'} \geq \tilde{F}_{Sz,Tz'}$.

Assume that $\tilde{F}_{Sz,Tz'} < 1$. Then by (3.3) for $x = z$ & $y = z'$.

$\tilde{F}_{Sz,Ty} \geq \min\{F_{Az,Bz'},\ F_{Az,Sz},\ F_{Bz',Tz'},\ \frac{F_{Az,Tz'} + F_{Bz',Sz}}{2}\}$

$\geq \min\{F_{Az,Bz'},\ 1,\ 1,\ F_{Az,Bz'}\}$

$= F_{Az,Bz'}$, a contradiction.

Hence $z = z'$. Thus, A, B, S & T have a unique common fixed point.

Theorem 3.4 Let $(X, \mathcal{F}, *)$ be a menger space. Let $A, B : X \to X$ and $S, T: X \to \Omega$ such that the pairs $\{A, S\}$ & $\{B, T\}$ are owc. If

$$\tilde{F}_{Sx,Ty} \geq \min\{F_{Ax,By},\ \frac{F_{Ax,Sx} + F_{By,Ty}}{2},\ \frac{F_{Ax,Ty} + F_{By,Sx}}{2}\} \qquad (3.4)$$

for all $x, y \in X$ & $t > 0$. Then A, B, S & T have a unique common fixed point.

Proof. Clearly the result immediately follows.

Theorem 3.5 Let $(X, \mathcal{F}, *)$ be a menger space. Let $A, B : X \to X$ and $S, T: X \to \Omega$ such that the pairs $\{A, S\}$ & $\{B, T\}$ are owc. If

$$\tilde{F}_{Sx,Ty} \geq \min\{F_{Ax,By},\ F_{By,Sx}\left[\frac{1+F_{Ax,Sx}}{1+F_{By,Ty}}\right], F_{Ax,Ty}\left[\frac{1+F_{By,Ty}}{1+F_{Ax,Sx}}\right]\} \qquad (3.5)$$

for all $x, y \in X$ & $t > 0$. Then A, B, S & T have a unique common fixed point.

Proof. Since the pairs $\{A, S\}$ & $\{B, T\}$ are owc, therefore, there exist two elements $u, v \in X$ such that $Au \in Su, ASu \subseteq SAu$ and $Bv \in Tv, BTv \subseteq TBv$. As $Au \in Su$ so $AAu \subset ASu \subset SAu, Bv \in Tv\ so\ BBv \subset BTv \subset TBv$

First we prove that $Au = Bv$.

We have $F_{A^2u,B^2v} \geq \tilde{F}_{SAu,TBv}$.

Suppose that $\tilde{F}_{SAu,TBv} < 1$. Then by (3.5)

Put $x = Au, y = Bv$

$$\tilde{F}_{SAu,TBv} \geq \min\{F_{AAu,BBv},\quad F_{BBv,SAu}\left[\frac{1 + F_{AAu,SAu}}{1 + F_{BBv,TBv}}\right], F_{AAu,TBv}\left[\frac{1 + F_{BBv,TBv}}{1 + F_{AAu,SAu}}\right]\}$$

$$\geq \min\{\tilde{F}_{SAu,TBv},\ \tilde{F}_{SAu,TBv},\ \tilde{F}_{SAu,TBv}\}$$

$= \tilde{F}_{SAu,TBv}$, a contradiction.

Hence $Au = Bv$.

Also, $F_{A^2u,Au} = F_{A^2u,Bv} \geq \tilde{F}_{SAu,Tv}$

Now we claim that $A^2u = Au$. If not, then $\tilde{F}_{SAu,Tv} < 1$.

Considering (3.5) for $Au = x, y = v$

$$\tilde{F}_{SAu,Tv} \geq \min\{F_{AAu,Bv}, \quad F_{Bv,SAu}\left[\frac{1 + F_{AAu,SAu}}{1 + F_{Bv,Tv}}\right], F_{AAu,Tv}\left[\frac{1 + F_{Bv,Tv}}{1 + F_{AAu,SAu}}\right]\}$$

$$\geq \min\{\tilde{F}_{SAu,Tv}, \ \tilde{F}_{SAu,Tv}, \ \tilde{F}_{SAu,Tv}\}$$

$= \tilde{F}_{SAu,Tv}$, which is again a contradiction and hence $A^2u = Au$.

Similarly, we can get $B^2v = Bv$. If $Au = Bv = z$ then $Az = z = Bz$, $z \in Sz$ & $z \in Tz$.

Therefore z is the common fixed point of A, B, S & T . Now suppose that A, B, S & T have another common fixed point $z' \neq z$. Then $F_{z,z'} = F_{Az,Bz'} \geq \tilde{F}_{Sz,Tz'}$.

