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APPROXIMATING A COMMON FIXED POINT FOR FINITE FAMILY OF DEMIMETRIC MAPPINGS IN CAT(0) SPACE

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Abstract. In this paper, we study a modified Halpern-type algorithm for approximating a common fixed point of demimetric mappings and prove a strong convergence theorem for finding a common element of the set of common fixed points for a finite family of demimetric mappings in a complete CAT(0) space.

Keywords: Demimetric mapping; common fixed point; $\triangle$ convergence; Strong convergence; CAT(0) space.

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1. Introduction

Let C be a nonempty, closed and convex subset of a real Hilbert space H and let T be any mapping on C, denote by $F(T) := \{x \in C : Tx = x\}$ the set of all fixed of point of T.

Definition 1.1 A mapping $T : C \to H$ is said to be:

(1) a nonexpansive mapping, if $||Tx - Ty|| \leq ||x - y||$ for all $x, y \in C$;

(2) a quasi-nonexpansive mapping if $F(T) \neq \emptyset$ and $||Tx - p|| \leq ||x - p||$, for any $x \in C$ and $p \in F(T)$;

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(3) a k-strict pseudo-contraction in the sense of Browder and Petryshyn [4] if there exists $k \in [0,1)$ such that

$$||Tx - Ty||^2 \leq ||x-y||^2 + k||x - Tx - (y - Ty)||^2, \text{ for all } x, y \in C; \tag{1.1}$$

(4) a generalized hybrid if there exist $\alpha, \beta \in \mathbb{R}$ such that, for all $x, y \in C$

$$\alpha||Tx - Ty||^2 + (1-\alpha)||x - Ty||^2 \leq \beta||Tx - y||^2 + (1-\beta)||x-y||^2.$$

Recently, Takahashi [17] introduced the notion of new nonlinear mappings in smooth, strictly convex and reflexive Banach space as follows:

Definition 1.2 Let E be a smooth, strictly convex and reflexive Banach space, let K be a nonempty, closed and convex subset of E and let η be a real number with $\eta \in (-\infty, 1)$. Then a mapping $T : K \to E$ with $F(T) \neq \emptyset$ is called $\eta - demimetric$ [17] if,

$$\langle x - q, J(x - Tx) \rangle \geq \frac{1-\eta}{2}||x - Tx||^2,$$

for any $x \in K$ and $q \in F(T)$, where J is the duality mapping on E. In a Hilbert space H, the above definition is as follows: A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called $\eta - demimetric$ if

$$\langle x - q, x - Tx \rangle \geq \frac{1-\eta}{2}||x - Tx||^2,$$

for any $x \in C$ and $q \in F(T)$. In [13], Komiya and Takahashi observed that the class of η-demimetric mapping covers strict pseudo-contraction and generalized hybrid mappings.

Very recently Takahashi et al. [18] proved a strong convergence theorem for finding a common element of the set of common fixed points for a finite family of these new mappings and the set of common solutions of variational inequality problems for a finite family of inverse-strongly monotone mappings in a Hilbert space. Also in 2018, Song [16], studied the infinite family of demimetric mappings and establish the following Lemma:

Lemma 1.3 (Song [16]) Let H be a Hilbert space and C be nonempty convex subset of H. Assume that $\{T_i\}_{i=1}^{\infty} : C \to H$ be an infinite family of $k_i - demimetric$ mappings with $\sup\{k_1 : i \in \mathbb{N}\} < 1$ such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Assume that $\{\eta_i\}_{i=1}^{\infty}$ is a positive sequence such that $\sum_{i=1}^{\infty} \eta_i = 1$. Then $\sum_{i=1}^{\infty} \eta_i T_i : C \to H$ is a k-demimetric mapping with $k = \sup\{k_i : i \in \mathbb{N}\}$ and $F(\sum_{i=1}^{\infty} \eta_i T_i) = \bigcap_{i=1}^{\infty} F(T_i)$.

Let (X,d) be a metric space and $x,y \in X$ with $d(x,y) = l$. A geodesic path from x to y is an isometry $c : [0,l] \to X$ such that $c(0) = x$ and $c(l) = y$. The image of a geodesic path is called a *geodesic segment*. A metric space X is a (uniquely) *geodesic space*, if every two points of X are joined by only one geodesic segment. A geodesic triangle $\triangle(x_1,x_2,x_3)$ in a geodesic space X consists of three points x_1,x_2,x_3 of X and three geodesic segments joining each pair of vertices. A comparison triangle of a geodesic triangle $\triangle(x_1,x_2,x_3)$ is the triangle $\bar{\triangle}(x_1,x_2,x_3) := \triangle(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in the Euclidean space $\mathbb{R}^2$ such that

$$d(x_i,x_j) = d_{\mathbb{R}^2}(\bar{x}_i,\bar{y}_j),\ \forall i,j = 1,2,3.$$

A geodesic space X is a CAT(0) space, if for each geodesic triangle $\triangle(x_1,x_2,x_3)$ in X and its comparison triangle $\bar{\triangle} := \triangle(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in $\mathbb{R}^2$, the CAT(0) inequality $d(x,y) \le d_{\mathbb{R}^2}(\bar{x},\bar{y})$ is satisfied for all $x,y \in \triangle$ and $\bar{x},\bar{y} \in \bar{\triangle}$.

