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A COMMON FIXED POINT OF GENERALIZED (ψ, ϕ)-WEAKLY CONTRACTIVE MAPS WHERE ϕ IS NONDECREASING (NOT NECESSARILY CONTINUOUS OR LOWER SEMICONTINUOUS)

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Abstract. In this paper, we introduce generalized $(\psi,\ \phi)$-weakly contractive condition for four selfmaps in which ψ is continuous and nondecreasing and ϕ is nondecreasing but not necessarily either continuous or lower semicontinuous, and we prove a common fixed point result for four selfmaps in a complete metric space. An example is given in support of the main result of the paper.

Keywords: common fixed point, generalized weak contraction, complete metric space; weakly compatible maps.

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1. Introduction

Alber and Guerre-Delabriere [1] introduced weakly contractive maps in Hilbert spaces as a generalization of contraction maps, and established a fixed point theorem in Hilbert space setting. Rhoades [11] extended this idea to Banach spaces and proved the existence of fixed points of weakly contractive selfmaps in Banach space setting. Different types of

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weakly contractive maps have been considered in several works by different researchers in [1], [2], [3], [4], [5], [6], [11] and [13] in order to establish the existence of fixed points. Rhoades [10] can be taken as a good reference for a comprehensive work in different types of contractive maps.

Definition 1.1. (*Rhoades* [11]*) Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be weakly contractive if $d(Tx, Ty) \leq d(x, y) - \phi(d(x, y))$ for all $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ satisfying ϕ is nondecreasing, continuous and $\phi(t) = 0$ if and only if $t = 0$.*

Theorem 1.2. (*Dutta and Choudhury* [6]) *Let (X, d) be a complete metric space and $T : X \to X$ be a selfmap satisfying the inequality*
$\psi(d(Tx,\ Ty)) \leq \psi(d(x,\ y)) - \phi(d(x,\ y))$, where $\psi,\ \phi : [0, \infty) \to [0, \infty)$ are both continuous and monotone nondecreasing functions with
$\psi(t) = 0 = \phi(t)$ if and only if $t = 0$. Then T has a unique fixed point.

Theorem 1.3. (*Doric* [5]) *Let (X, d) be a complete metric space and*
$T,\ S : X \to X$ be two selfmaps such that for all $x, y \in X$
$\psi(d(Tx,\ Sy)) \leq \psi(M(x,\ y)) - \phi(M(x,\ y))$,
where

(a) $\psi : [0, \infty) \to [0, \infty)$ is a continuous, monotone nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$,

(b) $\phi : [0, \infty) \to [0, \infty)$ is a lower semicontinuous function with $\phi(t) = 0$ if and only if $t = 0$,

(c) $M(x, y) = max\{d(x,\ y), d(Tx,\ x), d(Sy,\ y), \frac{1}{2}[d(y,\ Tx) + d(x,\ Sy)]\}$.

Then there exists a unique $u \in X$ such that $u = Tu = Su$.

Definition 1.4. (*Choudhury et al.*[4]) . *Let (X, d) be a metric space and T be a selfmap of X. T is said to be a generalized weakly contractive map if there exist maps $\psi : [0, \infty) \to [0, \infty)$ satisfying ψ is nondecreasing, continuous and $\psi(t) = 0$*

if and only if $t = 0$ and

$\phi : [0, \infty) \to [0, \infty)$ satisfying ϕ is continuous and $\phi(t) = 0$ if and only if $t = 0$ such that

$d(Tx, Ty) \leq \psi(M(x, y)) - \phi(max\{d(x, y),\ d(y, Ty)\})$ *for all* $x, y \in X$, *where*
$M(x, y) = max\{d(x, y),\ d(x, Tx),\ d(y, Ty),\ \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}.$

Note: A mapping ψ mentioned in Theorems 1.2, 1.3 and 1.4 is called an altering distance function. For more information on altering distance functions, we refer [9, 12].

Definition 1.5. (*Jungck* [7]) *Let* f *and* g *be selfmaps of a metric space* (X, d). *A point* $x \in X$ *is said be a coincidence point of* f *and* g *if* $fx = gx$.

Definition 1.6. (*Jungck and Rhoades* [8]) *Let* f *and* g *be selfmaps of a metric space* (X, d). *The pair* (f, g) *is said to be weakly compatible if they commute at their coincidence point, i.e.,* $fgx = gfx$ *whenever* $gx = fx$, $x \in X$.

Theorem 1.7. (*Choudhury et al.* [4]) *Let* (X, d) *be a complete metric space and* T *a generalized weakly contractive mapping of* X. *Then* T *has a unique fixed point.*

Theorem 1.8. (*Choudhury et al.* [4]) *Let* (X, d) *be a complete metric space. Let* f *and* g *be selfmaps of* X. *Suppose that there exist maps*
$\psi : [0, \infty) \to [0, \infty)$ *satisfying* ψ *is nondecreasing continuous and* $\psi(t) = 0$

if and only if $t = 0$ *and* $\phi : [0, \infty) \to [0, \infty)$ satisfying ϕ *is*

continuous and $\phi(t) = 0$ *if and only if* $t = 0$ *such that*

$d(fx, gy) \leq \psi(M(x, y)) - \phi(m(x, y)\})$, *for all* $x, y \in X$, *where*
$M(x, y) = max\{d(x, y),\ d(x, fx),\ d(y, gy),\ \frac{1}{2}[d(x, gy) + d(y, fx)]\}$ *and*
$m(x, y) = max\{d(x, y),\ d(x, fx),\ d(y, gy)\}$, *then* f *and* g *have a unique common fixed point. Moreover any fixed point of* f *is a fixed point of* g *and conversely.*