Assume that $\tilde{F}_{Sz,Tz'} < 1$. Then by (3.5) for $x = z$ & $y = z'$.

$$\tilde{F}_{Sz,Tz'} \geq \min\{F_{Az,Bz'}, \quad F_{Bz',Sz}\left[\frac{1 + F_{Az,Sz}}{1 + F_{Bz',Tz'}}\right], F_{Ax,Tz'}\left[\frac{1 + F_{Bz',Tz'}}{1 + F_{Az,Sz}}\right]\}$$

$$\geq \min\{F_{Az,Bz'}, \ F_{Az,Bz'}, \ F_{Az,Bz'}\}$$

$= F_{Az,Bz'}$, a contradiction.

Hence $z = z'$. Thus, A, B, S & T have a unique common fixed point.

Theorem 3.6 Let $(X, \mathcal{F}, *)$ be a menger space. Let $A, B : X \to X$ and $S, T: X \to \Omega$ such that the pairs $\{A, S\}$ & $\{B, T\}$ are owc. If

$$\tilde{F}_{Sx,Ty} \geq \frac{\{F_{Ax,By} + F_{Ax,By} . F_{Ax,Sx} + F_{By,Ty} . F_{By,Sx}\}}{3} \qquad (3.6)$$

for all $x, y \in X$ & $t > 0$. Then A, B, S & T have a unique common fixed point.

Proof. Clearly the result immediately follows.

Theorem 3.7 Let $(X, \mathcal{F}, *)$ be a menger space. Let $A, B : X \to X$ and $S, T: X \to \Omega$ such that the pairs $\{A, S\}$ & $\{B, T\}$ are owc. If

$$\tilde{F}_{Sx,Ty} \geq \alpha\, F_{Ax,By} + (1 - \alpha)\frac{[\, F_{Ax,Sx} \, . \, F_{By,Sx} + F_{Ax,Ty} \, . \, F_{By,Ty}]}{2} \qquad (3.7)$$

for all $x, y \in X, \alpha > 0$ & $t > 0$. Then A, B, S & T have a unique common fixed point.

Proof. Since the pairs $\{A,S\}$ & $\{B,T\}$ are owc, therefore, there exist two elements $u,v \in X$ such that $Au \in Su, ASu \subseteq SAu$ and $Bv \in Tv, BTv \subseteq TBv.$ As $Au \in Su$ so $AAu \subset ASu \subset SAu, Bv \in Tv$ so $BBv \subset BTv \subset TBv$

First we prove that $Au = Bv$.

We have $F_{A^2u,B^2v} \geq \tilde{F}_{SAu,TBv}$.

Suppose that $\tilde{F}_{SAu,TBv} < 1.$ Then by (3.7)

Put $x = Au, y = Bv$

$$\tilde{F}_{SAu,TBv} \geq \alpha\, F_{AAu,BBv} + (1-\alpha)\frac{[\ F_{AAu,SAu}\cdot F_{BBv,SAu} + F_{AAu,TBv}\cdot F_{BBv,TBv}]}{2}$$

$$\geq \alpha\tilde{F}_{SAu,TBv} + (1-\alpha)\tilde{F}_{SAu,TBv}$$

$$= \tilde{F}_{SAu,TBv}$$, a contradiction.

Hence $Au = Bv$.

Also, $$F_{A^2u,Au} = F_{A^2u,Bv} \geq \tilde{F}_{SAu,Tv}$$

Now we claim that $A^2u = Au$. If not, then $\tilde{F}_{SAu,Tv} < 1$.

Considering (3.7) for $Au = x, y = v$

$$\tilde{F}_{SAu,Tv} \geq \alpha\, F_{AAu,Bv} + (1-\alpha)\frac{[\ F_{AAu,SAu}\cdot F_{Bv,SAu} + F_{AAu,Tv}\cdot F_{Bv,Tv}]}{2}$$

$$= \tilde{F}_{SAu,Tv}$$, which is again a contradiction and hence $A^2u = Au$.

Similarly, we can get $B^2v = Bv$. If $Au = Bv = z$ then $Az = z = Bz$, $z \in Sz$ & $z \in Tz$.

Therefore z is the common fixed point of A, B, S & T . Now suppose that A, B, S & T have another common fixed point $z' \neq z$. Then $F_{z,z'} = F_{Az,Bz'} \geq \tilde{F}_{Sz,Tz'}$.

Assume that $\tilde{F}_{Sz,Tz'} < 1.$ Then by (3.7) for $x = z$ & $y = z'$.

$$\tilde{F}_{Sz,Tz'} \geq \alpha\, F_{Az,Bz'} + (1-\alpha)\frac{[\ F_{Az,Sz}\cdot F_{Bz',Sz} + F_{Az,Tz'}\cdot F_{Bz',Tz'}]}{2}$$

$$= F_{Az,Bz'}$$, a contradiction.

Hence $z = z'$. Thus, A, B, S & T have a unique common fixed point.

4. Conclusion:

In this paper, we define various results in Menger space for occasionally weakly compatible mappings. Our theorems extend and unify the existing results in the recent literature. Example is constructed to support our result.

Conflict of Interests

The authors declare that there is no conflict of interests.

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