A thorough discussion of these spaces and their important role in various branches of Mathematics are given in [3,5]. Let $x,y \in X$ and $\lambda \in [0,1]$, we write $\lambda x \oplus (1-\lambda)y$ for the unique point z in the geodesic segment joining from x to y such that

$$d(z,x) = (1-\lambda)d(x,y) \quad \text{and} \quad d(z,y) = \lambda d(x,y). \tag{1.2}$$

We also denote by $[x,y]$ the geodesic segment joining from x to y, that is, $[x,y] = \{\lambda x \oplus (1-\lambda)y : \lambda \in [0,1]\}$. A subset C of a CAT(0) space is convex if $[x,y] \subseteq C$ for all $x,y \in C$. Berg and Nikolaev [2] introduced the concept of *quasilinearization* in a metric space X. Let denote a pair $(a,b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. The quasilinearization is a map $\langle .,. \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\Big(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)\Big), \quad \forall a,b,c,d \in X. \tag{1.3}$$

It is easily seen that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ for all $a,b,c,d \in X$. We say that X satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \le d(a,b)d(c,d) \tag{1.4}$$

for all $a,b,c,d \in X$. It is known that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality (see [2]).

Let C be a nonempty subset of a complete CAT(0) space X. Then a mapping $T : C \to X$ is called k-demicontractive mapping if $F(T) \neq \emptyset$ and there exist $k \in [0,1)$ such that

$$d^2(Tx,p) \leq d(x,p) + kd^2(x,Tx), \text{ for all } x \in X \text{ and } p \in F(T). \tag{1.5}$$

Using (1.3) and (1.5), Aremu et al. [1] defined demimetric mapping in CAT(0) space as follows: Let C be a nonempty subset of a complete CAT(0) space X. Then a mapping $T : C \to X$ is called k-demimetric mapping if $F(T) \neq \emptyset$ there exist $k \in (-\infty,1)$ such that

$$\langle \overrightarrow{xp}, \overrightarrow{xTx} \rangle \geq \frac{1-k}{2} d^2(x,Tx), \text{ for all } x \in X \text{ and } p \in F(T). \tag{1.6}$$

Furthermore, $T : C \to X$ is said to be generalized hybrid mapping, if there exists $\alpha, \beta \in \mathbb{R}$ such that for all $x,y \in C$

$$\alpha d^2(Tx,Ty) + (1-\alpha)d^2(x,Ty) \leq \beta d^2(Tx,y) + (1-\beta)d^2(x,y). \tag{1.7}$$

If $F(T) \neq \emptyset$, then for any $p \in F(T)$ and $x \in C$ from (1.7), we obtain
$d^2(Tx,p) \leq d^2(x,p)$, which implies from (1.3) that

$$\langle \overrightarrow{xp}, \overrightarrow{xTx} \rangle \geq \frac{1-0}{2} d^2(x,Tx). \tag{1.8}$$

Hence, every generalized hybrid mapping T on C with $F(T) \neq \emptyset$ is $0-$demimetric mapping.
Also, a mapping $T : C \to H$ is said to be firmly nonexpansive if

$$d^2(Tx,Ty) \leq \langle \overrightarrow{xy}, \overrightarrow{TxTy} \rangle, \text{ for all } x,y \in C \tag{1.9}$$

and if $F(T) \neq \emptyset$, then for any $p \in F(T)$ and $x \in C$, from (1.9), we obtain

$$d^2(Tx,p) \leq \langle \overrightarrow{xp}, \overrightarrow{Txp} \rangle, \text{ for all } x,y \in C. \tag{1.10}$$

It follows from (1.10) and properties of quasilinearization that

$$\langle \overrightarrow{xp}, \overrightarrow{xTx} \rangle \geq \frac{1-(-1)}{2} d^2(x,Tx), \tag{1.11}$$

(see [1] for more details). Hence, every firmly nonexpansive mapping T on C with $F(T) \neq \emptyset$ is $(-1)-$demimetric mapping.
Motivated by work of Takahashi et al. [18] and Song [16], we study the version of Lemma 1.3 in CAT(0) space for a finite family of demimetric mappings. Furthermore, we study a modified

Halpern-type algorithm for approximating a common fixed point of demimetric mappings and prove a strong convergence theorem for finding a common element of the set of common fixed points for a finite family of these demimetric mappings in a complete CAT(0) space.

2. Preliminaries

Lemma 2.1 [10] Let X be a CAT(0) space, $x,y,z \in X$ and $\lambda \in [0,1]$. Then

(i) $d(\lambda x \oplus (1-\lambda)y, z) \leq \lambda d(x,z) + (1-\lambda)d(y,z)$;

(ii) $d^2(\lambda x \oplus (1-\lambda)y, z) \leq \lambda d^2(x,z) + (1-\lambda)d^2(y,z) - \lambda(1-\lambda)d^2(x,y)$.

Lemma 2.2 [20] Let X be a CAT(0) space. Then for all $u,x,y \in X$, the following inequality hold:

$$d^2(x,u) \leq d^2(y,u) + 2\langle \overrightarrow{xy}, \overrightarrow{xu} \rangle.$$

Lemma 2.3 [20] Let X be a CAT(0) space. For any $u,v, \in X$ and $t \in (0,1)$, let $u_t = tu \oplus (1-t)v$. Then for all $x,y \in X$,

(i) $\langle \overrightarrow{u_t x}, \overrightarrow{u_t y} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{u_t y} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{u_t y} \rangle$;

(ii) $\langle \overrightarrow{u_t x}, \overrightarrow{uy} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{uy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{uy} \rangle$
and $\langle \overrightarrow{u_t x}, \overrightarrow{vy} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{vy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{vy} \rangle$.