Definition 1.9. (*Babu, Nageswara Rao and Alemayehu* [2]) *Let* f, g, S *and* T *be selfmaps of a metric space* (X, d). *We say that the pair* (f, g) *is* (S, T) *generalized weakly contractive if there exists a function*
$\phi : [0, \infty) \to [0, \infty)$ *satisfying* ϕ *is lower semicontinuous and* $\phi(t) = 0$ *if and only if* $t = 0$, *such that*
$d(fx, gy) \leq M(x, y) - \phi(M(x, y))$ *for all* x, y *in* X,
where
$M(x, y) = max\{d(Sx, Ty),\ d(fx, Sx),\ d(gy, Ty),\ \frac{1}{2}[d(Sx, gy) + (fx, Ty)]\}.$

Theorem 1.10. (*Babu, Nageswara Rao and Alemayehu* [2]) *Let* f, g, S *and* T *be selfmaps* *of a complete metric space* (X,d) *such that* $fX \subseteq TX$ *and* $gX \subseteq SX$ *and* (f,g) *is* (S,T) *generalized weakly contractive pair. If one of the ranges* fX, gX, SX *and* TX *is closed, then* f, g, S *and* T *have a unique common fixed point in* X.

In all the above mentioned results, the authors used either continuity or lower semicontinuity of ϕ in proving the fixed point results. Now the following question arises: " Can we replace the continuity or lower semicontinuity of ϕ by nondecreasing nature of ϕ ? " In this paper we answer this question affirmatively.

Throughout this paper we denote by

$\Psi = \{\psi : [0,\infty) \to [0,\infty)$ such that ψ is continuous and nondecreasing$\}$

$\Phi = \{\phi : [0,\infty) \to [0,\infty)$ such that ϕ is nondecreasing and $\phi(t) = 0$

if and only if $t = 0\}$.

In this paper we introduce the following definition.

Definition 1.11 Let f, g, S and T be four selfmaps of a metric space (X,d). If there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that

$\psi(d(fx,gy)) \leq \psi(M(x,y)) - \phi(m(x,y))$ for all x, y in X, where

$M(x,y) = \max\{d(Sx,Ty),\ d(fx,Sx),\ d(gy,Ty),\ \frac{1}{2}[d(Sx,gy) + d(fx,Ty)]\}$,

and

$m(x,y) = \max\{d(Sx,Ty),\ d(fx,Sx),\ d(gy,Ty)\}$.

Then the maps f, g, S and T are said to satisfy generalized (ψ,ϕ)- weakly contractive condition.

In section 2 we prove a common fixed point result for four selfmaps satisfying generalized $(\psi,\ \phi)$ - weakly contractive condition in which ϕ need not be either continuous or lower semicontinuous in a complete metric space.

An example is given in support of the main result of the paper.

2. A common fixed point of two pairs of weakly contractive maps

Let f, g, S and T be selfmaps of a metric space (X, d) satisfying

$fX \subseteq TX$ and $gX \subseteq SX$. (A)

Let $x_0 \in X$. By using (A) we can choose $x_1 \in X$ such that

$y_0 = fx_0 = Tx_1$.

Corresponding to $x_1 \in X$ we can choose $x_2 \in X$ such that

$y_1 = gx_1 = Sx_2$, and so on.

In general, we can define sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$y_{2n} = fx_{2n} = Tx_{2n+1}$ and

$y_{2n+1} = gx_{2n+1} = Sx_{2n+2}, n = 0, 1, 2, \cdots$. (B)

Suppose there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that

$\psi(d(fx, gy)) \leq \psi(M(x, y)) - \phi(m(x, y))$ for all x, y in X, (A')

where

$M(x, y) = \max\{d(Sx, Ty),\ d(fx, Sx),\ d(gy, Ty),\ \frac{1}{2}[d(Sx, gy) + d(fx, Ty)]\}$,

and

$m(x, y) = \max\{d(Sx, Ty),\ d(fx, Sx),\ d(gy, Ty)\}$.

We denote: $F(f, S)$=$\{x \in X : f(x) = S(x) = x\}$ and

$F(g, T)$=$\{x \in X : g(x) = T(x) = x\}$.

Proposition 2.1. Let f, g, S and T be selfmaps of a metric space (X, d) such that $fX \subseteq TX$, $gX \subseteq SX$; and f, g, S and T are (ψ, ϕ) generalized weakly contractive maps. Assume that (f, S) and (g, T) are weakly compatible.

Then $F(f, S) \neq \emptyset$ if and only if $F(g, T) \neq \emptyset$.

In this case, f, g, S and T have a unique common fixed point.

proof. First we assume that $F(f, S) \neq \emptyset$. Let $z \in F(f, S)$, then

$$z = fz = Sz. \tag{2.1.1}$$

Now, we show that $z \in F(g, T)$.