In other to write a finite convex combination of elements in CAT(0) space, Dhompongsa et al. [7] introduced the following notation in CAT(0) space: Let $\{x_i : i = 1,2,\ldots,N\}$ be points in a CAT(0) space X and $\alpha_1, \alpha_2, \ldots, \alpha_N \in (0,1)$ with $\sum_{i=1}^N \alpha_i = 1$, then

$$\begin{aligned} \bigoplus_{i=1}^{N} \alpha_i x_i &:= (1-\alpha_N)\Big(\frac{\alpha_1}{1-\alpha_N}x_1 \oplus \frac{\alpha_2}{1-\alpha_N}x_2 \oplus \cdots \oplus \frac{\alpha_{N-1}}{1-\alpha_N}x_{N-1}\Big) \oplus \alpha_N x_N \\ &= (1-\alpha_N)\bigoplus_{i=1}^{N-1} \frac{\alpha_i}{1-\alpha_N}x_i \oplus \alpha_N x_N. \end{aligned} \tag{2.1}$$

Lemma 2.4 [6] Let C be a nonempty, closed and convex subset of CAT(0) space X. Let $\{x_i : i = 1,2,\ldots,N\}$ be in C, and $\alpha_1, \alpha_2, \ldots, \alpha_N \in (0,1)$ such that $\sum_{i=1}^N \alpha_i = 1$. Then the following inequalities hold:

(i) $d\Big(z, \bigoplus_{i=1}^N \alpha_i x_i\Big) \leq \sum_{i=1}^N \alpha_i d(z,x_i)$, for all $z \in C$.

(ii) $d^2\left(z, \bigoplus_{i=1}^N \alpha_i x_i\right) \leq \sum_{i=1}^N \alpha_i d^2(z,x_i) - \sum_{i,j=1,i\neq j}^N \alpha_i\alpha_j d^2(x_i,x_j)$, for all $z \in C$.

Let $\{x_n\}$ be a bounded sequence in a complete CAT(0) space X. For $x \in X$, we set

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\}.$$

It is well known that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point see (Proposition 7 of [9]).

Lemma 2.5 [12] Every bounded sequence in a complete CAT(0) space always has a $\triangle-$convergent subsequence.

Lemma 2.6 [8] If C is a nonempty, closed and convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.

Lemma 2.7 [15] If C is a nonempty, closed and convex subset of a complete CAT(0) space X and $\{x_n\}$ be a bounded sequence in C. Then $\triangle - \lim_{n\to\infty} x_n = p$ implies that $\{x_n\} \rightharpoonup p$.

Lemma 2.8 [11] Let X be a complete CAT(0) space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ $\triangle-$converges to x if and only if $\limsup_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle \leq 0$ for all $y \in C$.

Lemma 2.9 [1] Let X be a CAT(0) space and $T : X \to X$ be a k-demimetric mapping with $k \in (-\infty,\lambda)$ and $\lambda \in (0,1)$ such that $F(T) \neq \emptyset$. Suppose that $T_\lambda x := (1-\lambda)x \oplus \lambda Tx$. Then T_λ is quasi-nonexpansive mapping and $F(T_\lambda) = F(T)$.

Lemma 2.10 [14] Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$.

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

Lemma 2.11 (Xu, [19]) Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, n \geq 0,$$

where, (i) $\{\alpha_n\} \subset [0,1]$, $\sum \alpha_n = \infty$; (ii) $\lim sup\ \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$; $(n \geq 0)$, $\sum \gamma_n < \infty$. Then, $a_n \to 0$ as $n \to \infty$.

3. Main results

Lemma 3.1. Let C be a nonempty, convex subset of CAT(0) space X. Let $\{u_i : i = 1, 2, \ldots, N\} \subset C$, and $\alpha_1, \alpha_2, \ldots, \alpha_N \in (0,1)$ such that $\sum_{i=1}^N \alpha_i = 1$. Then the following inequalities hold:

$$\begin{aligned} \langle \overrightarrow{\bigoplus_{i=1}^N \alpha_i u_i x}, \overrightarrow{xy} \rangle &\leq \sum_{i=1}^N \alpha_i \langle \overrightarrow{u_i x}, \overrightarrow{xy} \rangle + \frac{1}{2}\Big(\sum_{i=1}^N \alpha_i d^2(u_i, x) - d^2(\bigoplus_{i=1}^N \alpha_i u_i, x) \Big) \\ &\leq \sum_{i=1}^N \alpha_i \langle \overrightarrow{u_i x}, \overrightarrow{xy} \rangle + \frac{1}{2} \sum_{i=1}^N \alpha_i d^2(u_i, x). \end{aligned} \tag{3.2}$$

Proof. From (1.3) and Lemma 2.4, we obtain

$$\begin{aligned} 2\langle \overrightarrow{\bigoplus_{i=1}^N \alpha_i u_i x}, \overrightarrow{xy} \rangle &= d^2(\bigoplus_{i=1}^N \alpha_i u_i, y) + d^2(x,x) - d^2(\bigoplus_{i=1}^N \alpha_i u_i, x) - d^2(x,y) \\ &\leq \sum_{i=1}^N \alpha_i d^2(u, y) - \sum_{i,j=1, i\neq j}^N \alpha_i \alpha_j d^2(u_i, u_j) \\ &- d^2(\bigoplus_{i=1}^N \alpha_i u_i, x) - d^2(x,y) \\ &= \sum_{i=1}^N \alpha_i [d^2(u_i, y) - d^2(u_i, x) - d^2(x,y)] \\ &- \sum_{i,j=1, i\neq j}^N \alpha_i \alpha_j d^2(u_i, u_j) - d^2(\bigoplus_{i=1}^N \alpha_i u_i, x) \\ &\leq 2\sum_{i=1}^N \alpha_i \langle \overrightarrow{u_i x}, \overrightarrow{xy} \rangle + \sum_{i=1}^N \alpha_i d^2(u_i, x) - d^2(\bigoplus_{i=1}^N \alpha_i u_i, x) \end{aligned}$$

therefore

$$\langle\overrightarrow{\bigoplus_{i=1}^{N}\alpha_i u_i x, \overrightarrow{xy}}\rangle \leq \sum_{i=1}^{N}\alpha_i\langle\overrightarrow{u_i x},\overrightarrow{xy}\rangle+\frac{1}{2}\Big(\sum_{i=1}^{N}\alpha_i d^2(u_i,x)-d^2(\bigoplus_{i=1}^{N}\alpha_i u_i,x)\Big)$$
$$\leq \sum_{i=1}^{N}\alpha_i\langle\overrightarrow{u_i x},\overrightarrow{xy}\rangle+\frac{1}{2}\sum_{i=1}^{N}\alpha_i d^2(u_i,x).$$

Lemma 3.2. Let X be a CAT(0) space and C a nonempty convex subset of X. Assume that $\{T_i\}_{i=1}^{N} : C \to X$ is a finite family of $k_i - demimetric$ mappings with $k_i \in (-\infty, 1)$ for each $i \in \{1,2,\ldots,N\}$ such that $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{\alpha_i\}_{i=1}^{N}$ be a positive sequence with $\sum_{i=1}^{N}\alpha_i = 1$. Then $\bigoplus_{i=1}^{N}\alpha_i T_i : C \to X$ is a k-demimetric mapping if $k := \max\{k_i : i = 1,2,\ldots,N\} \leq 0$ and $F(\bigoplus_{i=1}^{N}\alpha_i T_i) = \bigcap_{i=1}^{N} F(T_i)$.

Proof. Let $x \in C$ and $W_N x := \bigoplus_{i=1}^{N}\alpha_i T_i x$, with $\sum_{i=1}^{N}\alpha_i = 1$. For any $p \in \bigcap_{i=1}^{N} F(T_i)$, from Lemma 3.1, (1.6) and Lemma 2.4, we obtain

$$\begin{aligned}
\langle\overrightarrow{xW_N x},\overrightarrow{xp}\rangle &= -\langle\overrightarrow{W_N x x},\overrightarrow{xp}\rangle\\
&\geq -\Big(\sum_{i=1}^{N}\alpha_i\langle\overrightarrow{T_i x x},\overrightarrow{xp}\rangle+\frac{1}{2}\sum_{i=1}^{N}\alpha_i d^2(T_i x,x)-\frac{1}{2}d^2(W_N x,x)\Big)\\
&\geq \sum_{i=1}^{N}\alpha_i\langle\overrightarrow{xT_i x},\overrightarrow{xp}\rangle-\frac{1}{2}\sum_{i=1}^{N}\alpha_i d^2(T_i x,x)+\frac{1}{2}d^2(W_N x,x)\\
&\geq \sum_{i=1}^{N}\alpha_i\frac{1-k_i}{2}d^2(x,T_i x)-\frac{1}{2}\sum_{i=1}^{N}\alpha_i d^2(T_i x,x)+\frac{1}{2}d^2(W_N x,x)\\
&\geq \sum_{i=1}^{N}\alpha_i\frac{1-k}{2}d^2(x,T_i x)-\frac{1}{2}\sum_{i=1}^{N}\alpha_i d^2(T_i x,x)+\frac{1}{2}d^2(W_N x,x)\\
&= \frac{-k}{2}\sum_{i=1}^{N}\alpha_i d^2(T_i x,x)+\frac{1}{2}d^2(W_N x,x)\\
&\geq \frac{-k}{2}d^2(W_N x,x)+\frac{1}{2}d^2(W_N x,x)\\
&= \frac{1-k}{2}d^2(W_N x,x).
\end{aligned}$$

Therefore

$$\langle\overrightarrow{xW_N x},\overrightarrow{xp}\rangle \geq \frac{1-k}{2}d^2(W_N x,x).$$

Hence, W_N is a $k-$demimetric mapping with $k := \max\{k_i : i = 1,2,\ldots,N\} \leq 0$.

Next, we show that $F(W_N) = \bigcap_{i=1}^{N} F(T_i)$. Let $x = W_N x$, it suffices to show that $x \in \bigcap_{i=1}^{N} F(T_i)$.