Since $fX \subseteq TX$ there exists $w \in X$ such that

$$fz = Tw. \tag{2.1.2}$$

Then from (2.1.1) and (2.1.2) we get

$$fz = Tw = Sz = z. \tag{2.1.3}$$

Next we show that $gw = z$.

Now by using (A') we have

$$\psi(d(z, gw)) = \psi(d(fz, gw)) \leq \psi(M(z, w)) - \phi(m(z, w)), \tag{2.1.4}$$

where

$$\begin{aligned} M(z, w) = \; & max\{d(Sz, Tw),\; d(fz, Sz),\; d(gw, Tw), \frac{1}{2}[d(Sz, gw) + d(fz, Tw)]\} \\ = \; & max\{0,\; 0,\; d(gw, z),\; \frac{1}{2}d(z, gw)\} = d(z, gw), \end{aligned}$$

hence

$$M(z, w) = d(z, gw). \tag{2.1.5}$$

and

$$\begin{aligned} m(z, w) \; &= \; max\{d(Sz, Tw),\; d(fz, Sz),\; d(gw, Tw)\} \\ &= \; max\{0,\; 0,\; d(gw, z)\} = d(z, gw), \end{aligned}$$

so that

$$m(z, w) = d(z, gw). \tag{2.1.6}$$

Using (2.1.5) and (2.1.6) in (2.1.4), we have

$$\psi(d(z, gw)) \leq \psi(d(z, gw)) - \phi(d(z, gw)),$$

which implies that

$$\phi(d(z, gw)) = 0.$$

Hence

$$z = gw. \tag{2.1.7}$$

From (2.1.3) and (2.1.7) it follows that

$$gw = Tw = z. \tag{2.1.8}$$

Since g and T are weakly compatible, by (2.1.8) we have

$gz = gTw = Tgw = Tz$. Hence

$$gz = Tz. \tag{2.1.9}$$

Now, we show that

$$gz = z.$$

From (A') we have

$$\psi(d(z, gz)) = \psi(d(fz, gz)) \leq \psi(M(z, z)) - \phi(m(z, z)), \tag{2.1.10}$$

where

$$\begin{aligned} M(z, z) = & \ max\{d(Sz, Tz),\ d(fz, Sz),\ d(gz, Tz),\ \frac{1}{2}[d(Sz, gz) + d(fz, Tz)]\} \\ = & \ max\{d(z, gz),\ 0,\ 0,\ \frac{1}{2}[d(z, gz) + d(z, gz)]\} \\ = & \ d(z, gz), \end{aligned}$$

so that

$$M(z, z) = d(z, gz). \tag{2.1.11}$$

Also, it is easy to see that

$$m(z, z) = d(z, gz). \tag{2.1.12}$$

Therefore using (2.1.11) and (2.1.12) in (2.1.10), we have

$$\psi(d(z, gz)) \leq \psi(d(z, gz)) - \phi(d(z, gz)),$$

which implies that

$$\phi(d(z, gz)) = 0$$

i.e.,

$$z = gz. \tag{2.1.13}$$

Hence from (2.1.9) and (2.1.13) we have

$$z = gz = Tz.$$

Therefore

$$F(g,T) \neq \emptyset \ . \tag{2.1.14}$$

Hence, from (2.1.1) and (2.1.14),we have

$$F(f,S) \subseteq F(g,T). \tag{2.1.15}$$

Conversely assume that

$$F(g,T) \neq \emptyset.$$

Let $z \in F(g,T)$, then

$$gz = Tz = z. \tag{2.1.16}$$

On using similar steps as above we can show that

$$z \in F(f,S). \tag{2.1.17}$$

Thus from (2.1.16) and (2.1.17) we get

$$F(g,T) \subseteq F(f,S). \tag{2.1.18}$$

Therefore from (2.1.15) and (2.1.18) we have $F(f,S) = F(g,T)$, and f, g, S and T have a unique common fixed point.

Proposition 2.2. Let f, g, S and T be selfmaps of a metric space (X,d) such that $fX \subseteq TX$, $gX \subseteq SX$; and f, g, S and T are $(\psi, \ \phi)$ generalized weakly contractive maps. Then for each $x_0 \in X$ the sequence $\{ \ y_n\}$ defined by (B) is Cauchy in X.

proof. Let $x_0 \in X$ and $\{y_n\}$ be a sequence defined by (B). First we suppose that $y_n = y_{n+1}$ for some n .

Let $n=2m$, then $y_{2m}=y_{2m+1}$.