Then, for any $p \in \bigcap_{i=1}^{N} F(T_i)$, from Lemma 3.1 and (1.6), we obtain

$$\begin{aligned} d^2(x,p) &= \langle \overrightarrow{px}, \overrightarrow{xp} \rangle = \langle \overrightarrow{W_N x x}, \overrightarrow{xp} \rangle \\ &= \langle \overrightarrow{W_N x x}, \overrightarrow{xp} \rangle + \langle \overrightarrow{xp}, \overrightarrow{xp} \rangle \\ &\leq d^2(x,p) + \sum_{i=1}^{N} \alpha_i \langle \overrightarrow{T_i x x}, \overrightarrow{xp} \rangle + \frac{1}{2} \sum_{i=1}^{N} \alpha_i d^2(T_i x, x) - \frac{1}{2} d^2(W_N x, x) \\ &\leq d^2(x,p) + \sum_{i=1}^{N} \alpha_i \frac{k_i - 1}{2} d^2(T_i x, x) + \frac{1}{2} \sum_{i=1}^{N} \alpha_i d^2(T_i x, x) \\ &\leq d^2(x,p) + \sum_{i=1}^{N} \alpha_i \frac{k_i}{2} d^2(T_i x, x) \\ &\leq d^2(x,p) + \frac{k}{2} \sum_{i=1}^{N} \alpha_i d^2(T_i x, x). \end{aligned}$$

Therefore

$$\frac{-k}{2} \sum_{i=1}^{N} \alpha_i d^2(T_i x, x) \leq 0.$$

Since, $k = \max\{k_i : 1 \leq i \leq N\} \leq 0$, we obtain that $x = T_i x$ for each $i \in \{1,2,\ldots,N\}$. Hence $x \in \bigcap_{i=1}^{N} F(T_i)$.

Theorem 3.3. *Let X be a complete CAT(0) space and let C be a nonempty, closed and convex subset of X. Let $\{T_i\}_{i=1}^{N} : C \to X$ be a finite family of k_i-demimetric mapping and $\triangle-$demiclosed at 0 with $k_i \in (-\infty, 1)$ for each $i \in \{1,2,\ldots,N\}$ and $k = \max\{k_i : 1 \leq i \leq N\} \leq 0$. Assume $\Gamma := \bigcap_{i=1}^{N} F(T_i)$ is nonempty and $u \in C$ is fixed, let $\{\alpha_i\}$ for each $i \in \{1,2,\ldots,N\}$ and $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $(0,1)$ and suppose that the following conditions are satisfied:*

- (C1) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;
- (C2) $\sum_{i=1}^{N} \alpha_i = 1$;
- (C3) $\gamma_n \in [a,b]$ *for all* $n \geq 1$ *and for some* $a,b \in (0,1)$.

For some fixed $\delta \in (0,1)$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively by chosen $x_1 \in C$ arbitrarily and

$$\begin{cases} z_n = (1-\gamma_n)x_n \oplus \gamma_n \bigoplus_{i=1}^{N} \alpha_i T_i x_n; \\ x_{n+1} = \beta_n u \oplus (1-\delta)(1-\beta_n)x_n \oplus \delta(1-\beta_n) z_n, \ n \geq 1. \end{cases} \tag{3.2}$$

Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to a point in Γ.

Proof. Let $W_N := \bigoplus_{i=1}^N \alpha_i T_i$, from Lemma 3.2, W_N is $k-$demimetric mapping and $V_N := (1-\gamma_n)I \oplus \gamma_n W_N$. Then from Lemma 2.9, V_N is quasi-nonexpansive mapping and $F(V_N) = F(W_N) = \bigcap_{i=1}^N F(T_i)$. Therefore, for any $p \in \Gamma$, from (3.2) and Lemma 2.4, we obtain

$$\begin{aligned} d(x_{n+1}, p) &= d(\beta_n u \oplus (1-\delta)(1-\beta_n)x_n \oplus \delta(1-\beta_n)z_n, p) \\ &\leq \beta_n d(u,p) + (1-\delta)(1-\alpha_n)d(x_n,p) + \delta(1-\beta_n)d(z_n,p) \\ &= \beta_n d(u,p) + (1-\delta)(1-\alpha_n)d(x_n,p) + \delta(1-\beta_n)d(V_N x_n,p) \\ &\leq \beta_n d(u,p) + (1-\delta)(1-\alpha_n)d(x_n,p) + \delta(1-\beta_n)d(x_n,p) \\ &= \beta_n d(u,p) + (1-\beta_n)d(x_n,p) \\ &\leq \max\{d(u,p), d(x_n,p)\}. \end{aligned}$$

By induction, we obtain

$$d(x_n, p) \leq \max\{d(u,p), d(x_1,p)\}.$$

Therefore, $\{x_n\}$ is bounded, hence $\{z_n\}$ and $\{T_i x_n\}$ are bounded for each $i \in \{1,2,\ldots,N\}$. Now, from Lemma 2.4, we obtain

$$\begin{aligned} d^2(x_{n+1}, p) &\leq \beta_n d^2(u,p) + (1-\delta)(1-\beta_n)d^2(x_n,p) \\ &+ \delta(1-\beta_n)d^2(z_n,p) - \delta(1-\delta)(1-\beta_n)^2 d^2(x_n,z_n) \\ &\leq \beta_n d^2(u,p) + (1-\delta)(1-\beta_n)d^2(x_n,p) \\ &+ \delta(1-\beta_n)d^2(x_n,p) - \delta(1-\delta)(1-\beta_n)^2 d^2(x_n,z_n) \\ &\leq \beta_n d^2(u,p) + (1-\beta_n)d^2(x_n,p) \\ &- \delta(1-\delta)(1-\beta_n)^2 d^2(x_n,z_n). \end{aligned}$$

Therefore

$$\delta(1-\delta)(1-\beta_n)^2 d^2(x_n,z_n) \leq d^2(x_n,p) - d^2(x_{n+1},p) + \beta_n d^2(u,p).$$

Since $\delta(1-\delta)(1-\beta_n)^2 > 0$, then

$$d^2(x_n,z_n) \leq d^2(x_n,p) - d^2(x_{n+1},p) + \beta_n d^2(u,p). \tag{3.3}$$

Now, we will consider two cases to complete the proof.

Case 1: Assume that $\{d^2(x_n,p)\}_{n=1}^{\infty}$ is a non-increasing sequence of real numbers. Since $\{x_n\}$ is bounded, then $\lim_{n\to\infty} d^2(x_n,p)$ exists and from (3.3) and (C1), we obtain

$$\lim_{n\to\infty} d(x_n,z_n)=0. \tag{3.4}$$

Also from (3.2), we obtain

$$d(x_{n+1},x_n)\le \beta_n d(u,x_n)+(1-\delta)(1-\beta_n)d(x_n,x_n)+\delta(1-\beta_n)d(z_n,x_n)$$

hence, it follows from (C1) and (3.4) that

$$\lim_{n\to\infty} d(x_{n+1},x_n)=0. \tag{3.5}$$

Furthermore, since T_i is k_i-demimetric mapping for each $i\in\{1,2,\dots,N\}$ with $k=\max\{k_i\}\le 0$, then from (3.2), Lemma 2.3, 3.1 and (1.6), for any $p\in\Gamma$, we obtain

$$\begin{aligned}
\langle\overrightarrow{x_nz_n},\overrightarrow{x_np}\rangle &= -\langle\overrightarrow{z_nx_n},\overrightarrow{x_np}\rangle\\
&= -\langle\overrightarrow{((1-\gamma_n)x_n\oplus\gamma_nW_Nx_n)x_n},\overrightarrow{x_np}\rangle\\
&\ge -(1-\gamma_n)\langle\overrightarrow{x_nx_n},\overrightarrow{x_np}\rangle-\gamma_n\langle\overrightarrow{W_Nx_nx_n},\overrightarrow{x_np}\rangle\\
&\ge -\gamma_n\langle\overrightarrow{W_Nx_nx_n},\overrightarrow{x_np}\rangle\\
&= -\gamma_n\langle\overrightarrow{\bigoplus_{i=1}^{N}\alpha_iT_ix_nx_n},\overrightarrow{x_np}\rangle\\
&\ge -\gamma_n\sum_{i=1}^{N}\alpha_i\langle\overrightarrow{T_ix_nx_n},\overrightarrow{x_np}\rangle-\frac{1}{2}\gamma_n\sum_{i=1}^{N}\alpha_id^2(T_ix_n,x_n)\\
&\ge \gamma_n\sum_{i=1}^{N}\frac{1-k_i}{2}\alpha_id^2(T_ix_n,x_n)-\frac{1}{2}\gamma_n\sum_{i=1}^{N}\alpha_id^2(T_ix_n,x_n)\\
&= \frac{-k_i}{2}\gamma_n\sum_{i=1}^{N}\alpha_id^2(T_ix_n,x_n)\\
&\ge \frac{-k}{2}\gamma_n\sum_{i=1}^{N}\alpha_id^2(T_ix_n,x_n).
\end{aligned}$$

Therefore

$$\frac{-k}{2}\gamma_n\sum_{i=1}^{N}\alpha_i d^2(T_ix_n,x_n)\le\langle\overrightarrow{x_nz_n},\overrightarrow{x_np}\rangle\le d(x_n,z_n)d(x_n,p) \tag{3.6}$$

since $\{x_n\}$ is bounded, $k\le 0$, and $\gamma_n,\alpha_i\in(0,1)$ for all $n\ge 1$ and $i\in\{1,2,\dots,N\}$, then from (3.4) and (3.6), we obtain

$$\lim_{n\to\infty}d(T_ix_n,x_n)=0,\qquad \text{for each}\quad i\in\{1,2,\dots,N\}. \tag{3.7}$$

Since $\{x_n\}$ is bounded and X is a complete CAT(0) space, then from Lemma 2.5, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\triangle-\lim\ x_{n_j}=z\in X$. From (3.7) and fact that T_i is $\triangle-$demiclosed at 0 for each $i\in\{1,2,\dots,N\}$, we obtain $z\in\Gamma$ and from Lemma 2.8, we have

$$\limsup_{n\to\infty}\langle\overrightarrow{uz},\overrightarrow{x_nz}\rangle\le 0. \tag{3.8}$$

Furthermore, from (3.5) and (3.8), we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle\overrightarrow{uz},\overrightarrow{x_{n+1}z}\rangle&=\limsup_{n\to\infty}\langle\overrightarrow{uz},\overrightarrow{x_{n+1}x_n}\rangle+\limsup_{n\to\infty}\langle\overrightarrow{uz},\overrightarrow{x_nz}\rangle\\&\le d(u,z)d(x_{n+1},x_n)+\limsup_{n\to\infty}\langle\overrightarrow{uz},\overrightarrow{x_nz}\rangle\le 0.\end{aligned} \tag{3.9}$$