Now, we have

$$\begin{aligned}
M(x_{2m+2},x_{2m+1}) &= max\{d(Sx_{2m+2},Tx_{2m+1}),\ d(fx_{2m+2},Sx_{2m+2}),\ d(gx_{2m+1},Tx_{2m+1}),\\
&\qquad \frac{1}{2}[d(Sx_{2m+2},gx_{2m+1})+d(fx_{2m+2},Tx_{2m+1})]\}\\
&= max\{d(y_{2m+1},y_{2m}),\ d(y_{2m+2},y_{2m+1}),\ d(y_{2m+1},y_{2m}),\\
&\qquad \frac{1}{2}[d(y_{2m+1},y_{2m+1})+(d(y_{2m+2},y_{2m})]\}\\
&= max\{0,\ d(y_{2m+2},y_{2m+1}),\ 0\ ,\frac{1}{2}[0+(d(y_{2m+2},y_{2m})]\}\\
&= max\{d(y_{2m+2},y_{2m+1}),\ \frac{1}{2}d(y_{2m+2},y_{2m})\}\\
&\le max\{d(y_{2m+2},y_{2m+1}),\ \frac{1}{2}[d(y_{2m+2},y_{2m+1})+d(y_{2m+1},y_{2m})]\}\\
&= max\{d(y_{2m+2},y_{2m+1}),\frac{1}{2}d(y_{2m+2},y_{2m+1})\}\\
&= d(y_{2m+2},y_{2m+1}).
\end{aligned}$$

Since

$$d(y_{2m+2},y_{2m+1})\le M(x_{2m+2},x_{2m+1})$$

we have

$$M(x_{2m+2},x_{2m+1})=\ d(y_{2m+2},y_{2m+1}). \tag{2.2.1}$$

Also

$$\begin{aligned}
m(x_{2m+2},x_{2m+1}) &= max\{d(Sx_{2m+2},Tx_{2m+1}),d(fx_{2m+2},Sx_{2m+2}),d(gx_{2m+1},Tx_{2m+1})\}\\
&= max\{d(y_{2m+1},y_{2m}),\ d(y_{2m+2},y_{2m+1}),\ d(y_{2m+1},y_{2m})\}\\
&= max\{0,\ d(y_{2m+2},y_{2m+1}),\ 0)\}\\
&= d(y_{2m+2},y_{2m+1}),
\end{aligned}$$

so that

$$m(x_{2m+2},x_{2m+1})=d(y_{2m+2},y_{2m+1}). \tag{2.2.2}$$

Now, from $(A^{'})$ we have

$$\begin{aligned}\psi(d(y_{2m+2}, y_{2m+1}) &= \psi(d(fx_{2m+2}, gx_{2m+1})) \\ &\leq \psi(M(x_{2m+2}, x_{2m+1})) - \phi(m(x_{2m+2}.x_{2m+1}))\end{aligned} \tag{2.2.3}$$

Using (2.2.1) and (2.2.2) in (2.2.3) we get

$$\psi(d(y_{2m+2}, y_{2m+1})) \leq \psi(d(y_{2m+2}, y_{2m+1})) - \phi(d(y_{2m+2}, y_{2m+1})),$$

which implies that

$$\phi(d(y_{2m+2}, y_{2m+1})) \leq 0.$$

Hence

$$d(y_{2m+2}, y_{2m+1}) = 0, i.e., y_{2m+2} = y_{2m+1}. \tag{2.2.4}$$

In a similar way it is easy to show that

$$y_{2m+3} = y_{2m+2}. \tag{2.2.5}$$

Hence from (2.2.4) and (2.2.5) we have $y_{n+1} = y_{n+2}$.

Now by applying induction it is easy to show that $y_n = y_{n+k}$ for all $k = 0, 1, 2, ...$

Therefore, $\{y_m\}$ is a constant sequence for $m \geq n$ and hence

it is a Cauchy sequence in X.

Now we suppose that

$$y_n \neq y_{n+1}. \tag{2.2.6}$$

for all n.

Then from $(A^{'})$ we have

$$\psi(d(y_{2n+2}, y_{2n+1})) \leq \psi(M(x_{2n+2}, x_{2n+1})) - \phi(m(x_{2n+2}, x_{2n+1})), \tag{2.2.7}$$

where

$$\begin{aligned}
M(x_{2n+2},x_{2n+1}) = &\ max\{d(Sx_{2n+2},\ Tx_{2n+1}),\ d(fx_{2n+2},Sx_{2n+2}),\ d(gx_{2n+1},Tx_{2n+1}) \\
&\qquad \frac{1}{2}[d(Sx_{2n+2},gx_{2n+1})+d(fx_{2n+2},Tx_{2n+1})]\} \\
= &\ max\{d(y_{2n+1},y_{2n}),\ d(y_{2n+2},y_{2n+1}),\ d(y_{2n+1},y_{2n}) \\
&\qquad \frac{1}{2}[d(y_{2n+1},y_{2n+1})+d(y_{2n+2},y_{2n})]\} \\
= &\ max\{d(y_{2n+1},y_{2n}),d(y_{2n+2},y_{2n+1}),\frac{1}{2}d(y_{2n+2},y_{2n})\} \\
\leq &\ max\{d(y_{2n+1},y_{2n}),\ d(y_{2n+2},y_{2n+1}), \\
&\qquad \frac{1}{2}[d(y_{2n+2},y_{2n+1})+d(y_{2n+1},y_{2n})]\} \\
\leq &\ max\{d(y_{2n+1},y_{2n}),d(y_{2n+2},y_{2n+1}),max\{d(y_{2n+2},y_{2n+1}), \\
&\qquad d(y_{2n+1},y_{2n})\}\} \\
= &\ \max\{d(y_{2n+1},y_{2n}),d(y_{2n+2},y_{2n+1})\}.
\end{aligned} \tag{2.2.8}$$