Finally, we show that $x_n\to z$. Let $y_n:=\beta_nz\oplus(1-\delta)(1-\beta_n)x_n\oplus\delta(1-\beta_n)z_n$, then from Lemma 2.2, 2.3 and Lemma 2.4, we obtain

$$\begin{aligned}d^2(x_{n+1},z)&\le d^2(y_n,z)+2\langle\overrightarrow{x_{n+1}y_n},\overrightarrow{x_{n+1}z}\rangle\\&\le(1-\delta)(1-\beta_n)d^2(x_n,z)+\delta(1-\beta_n)d^2(z_n,z)+2\langle\overrightarrow{x_{n+1}y_n},\overrightarrow{x_{n+1}z}\rangle\\&\le(1-\beta_n)d^2(x_n,z)+2\Big(\beta_n\langle\overrightarrow{uy_n},\overrightarrow{x_{n+1}z}\rangle+(1-\delta)(1-\beta_n)\langle\overrightarrow{x_ny_n},\overrightarrow{x_{n+1}z}\rangle\\&+\delta(1-\beta_n)\langle\overrightarrow{z_ny_n},\overrightarrow{x_{n+1}z}\rangle\Big)\\&\le(1-\beta_n)d^2(x_n,z)+2\beta_n\Big(\beta_n\langle\overrightarrow{uz},\overrightarrow{x_{n+1}z}\rangle+(1-\delta)(1-\beta_n)\langle\overrightarrow{ux_n},\overrightarrow{x_{n+1}z}\rangle\\&+\delta(1-\beta_n)\langle\overrightarrow{uz_n},\overrightarrow{x_{n+1}z}\rangle\Big)+2(1-\delta)(1-\beta_n)\Big(\beta_n\langle\overrightarrow{x_nz},\overrightarrow{x_{n+1}z}\rangle\\&+(1-\delta)(1-\beta_n)\langle\overrightarrow{x_nx_n},\overrightarrow{x_{n+1}z}\rangle+\delta(1-\beta_n)\langle\overrightarrow{x_nz_n},\overrightarrow{x_{n+1}z}\rangle\Big)\end{aligned}$$

$$+2\delta(1-\beta_n)\Big(\beta_n\langle\overrightarrow{z_nz},\overrightarrow{x_{n+1}z}\rangle+(1-\delta)(1-\beta_n)\langle\overrightarrow{z_nx_n},\overrightarrow{x_{n+1}z}\rangle$$

$$+\delta(1-\beta_n)\langle\overrightarrow{z_nz_n},\overrightarrow{x_{n+1}z}\rangle\Big)$$

$$=(1-\beta_n)d^2(x_n,z)+2\beta_n^2\langle\overrightarrow{uz},\overrightarrow{x_{n+1}z}\rangle+2(1-\delta)(1-\beta_n)\beta_n\langle\overrightarrow{ux_n},\overrightarrow{x_{n+1}z}\rangle$$

$$+2\delta(1-\beta_n)\beta_n\langle\overrightarrow{uz_n},\overrightarrow{x_{n+1}z}\rangle$$

$$+2(1-\delta)(1-\beta_n)\beta_n\langle\overrightarrow{x_nz},\overrightarrow{x_{n+1}z}\rangle+2\delta(1-\delta)(1-\beta_n)^2\langle\overrightarrow{x_nz_n},\overrightarrow{x_{n+1}z}\rangle$$

$$(1-\beta_n)\beta_n\langle\overrightarrow{z_nz},\overrightarrow{x_{n+1}z}\rangle-2\delta(1-\delta)(1-\beta_n)^2\langle\overrightarrow{x_nz_n},\overrightarrow{x_{n+1}z}\rangle$$

$$=(1-\beta_n)d^2(x_n,z)+2\beta_n^2\langle\overrightarrow{uz},\overrightarrow{x_{n+1}z}\rangle+2(1-\delta)(1-\beta_n)\beta_n[\langle\overrightarrow{uz},\overrightarrow{x_{n+1}z}\rangle$$

$$+\langle\overrightarrow{zx_n},\overrightarrow{x_{n+1}z}\rangle]+2\delta\beta_n(1-\beta_n)[\langle\overrightarrow{uz},\overrightarrow{x_{n+1}z}\rangle+\langle\overrightarrow{zz_n},\overrightarrow{x_{n+1}z}\rangle]$$

$$-2(1-\delta)(1-\beta_n)\beta_n\langle\overrightarrow{zx_n},\overrightarrow{x_{n+1}z}\rangle-2\delta\beta_n\langle\overrightarrow{zz_n},\overrightarrow{x_{n+1}z}\rangle$$

$$=(1-\beta_n)d^2(x_n,z)+2\beta_n\langle\overrightarrow{uz},\overrightarrow{x_{n+1}z}\rangle.$$

Therefore

$$d^2(x_{n+1},z)\le(1-\beta_n)d^2(x_n,z)+2\beta_n\langle\overrightarrow{uz},\overrightarrow{x_{n+1}z}\rangle. \tag{3.10}$$

It follows from (3.9) and Lemma 2.11 that $d(x_n,z)\to 0$ as $n\to\infty$, that is $x_n\to z$ as $n\to\infty$.