Also we have

$$m(x_{2n+2},y_{2n+1}) = \ max\{d(y_{2n+1},y_{2n}),d(y_{2n+2},y_{2n+1})\}. \tag{2.2.9}$$

Hence from (2.2.8) and (2.2.9) we get

$$M(x_{2n+2},x_{2n+1}) = m(x_{2n+2},x_{2n+1}).$$

If

$$max\{d(y_{2n+1},y_{2n}),d(y_{2n+2},y_{2n+1})\} = d(y_{2n+2},y_{2n+1}), \tag{2.2.10}$$

then using (2.2.10) in (2.2.7) we get

$$\psi(d(y_{2n+2},y_{2n+1})) \leq \psi(d(y_{2n+2},y_{2n+1})) - \phi(d(y_{2n+2},y_{2n+1})),$$

which implies that

$$\phi(d(y_{2n+2},y_{2n+1})) \leq 0.$$

It follows that

$$y_{2n+2} = y_{2n+1},$$

which is a contradiction with (2.2.6) Therefore,

$$max\{d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1})\} = d(y_{2n+1}, y_{2n})$$

and

$$\psi(d(y_{2n+2}, y_{2n+1})) \leq \psi(d(y_{2n+1}, y_{2n})) - \phi(d(y_{2n+1}, y_{2n})) < \psi(d(y_{2n+1}, y_{2n})).$$

Since ψ is nondecreasing we have

$$d(y_{2n+2}, y_{2n+1}) < d(y_{2n+1}, y_{2n}). \tag{2.2.11}$$

Similarly we can show that

$$d(y_{2n+3}, y_{2n+2}) < d(y_{2n+2}, y_{2n+1}). \tag{2.2.12}$$

Therefore, from (2.2.11) and (2.2.12) we have

$$d(y_{n+2}, y_{n+1}) < d(y_{n+1}, y_n)$$

for $n = 0, 1, 2, 3, ...$

Hence the sequence $d\{(y_{n+1}, y_n)\}$ is a nonincreasing sequence of nonnegative real numbers and hence it converges to some real number δ (say), $\delta \geq 0$.

Now, we show that $\delta = 0$. Suppose

$$\delta > 0. \tag{2.2.13}$$

Since

$$M(x_{2n+2}, x_{2n+1}) = \ m(x_{2n+2}, x_{2n+1}) = \ d(y_{2n+1}, y_{2n})$$

from (2.2.7) we have

$$\psi(d(y_{2n+2}, y_{2n+1})) \leq \psi(d(y_{2n+1}, y_{2n})) - \phi(d(y_{2n+1}, y_{2n})), \tag{2.2.14}$$

which implies that

$$\phi(d(y_{2n+1}, y_{2n})) \leq \psi(d(y_{2n+1}, y_{2n})) - \psi(d(y_{2n+2}, y_{2n+1})). \tag{2.2.15}$$

Since the sequence $\{d(y_{n+1}, y_n)\}$ is nonincreasing it follows that

$$\delta \leq d(y_{2n+1}, y_{2n}),$$

for all n. Since ϕ is nondecreasing, $\phi(\delta) \leq \phi(d(y_{2n+1}, y_{2n}))$ for all n. Therefore we have

$$0 \leq \phi(\delta) \leq \phi(d(y_{2n+1}, y_{2n})) \leq \psi(d(y_{2n+1}, y_{2n})) - \psi(d(y_{2n+2}, y_{2n+1})). \tag{2.2.16}$$

On taking limits as $n \to \infty$ in (2.2.16)and using the continuity of ψ we get

$$0 \leq \phi(\delta) \leq \lim_{n\to\infty} \phi(d(y_{2n+1}, y_{2n})) \leq 0. \tag{2.2.17}$$

Thus we have

$\phi(\delta) = 0$ which implies that $\delta = 0$, a contradiction with (2.2.13). Therefore

$$\delta = 0. \tag{2.2.18}$$

Next, we show that the sequence $\{y_n\}$ is a Cauchy sequence in X. It suffices to show that $\{y_{2n}\}$ is a Cauchy sequence in X. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$, and sequences of even positive integers $\{2m_k\}$, $\{2n_k\}$ with $2m_k > 2n_k > k$ for each positive integer k such that

$$d(y_{2m_k}, y_{2n_k}) \geq \epsilon. \tag{2.2.19}$$

Let $2m_k$ be the least positive integer exceeding $2n_k$ and satisfying (2.2.19). Then we have

$$d(y_{2m_k}, y_{2n_k}) \geq \epsilon,$$

and

$$d(y_{2m_k-2}, y_{2n_k}) < \epsilon. \tag{2.2.20}$$

Now, we prove that

(i) $\lim_{k\to\infty} d(y_{2m_k}, y_{2n_k}) = \epsilon$, (ii) $\lim_{k\to\infty} d(y_{2m_k+1}, y_{2n_k}) = \epsilon$,

(iii) $\lim_{k\to\infty} d(y_{2m_k}, y_{2n_k-1}) = \epsilon$, and (iv) $\lim_{k\to\infty} d(y_{2m_k+1}, y_{2n_k-1}) = \epsilon$.

Since the proof in each case is similar, we prove (i). By (2.2.19), we have

$$\epsilon \leq d(y_{2m_k}, y_{2n_k})$$

for all k, we have

$$\epsilon \leq \lim_{k\to\infty} d(y_{2m_k}, y_{2n_k}). \tag{2.2.21}$$

For each positive integer k, by the triangle inequality and (2.2.18) we get

$$\begin{aligned} d(y_{2m_k}, y_{2n_k}) &\leq d(y_{2m_k}, y_{2m_k-1}) + d(y_{2m_k-1}, y_{2m_k-2}) + d(y_{2m_k-2}, y_{2n_k}) \\ &\leq d(y_{2m_k}, y_{2m_k-1}) + d(y_{2m_k-1}, y_{2m_k-2}) + \epsilon. \end{aligned}$$

On taking limits as $k \to \infty$ we have

$$\lim_{k\to\infty} d(y_{2m_k}, y_{2n_k}) \leq \epsilon. \tag{2.2.22}$$

Therefore, from (2.2.21) and (2.2.22) $\lim_{k\to\infty} d(y_{2m_k}, y_{2n_k}) = \epsilon$. Now we have

$$\begin{aligned} M(x_{2n_k}, x_{2m_k+1}) = max\{&d(Sx_{2n_k}, Tx_{2m_k+1}),\ d(fx_{2n_k}, Sx_{2n_k}),\ d(gx_{2m_k+1}, Tx_{2m_k+1}), \\ &\frac{1}{2}[d(Sx_{2n_k}, gx_{2m_k+1}) + d(fx_{2n_K}, Tx_{2m_k+1})]\} \\ = max\{&d(y_{2n_k-1}, y_{2m_k}),\ d(y_{2n_k}, y_{2n_{k-1}}),\ d(y_{2m_{k+1}}, y_{2m_k}), \\ &\frac{1}{2}[d(y_{2n_k-1}, y_{2m_k+1}) + d(y_{2n_k}, y_{2m_k})]\} \\ = max\{&d(y_{2n_k-1}, y_{2m_k}),\ d(y_{2n_k}, y_{2n_k-1}),\ d(y_{2m_k+1}, y_{2m_k}), \\ &\frac{1}{2}[d(y_{2n_k-1}, y_{2m_k+1}) + d(y_{2n_k}, y_{2m_k})]\}. \end{aligned}$$

On taking limits as $k \to \infty$ we get

$$\lim_{k\to\infty} M(x_{2n_k}, x_{2m_k+1}) = max\{\epsilon, 0, 0, \epsilon\} = \epsilon. \tag{2.2.23}$$

Similarly we can show that

$$\lim_{k\to\infty} m(x_{2n_k}, x_{2m_k+1}) = \epsilon. \tag{2.2.24}$$

Now putting $x = x_{2n_k}$ and $y = x_{2m_k+1}$ in (A') we obtain

$$\psi(d(y_{2n_k}, y_{2m_k+1})) = \psi(d(fx_{2n_k}, gx_{2m_k+1})) \leq \psi(M(x_{2n_k}, x_{2m_k+1})) - \phi(m(x_{2n_k}, x_{2m_k+1})).$$

This implies that

$$\phi(m(x_{2n_k}, x_{2m_k+1})) \leq \psi(M(x_{2n_k}, x_{2m_k+1})) - \psi(d(y_{2n_k}, y_{2m_k+1})).$$

Since $\lim\limits_{k\to\infty} M(x_{2n_k}, x_{2m_k+1}) = \epsilon$, $\lim\limits_{k\to\infty} d(y_{2m_k+1}, y_{2n_k}) = \epsilon$,
and $\lim\limits_{k\to\infty} m(x_{2n_k}, x_{2m_k+1}) = \epsilon$, we have $\frac{1}{2}\epsilon \le m(x_{2n_k}, x_{2m_k+1})$ for sufficiently large k.
Since ϕ is nondecreasing we have $0 \le \phi(\frac{1}{2}\epsilon) \le \phi(m(x_{2n_k}, x_{2m_k+1}))$ for sufficiently large k.
Hence we have

$$0 \le \phi(\frac{1}{2}\epsilon) \le \phi(m(x_{2n_k}), x_{2m_k+1})) \le \psi(M(x_{2n_k}, x_{2m_k+1})) - \psi(d(y_{2n_k}, y_{2m_k+1}))$$

for sufficiently large k.
On taking limits as $k \to \infty$ and using (2.2.23), (2.2.24) and the continuity of ψ in the last inequality we get

$$\begin{aligned} 0 &\le \phi(\frac{1}{2}\epsilon) \le \lim_{k\to\infty} \phi(m(x_{2n_k}, x_{2m_k+1})) \\ &\le \lim_{k\to\infty} (\psi(M(x_{2n_k}, x_{2m_k+1})) - \psi(d(y_{2n_k}, y_{2m_k+1})) \\ &= \psi(\epsilon) - \psi(\epsilon) = 0. \end{aligned}$$

Hence we have

$$\phi(\frac{1}{2}\epsilon) = 0.$$

Hence by the properity of ϕ, we have $\epsilon = 0$, a contradiction with $\epsilon > 0$.
Therefore $\{y_{2n}\}$ is a Cauchy sequence so that $\{y_n\}$ is a Cauchy sequence.