Case 2: Assume that $\{d^2(x_n,z)\}_{n=1}^{\infty}$ is non-decreasing sequence. Now, there exists a subsequence n_j of $\{n\}$ such that

$$d(x_j,z)<d(x_{n_j+1})$$

for all $j\in\mathbb{N}$ by Lemma 2.10, there exists an increasing sequence $\{m_\tau\}_{\tau\ge 1}$ such that $m_\tau\to\infty$, $d(x_{m_\tau},z)\le d(x_{m_\tau+1},z)$ and $d(x_\tau,z)\le d(x_{m_\tau+1},z)$ for all $\tau\ge 1$. Also from (3.3), we have

$$d^2(x_{m_\tau},z_{m_\tau})\le d^2(x_{m_\tau},z)-d^2(x_{m_\tau+1},z)+\beta_{m_\tau}d^2(u,z)$$

using the fact that $\beta_{m_\tau}\to\infty$, we obtain $d(x_{m_\tau},x_{m_\tau})\to 0$ as $\tau\to\infty$. Thus as in Case 1, we obtain $d(x_{m_\tau},T_ix_{m_\tau})\to 0$ as $\tau\to\infty$ for each $i\in\{1,2,\dots,N\}$. Following arguments similar to those in the proof of Case 1, we get $\limsup\langle\overrightarrow{uz},\overrightarrow{x_{m_\tau+1}z}\rangle\le 0$. Also from from (3.10), we obtain

$$d^2(x_{m_\tau+1},z)\le(1-\beta_{m_\tau})d^2(x_{m_\tau},z)+2\beta_{m_\tau}\langle\overrightarrow{uz},\overrightarrow{x_{m_\tau+1}z}\rangle \tag{3.11}$$

it follows that

$$\beta_{m_\tau}d^2(x_{m_\tau},z)\le d^2(x_{m_\tau},z)-d^2(x_{m_\tau+1})+2\beta_{m_\tau}\langle\overrightarrow{uz},\overrightarrow{x_{m_\tau+1}z}\rangle.$$

Since $d^2(x_{m_\tau},z) \leq d^2(x_{m_\tau+1})$ and $\beta_{m_\tau} > 0$, then

$$d^2(x_{m_\tau},z) \leq 2\langle \overrightarrow{uz}, \overrightarrow{x_{m_\tau+1}z}\rangle.$$

Using $\limsup\langle \overrightarrow{uz}, \overrightarrow{x_{m_\tau+1}z}\rangle \leq 0$, we obtain $d(x_{m_\tau},z) \to 0$ as $\tau \to \infty$. So from (3.11) , we have $d(x_{m_\tau+1},z) \to 0$. But $d(x_\tau,z) \leq d(x_{m_\tau+1})$, for all $\tau \geq 0$. Thus, we obtain $x_\tau \to z$ as $\tau \to \infty$.

This completes the proof.

Corollary 3.4. *Let X be a complete CAT(0) space and let C be a nonempty, closed and convex subset of X. Let $\{T_i\}_{i=1}^N : C \to X$ be a finite family of generalized hybrid mapping and $\triangle-$demiclosed at 0 for each $i \in \{1,2,\ldots,N\}$. Assume $\Gamma := \bigcap_{i=1}^N F(T_i)$ is nonempty and $u \in C$ is fixed, let $\{\alpha_i\}$ for each $i \in \{1,2,\ldots,N\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ and suppose that the following conditions are satisfied:*

(C1) $\lim_{n\to\infty}\beta_n = 0$ *and* $\sum_{n=1}^\infty \beta_n = \infty$;
(C2) $\sum_{i=1}^N \alpha_i = 1$;

For some fixed $\delta \in (0,1)$, let $\{x_n\}_{n=1}^\infty$ be a sequence defined iteratively by chosen $x_1 \in C$ arbitrarily and

$$\begin{cases} z_n = \bigoplus_{i=1}^N \alpha_i T_i x_n; \\ x_{n+1} = \beta_n u \oplus (1-\delta)(1-\beta_n)x_n \oplus \delta(1-\beta_n)z_n,\ n \geq 1. \end{cases}$$

Then, $\{x_n\}_{n=1}^\infty$ converges strongly to a point in Γ.

Corollary 3.5. *Let X be a complete CAT(0) space and let C be a nonempty, closed and convex subset of X. Let $\{T_i\}_{i=1}^N : C \to X$ be a finite family of nonexpansive mapping and $\triangle-$demiclosed at 0 for each $i \in \{1,2,\ldots,N\}$. Assume $\Gamma := \bigcap_{i=1}^N F(T_i)$ is nonempty and $u \in C$ is fixed, let $\{\alpha_i\}$ for each $i \in \{1,2,\ldots,N\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ and suppose that the following conditions are satisfied:*

(C1) $\lim_{n\to\infty}\beta_n = 0$ *and* $\sum_{n=1}^\infty \beta_n = \infty$;
(C2) $\sum_{i=1}^N \alpha_i = 1$;

For some fixed $\delta \in (0,1)$, *let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence defined iteratively by chosen* $x_1 \in C$ *arbitrarily and*

$$\begin{cases} z_n = \bigoplus_{i=1}^{N} \alpha_i T_i x_n; \\ x_{n+1} = \beta_n u \oplus (1-\delta)(1-\beta_n)x_n \oplus \delta(1-\beta_n)z_n,\ n \geq 1. \end{cases}$$

Then, $\{x_n\}_{n=1}^{\infty}$ *converges strongly to a point in* Γ.

Conflict of Interests

The authors declare that there is no conflict of interests.

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