Theorem 2.3. *Let f, g, S and T be selfmaps of a complete metric space (X, d) such that $fX \subseteq TX$, $gX \subseteq SX$. Assume that f, g, S and T are generalized (ψ, ϕ) - weakly contractive maps. If the pairs (f, S) and (g, T) are weakly compatible and one of the ranges fX, gX, SX and TX is closed, then for each $x_0 \in X$ the sequence $\{y_n\}$ defined by (B) is Cauchy in X and $\lim\limits_{n\to\infty} y_n = z$* (say) *and z is a unique common fixed point of f, g, S and T.*

Proof. Let $x_0 \in X$. By proposition 2.2, the sequence $\{y_n\}$ befined by (B) is Cauchy in X. Since X is complete, there exists $z \in X$ such that $\lim\limits_{n\to\infty} y_n = z$.
Thus clearly

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z,$$

and

$$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z. \tag{2.3.1}$$

Case (i): Suppose that SX is closed.

In this case, z is in SX and hence there exists

$u \in X$ such that

$$Su = z. \tag{2.3.2}$$

Now, we show that $fu = z$. Suppose $fu \neq z$.

Now

$$\begin{aligned} M(u, x_{2n+1}) = \ & max\{d(Su, Tx_{2n+1}),\ d(fu, Su),\ d(gx_{2n+1}, Tx_{2n+1}), \\ & \frac{1}{2}[d(Su, gx_{2n+1}) + d(fu, Tx_{2n+1})]\} \end{aligned}$$

and on taking limits as $n \to \infty$ we have

$$\lim_{n\to\infty} M(u, x_{2n+1}) = d(fu, z) \tag{2.3.3}$$

Similarly it is easy to see that

$$\lim_{n\to\infty} m(u, x_{2n+1}) = d(fu, z). \tag{2.3.4}$$

Using $(A^{'})$, we have

$$\psi(d(fu, gx_{2n+1})) \leq \psi(M(u, x_{2n+1})) - \phi(m(u, x_{2n+1})), \tag{2.3.5}$$

which implies that

$$\phi(m(u, x_{2n+1})) \leq \psi(M(u, x_{2n+1})) - \psi(d(fu, gx_{2n+1})).$$

Since

$$\lim_{n\to\infty} m(u, x_{2n+1}) = d(fu, z),$$

we have

$$\frac{1}{2}d(fu, z) \leq m(u, x_{2n+1})$$

for sufficiently large n. Since ϕ is nondecreasing we get

$$0 \leq \phi(\frac{1}{2}d(fu, z)) \leq \phi(m(u, x_{2n+1}))$$

for sufficiently large n and hence we have

$$0 \leq \phi(\frac{1}{2}d(fu,z)) \leq \phi(m(u,x_{2n+1})) \leq \psi(M(u,x_{2n+1})) - \psi(d(fu,gx_{2n+1}))$$

for sufficiently large n. On taking limits as $n \to \infty$ using (2.3.3), (2.3.4) and the continuity of ψ we get,

$$0 \leq \phi(\frac{1}{2}d(fu,z)) \leq \lim_{n\to\infty} \phi(m(u,x_{2n+1})) \leq \lim_{n\to\infty}(\psi(M(u,x_{2n+1})) - \psi(d(fu,gx_{2n+1}))) = 0.$$

Hence we have

$$0 \leq \phi(\frac{1}{2}d(fu,z)) \leq 0.$$

It follows that $\phi(\frac{1}{2}d(fu,z)) = 0$ so that $d(fu,z)) = 0$, and hence $fu = z$, a contradiction with the assumption $fu \neq z$.

Therefore $fu = z$. Since f and S are weakly compatible we have
$fz = fSu = Sfu = Sz$. Therefore,

$$fz = Sz. \tag{2.3.6}$$

Now, we show that $fz = z$. Suppose that $fz \neq z$.

We have

$$\begin{aligned} M(z,x_{2n+1}) = \ & max\{d(Sz,Tx_{2n+1}), d(fz,Sz), (gx_{2n+1},Tx_{2n+1}), \\ & \frac{1}{2}[d(Sz,gx_{2n+1}) + d(fz,Tx_{2n+1})]\} \end{aligned}$$

and on taking limits as $n \to \infty$ we have

$$\lim_{n\to\infty} M(z,x_{2n+1}) = d(fz,z). \tag{2.3.7}$$

Also we have

$$\lim_{n\to\infty} m(z,x_{2n+1}) = d(fz,z). \tag{2.3.8}$$

Now, from (A') we have

$$\psi(d(fz,gx_{2n+1})) \leq \psi(M(z,x_{2n+1})) - \phi(m(u,x_{2n+1})) \tag{2.3.9}$$

which implies that

$$\phi(m(z,x_{2n+1})) \leq \psi(M(z,x_{2n+1})) - \phi(d(fz,gx_{2n+1})).$$

Since

$$\lim_{n\to\infty} m(u, x_{2n+1}) = d(fu, z),$$

it follows that

$$\frac{1}{2}d(fz, z) \leq m(z, x_{2n+1})$$

for sufficiently large n. Since ϕ is nondecreasing we have

$$0 \leq \phi(\frac{1}{2}d(fz, z)) \leq \phi(m(z, x_{2n+1}))$$

for sufficiently large n. So we have

$$0 \leq \phi(\frac{1}{2}d(fz, z)) \leq \phi(m(z, x_{2n+1})) \leq \psi(M(z, x_{2n+1})) - \psi(d(fz, gx_{2n+1}))$$

for sufficiently large n. On taking limits as $n \to \infty$ using (2.3.7), (2.3.8) and the continuity of ψ we get

$$0 \leq \phi(\frac{1}{2}d(fz, z)) \leq \lim_{n\to\infty} \phi(m(z, x_{2n+1})) \leq \lim_{n\to\infty} (\psi(M(z, x_{2n+1})) - \psi(d(fz, gx_{2n+1}))) = 0.$$

Hence we have

$$\phi(\frac{1}{2}d(fz, z)) = 0,$$

which implies that

$d(fz, z) = 0$, that is $fz = z$, a contradiction with $fz \neq z$.

Hence

$$fz = z. \tag{2.3.10}$$

Therefore from (2.3.6) and (2.3.10) we have $z = fz = Sz$.

By proposition 2.1. $F(g, T) \neq \emptyset$ and $z \in F(g, T)$.

Hence $z = fz = gz = Sz = Tz$.

Case (ii): Suppose that gX is closed.

In this case $z \in gX \subseteq SX$, which implies that $z \in SX$ and hence the proof follows as in case (i).

For the cases TX is closed and fX is closed we follow the arguments similar to the cases of SX is closed and gX is closed respectively.

This completes the proof of the theorem.

Corollary 2.4. *Let f, g, S and T be selfmaps of a complete metric space (X, d) satisfying $fX \subseteq TX$ and $gX \subseteq SX$. Assume that the maps f, g, S and T satisfy the following condition there exists $\phi \in \Phi$ such that*

$d(fx, gy) \leq M(x, y) - \phi(m(x, y))$ *for all x, y in X, where*

$M(x, y) = max\{d(Sx, Ty),\ d(fx, Sx),\ d(gy, Ty),\ \frac{1}{2}[d(Sx, gy) + d(fx, Ty)]\}$,

and

$m(x, y) = max\{d(Sx, Ty),\ d(fx, Sx),\ d(gy, Ty)\}$.

If the pairs (f, S) and (g, T) are weakly compatible and one of the ranges fX, gX, SX and TX is closed, then f, g, S and T have a unique common fixed point.

Proof. Follows by choosing ψ as the identity mapping on $[0,\ \infty)$ in Theorem 2.3.

Corollary 2.5. *Let f and g be selfmaps of a complete metric space (X, d). Suppose that there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$\psi(d(fx, gy)) \leq \psi(M(x, y)) - \phi(m(x, y))$ *for all x, y in X, where*

$M(x, y) = max\{d(x, y),\ d(x, fx),\ d(y, gy),\ \frac{1}{2}[d(x, gy) + d(y, fx)]\}$,

and

$m(x, y) = max\{d(x, y),\ d(x, fx),\ d(y, gy)\}$.

Then f and g have a unique common fixed point.

Proof. Follows by choosing $T = S = I_X$ (I_X, the identity map on X) in Theorem 2.3.

Now we give an example in support of Theorem 2.3.

Example 2.6. *Let $X = [0, 1]$ with the usual metric and let f, g, S and T be selfmaps on X defined as follows*

$$gx = \begin{cases} \frac{1}{2}, & if\ 0 \leq x \leq \frac{1}{2} \\ 1, & if\ \frac{1}{2} < x \leq 1, \end{cases} \qquad fx = \begin{cases} \frac{1}{2}, & if\ 0 \leq x < 1 \\ 1, & if\ x = 1, \end{cases}$$

$$Sx = \begin{cases} 0, & if\ 0 \leq x < \frac{1}{2}\ and\ \frac{3}{4} \leq x \leq 1 \\ \frac{1}{2}, & if\ x = \frac{1}{2} \\ 1, & if\ \frac{1}{2} < x < \frac{3}{4} \end{cases} \quad and\ Tx = \begin{cases} 1, & if\ 0 \leq x < \frac{1}{2} \\ \frac{1}{2}, & if\ x = \frac{1}{2} \\ \frac{1}{12}, & if\ \frac{1}{2} < x \leq 1. \end{cases}$$

We define $\psi,\ \phi : [0, \infty) \to\ [0, \infty)$ *by*

$\psi(t) = t^2$, $t \geq 0$ *and*

$$\phi(t) = \begin{cases} \frac{t}{4}, & if\ 0 \leq t < 1 \\ \frac{1}{3}, & if\ t = 1 \\ \frac{t^2}{2}, & if\ t > 1. \end{cases}$$

Then $\psi \in \Psi$ *and* $\phi \in \Phi$ *and the maps* f, g, S *and* T *are* (ψ, ϕ) *generalized weakly contractive so that* f, g, S *and* T *satisfy all the hypotheses of Theorem 2.3. and* f, g, S *and* T *have a unique common fixed point* $\frac{1}{2}$*. Here we note that* ϕ *is not a lower semicontinuous function.*

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